Handbook of Differential Equations

Ordinary Differential Equations

Volume 4

Edited by

Flaviano Battelli

Michal Fečkan

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VOLUME IV

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Preface

This book is the fourth volume in a series of the Handbook of Ordinary Differential Equations. This volume contains six contributions which are written by excellent mathematicians. We thank them for accepting our invitation to contribute to this volume and also for their effort and hard work on their papers. The scope of this volume is large. We hope that it will be interesting and useful for research, learning and teaching.

A brief survey of the volume follows. First, the contributions are presented in alphabetical authors' names. The paper by Balanov and Krawcewicz is devoted to the Hopf bifurcation occurring in dynamical systems admitting a certain group of symmetries. They use a so-called twisted equivariant degree method. Global symmetric Hopf bifurcation results are presented. Applications are given to several concrete problems. The contribution of Fabbri, Johnson and Zampogni lies in linear, nonautonomous, two-dimensional differential equation. For instance, they study the minimal subsets of the projective flow defined by these equations. They also discuss some recent developments in the spectral theory and inverse spectral theory of the classical Sturm–Liouville operator. The question of the genericity of the exponential dichotomy property is considered, as well, for cocycles generated by quasi-periodic, two-dimensional linear systems. The paper by Lailne is mainly devoted to considering growth and value distribution of meromorphic solutions of complex differential equations in the complex plane, as well as in the unit disc. Both linear and nonlinear equations are studied including algebraic differential equations in general and their relations to differential fields. A short presentation of algebroid solutions of complex differential equations is also given. The paper by Palmer deals with the existence of chaotic behaviour in the neighbourhood of a transversal periodic-to-periodic homoclinic orbit for autonomous ordinary differential equations. The concept of trichotomy is essential in this study. Also, a perturbation problem is considered when an unperturbed system has a nontransversal homoclinic orbit. Then it is shown that a perturbed system has a transversal orbit nearby provided that a certain Melnikov function has a simple zero. The contribution by A. Rontó and M. Miklós investigates the solvability and the approximate construction of solutions of certain types of regular nonlinear boundary value problems for systems of ordinary differential equations on a compact interval. Several types of problems are considered including periodic and multi-point problems. Parametrized and symmetric systems are considered as well. Most of theoretical results are illustrated by examples. Some historical remarks concerning the development and application of the method are presented. Finally, the paper by Żołądek is devoted to the local theory of analytic differential equations. Classification of linear meromorphic systems near regular and irregular singular point is described. Also, a local theory of nonlinear holomorphic equations is presented. Next, for-

mal classification of nilpotent singularities is given and analyticity of the Takens prenormal form is proved.

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CHAPTER 1

Symmetric Hopf Bifurcation: Twisted Degree Approach

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1. Introduction

Subject and goal

As it is clear from the title,

(i) the *subject* of this paper is the Hopf bifurcation occurring in dynamical systems admitting a certain group of symmetries;

(ii) the *method* to study the above phenomenon presented in this paper is based on the usage of the so-called twisted equivariant degree.

The *goal* of this paper is to explain why “(ii)” is an appropriate tool to attack “(i)”.

To get an idea of what the Hopf bifurcation is about, consider the simplest system of ODEs

$$\dot{x} = Ax \quad (x \in \mathbb{R}^2), \tag{1}$$

where $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator. Obviously, the origin is a stationary solution to (1), and it is a standard fact of any undergraduate course of ODEs that this system admits a non-constant periodic solution iff the characteristic equation $\det \Delta(\lambda) = 0$, where $\Delta(\lambda) := \lambda \operatorname{Id} - A$, has a pair of (conjugate) purely imaginary complex roots (in this case the origin is called a *center* for (1)).

Assume now that the system (1) is included into a one-parameter family of systems

$$\dot{x} = A(\alpha)x \quad (\alpha \in \mathbb{R},\ x \in \mathbb{R}^2), \tag{2}$$

where $A(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator (smoothly) depending on α, $A(\alpha_o) = A$ for some ‘critical’ value α_o and $(\alpha_o, 0)$ is an *isolated* center for (2) (i.e. it is the only center for α close to α_o). If a pair of complex roots λ of the characteristic equation crosses (for $\alpha = \alpha_o$) the imaginary axis, then the stationary solution $(\alpha, 0)$ changes its stability which results in appearance of non-constant periodic solutions. This phenomenon is called the Hopf bifurcation in (2). Similarly, one can speak about the occurrence of the Hopf bifurcation in a one-parameter family of n-dimensional linear systems of ODEs for $n > 2$ (does not matter that the corresponding purely imaginary roots may have multiplicities greater than one).

Next, consider a (nonlinear) autonomous system of ODEs of the type

$$\dot{x} = f(\alpha, x) \quad (\alpha \in \mathbb{R},\ x \in V := \mathbb{R}^n), \tag{3}$$

where $f : \mathbb{R} \oplus V \to V$ is a continuously differentiable function satisfying the condition that $f(\alpha, 0) = 0$ for all $\alpha \in \mathbb{R}$. Clearly, $(\alpha, 0)$ is a stationary solution to (3) for all α. In this situation, a change of stability simply means that some of the complex roots λ of the characteristic equation $\det \Delta_{(\alpha,0)}(\lambda) = 0$, where $\Delta_{(\alpha,0)}(\lambda) := \lambda \operatorname{Id} - D_x f(\alpha, 0)$, cross (for $\alpha = \alpha_o$) the imaginary axis. In particular, this means the existence of a purely imaginary characteristic root $i\beta_o$ for $\alpha = \alpha_o$, which (by the implicit function theorem) is a *necessary* condition for the Hopf bifurcation (i.e. for the appearance of non-constant small amplitude periodic solutions). However, simple examples show that, in contrast to the linear case,

this algebraic condition on the linearization is not enough, in general, for the occurrence of the Hopf bifurcation. Of course, this is not surprising: the classical Grobman–Hartman Theorem provides the local topological equivalence of an autonomous system to its linearization (near the origin) only under the assumption that the linearization matrix does not have eigenvalues on the imaginary axis. In particular, this means that studying the Hopf bifurcation phenomenon in parametrized families of *nonlinear* systems requires an additional *topological* argument.

A standard way of studying the Hopf bifurcation is the application of the Central Manifold theorem (allowing a two-dimensional reduction) and usage of the Poincaré section associated with the induced system (see [113] for a detailed exposition of this stream of ideas, see also [8,68]). However, this approach meets serious technical difficulties if the multiplicity of a purely imaginary characteristic root is greater than one. To overcome these and other technical difficulties, alternative methods were developed based on Lyapunov–Schmidt reduction, normal form techniques, integral averaging, etc. (cf. [33,34,64,111]). On the other hand, one should mention rational-valued homotopy invariants of "degree type" introduced by F.B. Fuller [57], E.N. Dancer [39] and E.N. Dancer and J.F. Toland [40, 42,41] as important tools to study the Hopf bifurcation phenomenon.

Observe that for many mathematical models of natural phenomena, very often, their closeness to the real world problems is reflected (on top of their non-linear character) in the presence of symmetries that are related to some physical or geometric regularities. For systems (3) this means: V is an (orthogonal) representation of a group Γ and f commutes with the Γ-action on V (i.e.

$$f(\alpha, \gamma x) = \gamma f(\alpha, x) \quad (\gamma \in \Gamma,\ x \in V), \tag{4}$$

in which case f is called Γ-equivariant (here Γ acts trivially on the parameter space)). In this way, we arrive at the following question: what is a link between symmetries of a system and symmetric properties of the actual dynamics? In the context relevant to our discussion, this question translates as the following *symmetric Hopf bifurcation problem*: how can one measure, predict and classify symmetric properties/minimal number of periodic solutions appearing as a result of the Hopf bifurcation?

It should be pointed out that in the symmetric setting, the characteristic roots almost always are not simple which causes significant difficulties for the application of the standard methods. To analyze the symmetric Hopf bifurcation problem, Golubitsky et al. (cf. [63, 65–67], see also [31,28,100,116,101,102,147]) suggested a method based on the Central Manifold/Lyapunov–Schmidt finite-dimensional reduction and further usage of a special singularity theory. On the other hand, if system (3) is Hamiltonian, one can use a wide spectrum of variational methods rooted in Morse theory/Lusternik–Schnirelman theory. Although very effective, these methods are not easy to use as they require a serious topological/analytical background. Also, when dealing with a concrete problem admitting a large group of symmetries, one would like to take advantage of using computer routines to handle a huge number of possible symmetry types of the bifurcating periodic solutions. From this viewpoint, it is not clear if the above methods are "open enough" to be computerized. It is our belief that the method presented in this paper is simple enough to be understood by applied mathematicians, and effective enough

(a) to be applied in a standard way to different types of symmetric dynamical systems,
(b) to provide a full topological information on the symmetric structure of the bifurcating branches of periodic solutions,
(c) to be transparent from the viewpoint of interpretation of its results, and
(d) "last but not least", to be completely computerized.

Topological degree approach

(i) *From Leray–Schauder degree to S^1-degree*. The Leray–Schauder degree theory proved itself as a powerful tool for the detection of single and multiple solutions in various types of differential equations. However, when dealing with the Hopf bifurcation phenomenon in autonomous systems through a functional analysis approach, it can only detect equilibria while it remains *blind* to non-constant periodic solutions. The reason for it can be easily explained: shifting the argument of periodic functions represents an S^1-action, which implies that finding periodic solutions to the associated operator equation constitutes an S^1-equivariant problem. By the obvious reason, the Leray–Schauder degree cannot "distinguish" between the zero-dimensional S^1-orbits (= equilibria) and the one-dimensional ones (= cycles), and one should look for a suitable S^1-equivariant homotopy invariant.

Speaking in a slightly more formal language, introduce the frequency β of the (unknown) periodic solution as an additional parameter and reformulate problem (3) as an operator equation in the first Sobolev space $W := H^1(S^1; V)$ as follows:

$$\mathfrak{F}(\alpha, \beta, u) = 0, \tag{5}$$

where $u \in W$, $\mathfrak{F} : \mathbb{R} \times \mathbb{R}_+ \times W \to W$ is given by

$$\mathfrak{F}(\alpha, \beta, u) := u - (L + K)^{-1}\left[Ku + \frac{1}{\beta} N_f\big(\alpha, j(u)\big)\right],$$

with $N_f(\alpha, u)(t) := f(\alpha, u(t))$, $K(u) := \frac{1}{2\pi}\int_0^{2\pi} u(s)\, \mathrm{d}s$ and $j : H^1(S^1; V) \to C(S^1, V)$ being a natural embedding into the space of continuous functions. Formula

$$(\mathrm{e}^{\mathrm{i}\tau} u)(t) := u(t + \tau) \quad (\mathrm{e}^{\mathrm{i}\tau} \in S^1,\ u \in W) \tag{6}$$

equips W with a structure of Hilbert S^1-representation and, moreover, $\mathfrak{F}$ is S^1-equivariant. Take $u \in W$, put

$$G_u := \{z \in S^1 \colon\ \mathrm{e}^{\mathrm{i}\tau} u = u\} \tag{7}$$

and call it a *symmetry* of u (commonly called the *isotropy* of u). It is easy to see that equilibria for (3) have the whole group S^1 as their symmetry, while for a non-constant periodic solution u, one has $G_u = \mathbb{Z}_l$ for some $l = 1, 2, \ldots$. Observe, by the way, that the presence of symmetry $\mathbb{Z}_l$ for a periodic solution u has a transparent geometric meaning: it clearly indicates that u is "l-folded". These simple observations suggest a natural candidate

for the range of values of the “right” invariant responsible for the *existence* of non-constant periodic solutions (in particular, for the occurrence of (non-symmetric) Hopf bifurcation) – it should take its values in the free $\mathbb{Z}$-module generated by $\mathbb{Z}_l$, $l = 1, 2, \dots$, rather than in the ring $\mathbb{Z}$ (as the Leray–Schauder degree does). The corresponding construction (called S^1-degree) was suggested in [45,46] (see also Subsections 3.5 and 3.6 of the present paper for an axiomatic approach and [79,81] for a more general setting).

(ii) *From S^1-degree to twisted degree*. Let us make one more step assuming system (3) to be Γ-symmetric (cf. (4)). Put $G := \Gamma \times S^1$. Then, W is a Hilbert G-representation with the G-action given by (cf. (6))

$$\big((\gamma, e^{i\tau})u\big)(t) := \gamma u(t+\tau) \quad (\gamma \in \Gamma,\ e^{i\tau} \in S^1,\ u \in W).$$

Moreover, this time $\mathfrak{F}$ is G-equivariant. One can easily verify that the symmetries $G_u := \{g \in G\colon\ gu = u\}$ of non-constant periodic functions u are the so-called *φ-twisted l-folded* subgroups $K^{\varphi,l}$ of G given by

$$K^{\varphi,l} := \big\{(\gamma, z) \in K \times S^1\colon\ \varphi(\gamma) = z^l\big\}$$

for K being a subgroup of Γ, $\varphi : K \to S^1$ a homomorphism and $l = 1, 2, \dots$ (cf. (7)). We have now a complete parallelism with the previous situation: in the same way as the Leray–Schauder degree was not enough to establish the *existence* of non-constant periodic solutions to (3), the S^1-degree is not enough to *classify symmetries* of these solutions. Clearly, the right G-equivariant homotopy invariant should take its values in the $\mathbb{Z}$-module $A_1^t(G)$ generated by φ-twisted l-folded subgroups (more precisely, by their conjugacy classes). The *twisted G-equivariant degree* (in short, twisted degree) is a topological tool precisely needed for the above purpose.

Roughly speaking, the G-equivariant twisted degree is an object that is only slightly more complicated than the usual Leray–Schauder degree. It is a finite sequence of integers, indexed (for the convenience of the user) by the conjugacy classes (H) of φ-twisted l-folded subgroups H of G. To be more specific, given a group $G = \Gamma \times S^1$, an isometric Banach G-representation W, an open invariant subset $\Omega \subset \mathbb{R} \oplus W$ and a continuous G-equivariant map $f : (\overline{\Omega}, \partial\Omega) \to (W, W \setminus \{0\})$, one can assign to f the twisted G-equivariant degree in the following form:

$$G\text{-Deg}^t(f, \Omega) = n_1(H_1) + n_2(H_2) + \cdots + n_k(H_k), \quad n_i \in \mathbb{Z}. \tag{8}$$

As we will see later on, $G\text{-Deg}^t(f, \Omega)$ satisfies all the properties expected from any reasonable “degree theory”, in particular, existence, homotopy invariance, excision, suspension, additivity, multiplicativity, etc. (adopted to the equivariant setting). Moreover, similarly to the Leray–Schauder degree, the twisted degree admits an axiomatic approach which allows applied mathematicians to use it without going into topological (homotopy theory, bordism theory) and analytical (equivariant transversality, normality) roots underlying its construction. Thus, it can be easily applied to equivariant settings in the same way as the Leray–Schauder degree is applied to non-symmetric situations.

(iii) *Application scheme of twisted degree*. In this survey article, we will explain how to apply the twisted G-equivariant degree to study various Hopf bifurcation problems (pa-

rameterized by $\alpha \in \mathbb{R}$) with a certain symmetry group Γ. To clarify the essence of our approach, we list below the main steps one should follow to attack problem (3).

(a) Let $(\alpha_o, 0)$ be an isolated center for (3) and let $\mathrm{i}\beta_o$ be the corresponding purely imaginary characteristic root.

(b) Take a small 'cylinder' $\Omega \subset \mathbb{R} \times \mathbb{R}_+ \times W$ around the point $(\alpha_o, \beta_o, 0)$, construct an *auxiliary* G-invariant function $\varsigma : \overline{\Omega} \to \mathbb{R}$ (see Definition 4.3), confining solutions to (5) to the inside of Ω, i.e. ς is negative on the 'trivial' solutions $(\alpha, \beta, 0)$ and it is positive on the 'exit' set from Ω.

(c) Consider the equation

$$\mathfrak{F}_\varsigma(\alpha, \beta, u) := \big(\varsigma, \mathfrak{F}(\alpha, \beta, u)\big) = 0 \tag{9}$$

(obviously, $\mathfrak{F}_\varsigma$ decreases only one dimension and any solution to (9) is also a solution to (5)).

(d) Define the *local bifurcation invariant* $\omega(\alpha_o, \beta_o)$ containing the topological information about the symmetric nature of the bifurcation, by

$$\omega(\alpha_o, \beta_o) := G\text{-Deg}^t(\mathfrak{F}_\varsigma, \Omega).$$

(e) Use the equivariant spectral properties of the linearized system at the point $(\alpha_o, \beta_o, 0)$ to extract data needed for the computation of $\omega(\alpha_o, \beta_o)$.

(f) Apply appropriate computer program (for example Maple© routines) in order to obtain *the exact value* of the local invariant $\omega(\alpha_o, \beta_o)$.

(g) Analyze $\omega(\alpha_o, \beta_o)$ in order to obtain the information describing possible branches of non-constant periodic solutions, their multiplicity and symmetric properties.

The fact that the equivariant degree approach to the symmetric Hopf bifurcation allows a computerization (based on the algebraic properties of the twisted degree) of many tedious technicalities related to algebraic nature of this problems (combined with the above mentioned axiomatic approach) constitutes one of the most significant advantages of the twisted degree method. Theoretically, this method supported by computer programming can be applied to any kind of Γ-symmetric Hopf bifurcation problem, with the group Γ being of *arbitrary size*.

(iv) *Historical roots of the twisted degree method*. Twisted equivariant degree is a part of the so-called equivariant degree, which was introduced by Ize et al. in [78] and rigorously studied in [81] for Abelian groups. The idea of the equivariant degree has emanated from different mathematical fields rooted in a variety of concepts and methods. The historical roots of the equivariant degree theory can be traced back to several mathematical fields:

(a) Borsuk–Ulam Theorems (cf. [25,9,24,38,55,81,86,99,121,127,138]; see also references in [99] and [138]);

(b) Fundamental Domains, Equivariant Retract Theory (cf. [9,81,99]; see also [3–7,82, 83,103,107,108]);

(c) Equivariant Obstruction Theory, Equivariant Bordisms, Equivariant Homotopy Groups of Spheres (cf. [37,43,87,105,130,135,139,141]; see also [12,17,18,81,99, 114,131]);

(d) Equivariant General Position Theorems (cf. [22,46,60,81,99]; see also [12,18,91,96, 115,150]);

(f) Generalized Topological Degree, Primary Degree, Topological Invariants of Equivariant Gradient Maps (cf. [39,46,57,59,62,79,81,93,99,126,132–134]);
(g) Geometric Obstruction Theory and J-Homomorphism in Multiparameter Bifurcations (cf. [2,76,77]).

In this article, we will discuss the construction of the twisted equivariant degree without entering into too much details. For more information and proofs, we refer to [19].

Overview

Let us discuss in more detail the contents of this article.

In Subsection 2.1, we include some preliminary results and the equivariant jargon frequently used later. In Subsection 2.2, we give a brief overview of some basic concepts and constructions from representation theory, explain our conventions and collect necessary information needed for the construction and usage of the equivariant degree techniques. In Subsection 2.3, we present some facts related to the notion of G-vector bundles modeled on Banach spaces and describe certain properties of (smooth) G-manifolds. In Subsection 2.4, we discuss the set of unbounded Fredholm operators of index zero $\mathfrak{F}_0(\mathbb{E},\mathbb{F})$ between two Banach spaces $\mathbb{E}$ and $\mathbb{F}$, and describe the topology on this set.

Section 3 naturally splits into four parts: (a) main ideas underlying the notion of twisted degree, (b) construction and basic properties, (c) practical computations, and (d) infinite dimensional extensions.

From the topological point of view, the (twisted) equivariant degree “measures” homotopy obstructions for an equivariant map to have equivariant extensions without zeros on a set composed of several orbit types. Therefore, in Subsection 3.1, we describe the topological ideas related to the construction of the twisted equivariant degree, i.e. the induction over orbit types, concept of fundamental domain and equivariant Kuratowski–Dugundji theorem. Since we follow the “differential viewpoint” to construct the twisted degree, in Subsection 3.2, the notions of normal and regular normal maps (i.e. equivariant replacements of “nice” representatives of homotopy classes) are introduced. In Subsection 3.3, we focus on algebraic properties of the $\mathbb{Z}$-module $A_1^t(G)$ – the range of values of the twisted degree. Namely, we discuss generators of $A_1^t(G)$ (i.e. conjugacy classes of φ-twisted l-folded subgroups of G) and, given two subgroups $L \subset H \subset G$, define purely algebraic quantities $n(L,H)$ allowing us to study an important *multiplication* structure on $A_1^t(G)$. Later on, this multiplication (“module structure”) is used in the same way as the usual multiplication in $\mathbb{Z}$ is used for the multiplicativity property of the Brouwer degree.

In Subsection 3.4, we present a construction of the twisted degree, showing that its coefficients n_i (see (8)) can be evaluated using the usual Brouwer degree of regular normal approximations restricted to fundamental domains. The properties of twisted degree providing an axiomatic approach to its usage, are listed in Subsection 3.5. The particular, nevertheless very important, case of the twisted S^1-degree is discussed in Subsection 3.6. The presented axiomatic definition of the twisted S^1-degree makes no use of the normality condition, therefore it is very close to the axiomatic definition of the Brouwer degree. What is probably more important, this definition opens a way for the practical computation of twisted degree.

To be more specific, in Subsection 3.7, based on the axiomatic definition of S^1-degree, we deduce some computational formulae for S^1-degree of mappings that naturally appear in studying the symmetric Hopf bifurcation phenomenon (cf. Problem 3.32, Theorem 3.39 and Corollary 3.40). In addition, we present the so-called Recurrence Formula (see Theorem 3.41) allowing a direct reduction of computations of the twisted degree to the ones of S^1-degree. Next, in Subsection 3.8, we introduce the concept of *basic maps* and *basic degrees*. These maps are the simplest ones having a non-trivial twisted degree and fitting our approach. Their twisted degrees, which are fully computable (based on the Recurrence Formula and the corresponding S^1-degree results), will constitute a 'library' that can be used by the computer programs in order to evaluate the twisted degree of more complicated maps. Finally, in Subsection 3.9, we establish the multiplicativity property of the twisted degree. This (functorial) property, which is similar to the multiplicativity of the Brouwer degree, plays a substantial role in developing the software needed for practical computations of twisted degree.

In Subsection 3.10, we extend in a standard way the (finite-dimensional) twisted degree to two important classes of equivariant vector fields on Banach G-representations: compact fields and condensing fields. These fields turn out to be enough to cover the applications to the Hopf bifurcation problems we are dealing with in this paper.

In Section 4, we lay down the standard steps of the twisted equivariant degree treatment for an autonomous 1-st order system of ODEs (without symmetries). As it was mentioned above, this system, in spite of having no symmetries, still leads to an S^1-equivariant fixed-point problem. In fact, we use the following strategy: in this section, we consider in detail the *occurrence* of the Hopf bifurcation in ODEs without symmetries (resp. S^1-degree approach) while, in subsequent sections, we analyze (trying to avoid any repetitions) the impact of Γ-symmetries in different classes of dynamical systems to *symmetric properties and minimal number* of bifurcating periodic solutions (resp. $\Gamma \times S^1$-equivariant twisted degree approach). In Subsection 4.1, we give a rigorous definition of the Hopf bifurcation. The corresponding setting in functional spaces is explained in Subsection 4.2. In Subsection 4.3, we introduce the local bifurcation invariant ($= S^1$-degree of an appropriate map) and establish (in its terms) necessary and sufficient conditions for the occurrence of Hopf bifurcation. To obtain an effective formula for the local bifurcation invariant, we apply (based on the properties of S^1-degree), in Subsection 4.4, appropriate equivariant deformations to simplify the operators in question. In Subsection 4.5, by introducing the concept of the so-called crossing number, we complete the computation of the local bifurcation invariant and formulate the Hopf bifurcation result in terms of the right-hand side of the considered system only.

Section 5 is devoted to a discussion of the Hopf bifurcation problem for a systems of ODEs *with symmetries*. A functional setting for a symmetric Hopf bifurcation and the local bifurcation invariant are described in Subsection 5.1. In particular, for the purpose of estimating *precise* symmetries and a minimal number of bifurcating periodic solutions, we introduce the concept of *dominating orbit types*. The computations of the local invariant based on a reduction to the so-called product formula, are presented in Subsection 5.2 (here we essentially use the multiplicativity property of the twisted degree). The concept of an *isotypical* crossing number and the refinement of the product formula are given in Subsection 5.3. The main steps related to the usage of the twisted equivariant degree method are

resumed in Subsection 5.4. In Subsection 5.5, we outline how to use the special Maple© routines, which were developed for the twisted degree computations, to compute the exact value of the local bifurcation invariant. These routines are available on the Internet through the link: http://krawcewicz.net/degree.

Section 6, based on the framework developed in the previous section, is dealing with a general symmetric Hopf bifurcation problem in functional differential equations. Following the standard steps, in Subsection 6.1, we analyze a parametrized system of symmetric delayed functional differential equations. In Subsection 6.2, we briefly discuss particularities related to the usage of the same method in symmetric neutral functional differential equations. Observe that the local bifurcation invariants can be effectively applied to study the problem of continuation of symmetric branches of non-constant periodic solutions (i.e. a global Hopf bifurcation problem) for a system of symmetric functional differential equations (see Subsection 6.3).

In Section 7, we extend the applicability of the twisted degree method to a system of functional parabolic differential equations. A general setting (as an abstract coincidence problem) in functional spaces, appropriate for studying symmetric FPDEs, is presented in Subsection 7.1. In Subsection 7.2, we discuss the Hopf bifurcation problem for a general FPDEs with symmetries. Although, basically the computations of the local bifurcation invariant follow the standard lines, several points specific to this setting are explained in Subsection 7.3.

In Section 8, we illustrate the twisted degree method by applying it to three concrete models. It should be pointed out that our choice of the examples (see the Contents) is motivated by their relative simplicity, practical meaning and the effectiveness of the analysis (via the twisted degree method) of the symmetric Hopf bifurcation phenomenon occurring in these systems. Other examples and more details can be found in [19].

In a short Appendix A, we collect the information about the groups (i.e. dihedral groups D_n and the icosahedral group A_5) and their representations, which are used in our examples presented in Section 8.

2. Auxiliary information

2.1. *Basic definitions and notations*

In this section, G stands for a compact Lie group.

DEFINITION 2.1. A *topological transformation group* is a triple (G, X, φ), where X is a Hausdorff topological space and $\varphi : G \times X \to X$ is a continuous map such that:

(i) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ for all $g,\ h \in G$ and $x \in X$;

(ii) $\varphi(e, x) = x$ for all $x \in X$, where e is the identity element of G.

The map φ is called a *G-action* on X and the space X, together with a given action φ of G, is called a *G-space* (or, more precisely, *left G-action* and *left G-space*, respectively). We shall use the notation $g(x)$, $g \cdot x$ or simply gx, for $\varphi(g, x)$. For $K \subset G$ and $A \subset X$, put $K(A) := \{gx\colon\ g \in K, x \in A\}$ and for $g \in G$ we write $gA := \{gx\colon\ x \in A\}$. A set $A \subset X$

is said to be G-*invariant* if $G(A) = A$. Notice that if A is a compact set, then so is $G(A)$. Observe that on any Hausdorff topological space X one can define the *trivial action* of G by $gx = x$ for all $g \in G$ and $x \in X$.

For any $x \in X$, the closed subgroup $G_x = \{g \in G\colon\ gx = x\}$ of G is called the *isotropy group* of x and the invariant subspace $G(x) := \{gx\colon\ g \in G\}$ of X is called the *orbit* of x. Also, we say that a G-action on X is *free* if $G_x = \{e\}$ for all $x \in X$.

It is easy to see that if H is the isotropy group of $x \in X$, then $G(x)$ is homeomorphic to G/H. Denote by X/G the set of all orbits in X and consider the canonical projection $\pi : X \to X/G$ given by $\pi(x) = G(x)$. The space X/G equipped with the quotient topology induced by π is called the *orbit space* of X under the action of G.

Recall that two closed subgroups H and K of G are said to be *conjugate* in G, if $H = gKg^{-1}$ for some $g \in G$. Clearly, the conjugacy relation is an equivalence relation; denote by (H) the equivalence class of H and call (H) the *conjugacy class* of H in G. Denote by $\mathcal{O}(G)$ the set of all conjugacy classes. The set $\mathcal{O}(G)$ is partially ordered by the relation $\leqslant$ defined as follows:

$$(H) \leqslant (K) \overset{\text{Def}}{\iff} \exists_{g \in G}\, gHg^{-1} \subset K. \tag{10}$$

Obviously, for a G-space X and $x \in X$, one has $G_{gx} = gG_xg^{-1}$. This gives rise to the notion of the *orbit type* of x defined as the conjugacy class (G_x).

Denote by $\mathcal{J}(X)$ the set of all orbit types occurring in X. Let us point out that according to the order relation (10), if an orbit type (G_x) is smaller than an orbit type (G_y), then the orbit $G(x)$ is “bigger” than the orbit $G(y)$. According to the famous result by Mann (cf. [112]), every action of a compact Lie group on a manifold with finitely generated homology groups has a finite number of orbit types.

The following result is known as Gleason Lemma (cf. [26]).

THEOREM 2.2. *Let X be a metric G-space such that $\mathcal{J}(X) = \{(H)\}$. Then, the orbit map $\pi : X \to X/G$ is a projection in a locally trivial fiber bundle with fiber G/H.*

Suppose that X is a finite-dimensional G-space and G is a compact Lie group. Using the definition of covering dimension (cf. [48]), the Gleason lemma and the Morita theorem (see [120]), which states that $\dim(K \times [0, 1]) = \dim K + 1$ for any metric space K, one can easily prove

PROPOSITION 2.3. *If X is a* (*metric*) *free G-space, then* $\dim(X/G) = \dim X - \dim G$.

For a G-space X and a closed subgroup H of G, we adopt the following notations:

$$X_H := \{x \in X\colon\ G_x = H\},$$
$$X^H := \{x \in X\colon\ G_x \supset H\},$$
$$X_{(H)} := \{x \in X\colon\ (G_x) = (H)\}.$$

We call X^H the H-*fixed-point subset* of X.

For a closed subgroup H of G, we use $N(H) := \{g\colon\ gHg^{-1} = H\}$ to denote the *normalizer* of H in G and $W(H) := N(H)/H$ to denote the *Weyl group* of H in G. Notice that $N(H)$ is a closed subgroup of G and X^H is $N(H)$-invariant. Consequently, X^H is $W(H)$-invariant, where the $W(H)$-action on X^H is given by $gH \cdot x := gx$ for $g \in N(H)$ and $x \in X^H$. It is also clear that $W(H)$ acts freely on X_H.

Let X and Y be two G-spaces. A continuous map $f : X \to Y$ is called G-*equivariant*, or simply *equivariant*, if

$$\forall g \in G\ \forall x \in X \quad f(gx) = gf(x).$$

If the G-action on Y is trivial, an equivariant map $f : X \to Y$ is called G-*invariant*, or simply *invariant*, i.e.

$$\forall g \in G\ \forall x \in X \quad f(gx) = f(x).$$

Since, for an equivariant map $f : X \to Y$, one has $G_{f(x)} \supset G_x$, it follows that $f(X^H) \subset Y^H$ for every subgroup $H \subset G$. Consequently, the maps $f^H : X^H \to Y^H$ with $f^H := f_{|X^H}$, are well-defined and $W(H)$-equivariant for every subgroup $H \subset G$.

2.2. *Elements of representation theory*

Finite-dimensional G-representations We start with the following

DEFINITION 2.4. Let W be a finite-dimensional real (resp. complex) vector space. We say that W is a real (resp. complex) *representation* of G (in short, G-*representation*), if W is a G-space such that the *translation map* $T_g : W \to W$, defined by $T_g(v) := gv$ for $v \in W$, is an $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) operator for every $g \in G$.

It is clear that for a G-representation W, the map $T : G \to GL(W)$, given by $T(g) = T_g : W \to W$, is a continuous homomorphism. It is convenient to identify the representation W with the homomorphism $T : G \to GL(W)$. For example, a continuous homomorphism $T : G \to GL(n;\mathbb{R})$ (resp. $T : G \to GL(n;\mathbb{C})$) is called a *real* (resp. *complex*) *matrix* G-*representation*. For two representations W_1 and W_2, if there is an equivariant isomorphism $A : W_1 \to W_2$, we say that W_1 and W_2 are *equivalent* and write $W_1 \cong W_2$. Let W be a real (resp. complex) G-representation. An inner product (resp. Hermitian inner product) $\langle\cdot,\cdot\rangle : W \oplus W \to \mathbb{R}$ (resp. $\langle\cdot,\cdot\rangle : W \oplus W \to \mathbb{C}$) is called G-*invariant* if $\langle gu, gv\rangle = \langle u, v\rangle$ for all $g \in G$, u, $v \in W$. A G-representation together with a G-invariant inner product is called an *orthogonal* (resp. *unitary*) G-*representation*. It is well-known that every real (resp. complex) G-representation is equivalent to an orthogonal (resp. unitary) representation $T : G \to O(n)$ (resp. $T : G \to U(n)$).

Notice that any G-representation W is a manifold with trivial homology groups in nonzero dimensions, therefore (cf. [112]), there are only finitely many orbit types in W.

An invariant linear subspace $\widetilde{W} \subset W$ is called a *subrepresentation* of W and we say that W is an *irreducible* representation if it has no subrepresentation different from $\{0\}$ and W. Otherwise, W is called *reducible*.

The *complete reducibility theorem* states that every (finite-dimensional) G-representation W is a (not necessarily unique) direct sum of irreducible subrepresentations of W, i.e. there exist irreducible subrepresentations $\mathcal{W}^1, \dots, \mathcal{W}^m$ of V such that

$$W = \mathcal{W}^1 \oplus \mathcal{W}^2 \oplus \cdots \oplus \mathcal{W}^m.$$

Let W_1 and W_2 be two G-representations. Denote by $L^G(W_1, W_2)$ the space of all linear G-equivariant maps $A : W_1 \to W_2$, and by $GL^G(W_1, W_2)$ its subspace of all (G-equivariant) isomorphisms. Also, put $L^G(W) := L^G(W, W)$ and $GL^G(W) := GL^G(W, W)$.

Let $\mathcal{W}^1$ and $\mathcal{W}^2$ be two real or complex irreducible G-representations. Then, the Schur lemma states that every equivariant linear map $A : \mathcal{W}^1 \to \mathcal{W}^2$ is either an isomorphism or zero. Assume $\mathcal{U}$ is a *complex* irreducible G-representation. Then, it follows from Schur Lemma that every equivariant linear map $A : \mathcal{U} \to \mathcal{U}$ satisfies $A = \lambda\,\mathrm{Id}$, for some $\lambda \in \mathbb{C}$, i.e. the G-representation $\mathcal{U}$ is *absolutely irreducible* (cf. [27]). Consequently, we have that $\dim_{\mathbb{C}} L^G(\mathcal{U}^1, \mathcal{U}^2) = 1$ or 0 (where $\mathcal{U}^1$ and $\mathcal{U}^2$ are two irreducible complex G-representations). Using this fact, it can be easily proved that every complex irreducible G-representation of an Abelian compact Lie group G is one-dimensional.

In the case $\mathcal{V}$ is a *real* irreducible G-representation, the set $L^G(\mathcal{V})$ is a finite-dimensional associative division algebra over $\mathbb{R}$, so it is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and we call $\mathcal{V}$ to be of *real*, *complex* or *quaternionic type*, respectively. Observe also that the type of a real irreducible G-representation is closely related to its *complexification* (cf. [27]).

REMARK 2.5 (Convention of notation). Let us explain our convention, which we use in connection to the complex and real G-representations. As long as it is possible, we use the letter V to denote a real G-representation, while the letter U is reserved for complex G-representations. In the case the type of a G-representation is not specified, we apply the letter W. Since, for a given compact Lie group G, there are only countably many irreducible G-representations (see [85,151]), we also assume that a complete list, indexed by numbers $0, 1, 2, 3, \dots,$ of these irreducible representation is available, and, in the case of real G-representations, we denote them by $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \dots$ (where $\mathcal{V}_0$ always stands for the trivial irreducible G-representation), in the case of complex G-representations, by $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \dots$ (where $\mathcal{U}_0$ is the trivial complex irreducible G-representation), and in the case the type of an irreducible G-representation is not clearly specified as real or complex, we denote them by $\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \dots$ (where again $\mathcal{W}_0$ is the trivial irreducible G-representation). Unspecified irreducible G-representations are denoted as follows: in the case of real representations $\mathcal{V}$, $\mathcal{V}^1$, or $\mathcal{V}^k$, in the case of complex representations $\mathcal{U}$, $\mathcal{U}^1$, or $\mathcal{U}^k$, and if the type is unknown $\mathcal{W}$, $\mathcal{W}^1$, or $\mathcal{W}^k$. We summarize our convention in Table 1.

REMARK AND NOTATION 2.6. Let Γ be a compact Lie group.

(i) Any (irreducible) complex Γ-representation can be converted in a natural way to an (irreducible) real $\Gamma \times S^1$-representation as follows. For any complex Γ-representation U and $l = 1, 2, \dots,$ define a $\Gamma \times S^1$-action on U by

$$(\gamma, z)w = z^l \cdot (\gamma w), \quad (\gamma, z) \in \Gamma \times S^1,\ w \in U, \tag{11}$$

Table 1
Notational convention for real and complex G-representations

	Real	Complex	Unspecified
G-representation	$V, \mathfrak{V}$	$U, \mathfrak{U}$	$W, \mathfrak{W}$
Irreducible G-representation	$\mathcal{V}$	$\mathcal{U}$	$\mathcal{W}$
List of all irreducible G-representations	$\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \dots$	$\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \dots$	$\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \dots$
Various irreducible G-representations	$\mathcal{V}, \mathcal{V}^1, \mathcal{V}^2, \mathcal{V}^j$	$\mathcal{U}, \mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^j$	$\mathcal{W}, \mathcal{W}^1, \mathcal{W}^2, \mathcal{W}^j$

where '$\cdot$' is the complex multiplication. This real $\Gamma \times S^1$-representation is denoted by ${}^lU$. Moreover, if U is a complex irreducible Γ-representation, then ${}^lU$ is a real irreducible $\Gamma \times S^1$-representation.

(ii) Following Remark 2.5, assume that $\{\mathcal{U}_j\}$ is a *complete* list of irreducible complex Γ-representations. The (real) irreducible $\Gamma \times S^1$-representation ${}^l\mathcal{U}_j$ is called the l-th irreducible $\Gamma \times S^1$-representation *associated* with $\mathcal{U}_j$ and denoted by $\mathcal{V}_{j,l}$.

Complexification and conjugation For a real vector space V, denote by V^c the *complexification* of V given by

$$V^c := \mathbb{C} \otimes_{\mathbb{R}} V.$$

Assume further that V is a real G-representation. Then, V^c has a natural structure of a complex G-representation defined by $g(z \otimes v) = z \otimes gv$, $z \in \mathbb{C}$, $v \in V$.

For a complex vector space U, denote by $\overline{U}$ the conjugate complex vector space structure on U, i.e. the space U with the complex multiplication given by $z \cdot v := \bar{z}v$, $z \in \mathbb{C}$, $v \in U$. Notice that $GL(U) = GL(\overline{U})$, therefore, if U is a complex G-representation, then so is $\overline{U}$, which is called the G-representation *conjugate* to U. In the case of a complex matrix G-representation $T : G \to GL(n, \mathbb{C})$, the G-representation conjugate to T is given by $\overline{T} : G \to GL(n, \mathbb{C})$, where $\overline{T}(g)$ denotes the matrix obtained from $T(g)$ by replacing its entries with their conjugates.

REMARK 2.7. For a (real) vector space V, an element x of its complexification V^c can be written in a unique way as $x = 1 \otimes u + \mathrm{i} \otimes v$, where $u, v \in V$. Using this representation of elements in V^c, it is easy to show that if V has a complex structure (i.e. an isomorphism $J : V \to V$ satisfying $J^2 = -\mathrm{Id}$), then the space V^c is isomorphic (as a complex vector space) to the direct sum $V \oplus \overline{V}$, where $\overline{V}$ stands for the (complex) vector space *conjugate* to V, i.e. the space V with the complex multiplication given by $z \cdot v := \bar{z}v$. Indeed, notice that for every $x \in V^c$,

$$\begin{aligned} x &= 1 \otimes u + \mathrm{i} \otimes v \\ &= 1 \otimes \frac{u + \mathrm{i}v}{2} + \mathrm{i} \otimes \frac{v - \mathrm{i}u}{2} + 1 \otimes \frac{u - \mathrm{i}v}{2} + \mathrm{i} \otimes \frac{v + \mathrm{i}u}{2}, \end{aligned}$$

so we obtain the following direct decomposition $V^c = V_1 \oplus V_2$, where

$$V_1 = \left\{ 1 \otimes \frac{u + \mathrm{i}v}{2} + \mathrm{i} \otimes \frac{v - \mathrm{i}u}{2} \colon u, v \in V \right\},$$
$$V_2 = \left\{ 1 \otimes \frac{u - \mathrm{i}v}{2} + \mathrm{i} \otimes \frac{v + \mathrm{i}u}{2} \colon u, v \in V \right\}.$$

It is easy to verify that V_1 is $\mathbb{C}$-isomorphic to V and V_2 to $\overline{V}$.

REMARK 2.8. It is well-known (see [27]) that for a real irreducible G-representation V, the complex G-representation V^c is irreducible if and only if V is of real type. Otherwise, it follows from Remark 2.7 that if V has a natural complex structure, then V^c, as a complex G-representation, is equivalent to $V \oplus \overline{V}$. In this case V is equivalent (as a complex G-representation) to $\overline{V}$, if and only if V is of quaternionic type.

Character of representation For a finite-dimensional real (resp. complex) G-representation W, with the corresponding homomorphism $T : G \to GL(W)$, the function $\chi_W : G \to \mathbb{R}$ (resp. $\chi_W : G \to \mathbb{C}$), defined by

$$\chi_W(g) = \mathrm{Tr}\big(T(g)\big), \quad g \in G,$$

where Tr stands for the trace, is called the *character* of W. It is well-known (see, for example, [27]) that $\chi_W(ghg^{-1}) = \chi_W(h)$, for $g,\ h \in G$, and for two G-representations (both being either real or complex) W_1 and W_2, we have $\chi_{W_1 \oplus W_2} = \chi_{W_1} + \chi_{W_2}$. Let us point out that any G-representation is completely determined by its character.

Haar integral Recall the notion of the *Haar integral* on G, denoted by $\int_G f(g)\,\mathrm{d}\mu(g)$, where $f : G \to \mathbb{R}$ is a continuous function. There exists a unique functional $f \to \int_G f(g)\,\mathrm{d}\mu(g)$, on the space of continuous functions on G, satisfying the following properties:

(i) $\int_G (f_1(g) + f_2(g))\,\mathrm{d}\mu(g) = \int_G f_1(g)\,\mathrm{d}\mu(g) + \int_G f_2(g)\,\mathrm{d}\mu(g)$;

(ii) $\int_G cf(g)\,\mathrm{d}\mu(g) = c\int_G f(g)\,\mathrm{d}\mu(g)$, where $c \in \mathbb{R}$;

(iii) $\int_G \mathrm{d}\mu(g) = 1$;

(iv) $\int_G f(h^{-1}g)\,\mathrm{d}\mu(g) = \int_G f(gh)\,\mathrm{d}\mu(g) = \int_G f(g^{-1})\,\mathrm{d}\mu(g) = \int_G f(g)\,\mathrm{d}\mu(g)$, for all $h \in G$;

(v) $\int_G f(g)\,\mathrm{d}\mu(g) \geqslant 0$ for $f(g) \geqslant 0$, $\forall g \in G$.

Isotypical decomposition Consider a real G-representation V. The representation V can be decomposed into a direct sum

$$V = \mathcal{V}^1 \oplus \mathcal{V}^2 \oplus \cdots \oplus \mathcal{V}^m \tag{12}$$

of irreducible subrepresentations $\mathcal{V}^i$ of V, some of them may be equivalent. This direct decomposition is not “geometrically” unique and is only defined up to isomorphism. Of

course, among these irreducible subrepresentations there may be distinct (non-equivalent) subrepresentations which we denote by $\mathcal{V}_{k_1}, \ldots, \mathcal{V}_{k_r}$, including possibly the trivial one-dimensional representation $\mathcal{V}_0$. Let V_{k_i} be the sum of all irreducible subspaces $\mathcal{V}^o \subset V$ equivalent to $\mathcal{V}_{k_i}$. Then,

$$V = V_{k_1} \oplus V_{k_2} \oplus \cdots \oplus V_{k_n} \tag{13}$$

and the direct sum (13) is called the *isotypical decomposition* of V. In contrast to the decomposition (12), the isotypical decomposition (13) is unique. The subspaces V_{k_i} are called the *isotypical* components of V (of type $\mathcal{V}_{k_i}$ or modeled on $\mathcal{V}_{k_i}$). It is also convenient to write the isotypical decomposition (13) in the form

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_r, \quad \text{for some } r \geqslant 0, \tag{14}$$

where the isotypical components V_i are modeled on $\mathcal{V}_i$, according to the complete list of irreducible G-representations (some V_i may be trivial subspaces).

The isotypical components V_i, $i = 0, 1, 2, \ldots, r$, can be also described in another way, which is more useful for infinite dimensional generalizations. Denote by $\chi_i : G \to \mathbb{R}$ the real character of the irreducible representation $\mathcal{V}_i$, $i = 0, 1, 2, \ldots, r$. Define the linear map $P_i : V \to V$ by

$$P_i x = n(\mathcal{V}_i) \int_G \chi_i(g) g x \, \mathrm{d}\mu(g), \quad x \in V, \tag{15}$$

where

$$n(\mathcal{V}_i) = \begin{cases} \dim_{\mathbb{R}} \mathcal{V}_i, & \text{if } \mathcal{V}_i \text{ is of real type}, \\ \dfrac{\dim_{\mathbb{R}} \mathcal{V}_i}{2}, & \text{if } \mathcal{V}_i \text{ is of complex type}, \\ \dfrac{\dim_{\mathbb{R}} \mathcal{V}_i}{4}, & \text{if } \mathcal{V}_i \text{ is of quaternionic type}. \end{cases}$$

The number $n(\mathcal{V}_i)$ defined above, is called the *intristic dimension* of $\mathcal{V}_i$. Then (see [93] for more details):

(a) $x \in V_i \Longleftrightarrow P_i(x) = x$;
(b) $x \in V_l$, $l \neq i \Longrightarrow P_i(x) = 0$;
(c) $P_i \circ P_i(x) = P_i(x)$ for all $x \in V$;
(d) $P_i : V \to V$ is G-equivariant.

Consequently, every $x \in V$ can be written as

$$x = \sum_{i=0}^{r} P_i(x),$$

where $P_i(x) \in V_i$, so $\mathrm{Id} = \sum_{i=0}^{r} P_i$.

In the case U is a complex G-representation, a similar *complex isotypical decomposition* of U can be constructed, i.e.

$$U = U_0 \oplus U_1 \oplus \cdots \oplus U_s, \quad \text{for some } s \geqslant 0,$$

where the isotypical component U_j is modeled on the complex irreducible G-representation $\mathcal{U}_j$, $j = 0, 1, \dots, s$.

Decomposition of $GL^G(V)$ Let V be an orthogonal G-representation. We have the following standard algebraic facts on a decomposition of $GL^G(V)$.

PROPOSITION 2.9. (Cf. [85].) *Let* (*cf.* (13))

$$V = V_{k_1} \oplus \cdots \oplus V_{k_r}, \tag{16}$$

be the G-isotypical decomposition, where a component V_{k_i} is modeled on an irreducible representation $\mathcal{V}_{k_i}$. Then:

(i) $GL^G(V) = \bigoplus_{i=1}^r GL^G(V_{k_i})$;

(ii) *for any isotypical component V_{k_i} from* (16), *we have $GL^G(V_{k_i}) \simeq GL(m, \mathbb{F})$, where $m = \dim V_{k_i} / \dim \mathcal{V}_{k_i}$ and $\mathbb{F} \simeq GL^G(\mathcal{V}_{k_i})$, i.e. $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, depending on the type of the irreducible representation $\mathcal{V}_{k_i}$.*

Banach G-representations Recall

DEFINITION 2.10. Let W be a real (resp. complex) Banach space. We say that W is a real (resp. complex) *Banach representation* of G (in short *Banach G-representation*) if the space W is a G-space such that the translation map $T_g : W \to W$, defined by $T_g(w) = gw$ for $w \in W$, is a *bounded* $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) operator for every $g \in G$.

Clearly, every finite-dimensional G-representation is a Banach G-representation.

We say that a Banach G-representation W is *isometric* if for each $g \in G$, the translation operator $T_g : W \to W$ is an isometry, i.e. $\|T\|_g w = \|w\|$ for all $w \in W$, and we call the norm $\|\cdot\|$ a *G-invariant* norm. Using the Haar integral, it can be proved that for every Banach G-representation W, it is possible to construct a new G-invariant norm on W equivalent to the initial one.

A closed G-invariant linear subspace of W is called a Banach subrepresentation. It is also important to notice that all irreducible Banach G-representations (i.e. representations that do not contain any proper non-trivial Banach G-subrepresentations) are finite-dimensional (see [85] or [93]). Notice that for a closed subgroup $H \subset G$, the set W^H is a closed linear subspace of W.

In the case W is a real (resp. complex) Hilbert space, the inner (resp. Hermitian inner) product $\langle \cdot, \cdot \rangle$ on W is called *G-invariant* if $\langle gv, gw \rangle = \langle v, w \rangle$ for all $g \in G$, $v, w \in W$. In this case, W is called an *isometric Hilbert* (resp. *unitary Hilbert*) *G-representation*.

Consider the complete list $\{\mathcal{V}_k \colon k = 0, 1, 2, \dots\}$ of all irreducible G-representations (which is a countable set [27]) and let $\chi_k : G \to \mathbb{R}$ be the corresponding character of

$\mathcal{V}_k$ for $k = 0, 1, 2, \ldots$. Take a real Banach G-representation and define the linear maps $P_k : W \to W$ by

$$P_k x = n(\mathcal{V}_k) \int_G \chi_k(g) g x \, \mathrm{d}\mu(g), \quad g \in G, \ x \in W, \ k = 0, 1, 2, \ldots, \tag{17}$$

where $n(\mathcal{V}_k)$ denotes the intristic dimension of $\mathcal{V}_k$.

Then, similarly to the finite-dimensional case, we have that $P_k : W \to W$ is a *G-equivariant* (i.e. $P_k(gw) = gP_k(w)$ for $g \in G$ and $w \in W$) bounded linear projection onto the subspace

$$W_k := P_k(W). \tag{18}$$

Also (see [93]), we have the following

THEOREM 2.11. *Let W be a real isometric Banach representation of a compact Lie group G, $\mathcal{V}$ a real irreducible representation of G and χ the character of $\mathcal{V}$. Then, the linear operator $P_{\mathcal{V}} : W \to W$ defined by*

$$P_{\mathcal{V}} x = n(\mathcal{V}) \int_G \chi(g) g x \, \mathrm{d}\mu(g), \quad x \in W,$$

is a bounded G-equivariant projection on the subspace $P_{\mathcal{V}}(W)$ satisfying the following properties:

(i) *If $x \in W$ belongs to a representation space of an irreducible subrepresentation of W that is equivalent to $\mathcal{V}$, then $P_{\mathcal{V}} x = x$;*

(ii) *If $x \in W$ belongs to a representation space of an irreducible subrepresentation of W that is not equivalent to $\mathcal{V}$, then $P_{\mathcal{V}} x = 0$.*

It is an immediate consequence of Theorem 2.11 that every irreducible subrepresentation of W, which is equivalent to $\mathcal{V}_k$, is contained in W_k. The G-invariant subspace W_k is called the *isotypical component* of W corresponding to $\mathcal{V}_k$. Define the subspace

$$W_\infty := \bigoplus_k W_k \tag{19}$$

which is clearly dense in W, i.e. we have $\overline{W_\infty} = W$. Consequently, the decomposition

$$W = \overline{\bigoplus_k W_k} \tag{20}$$

is called the *isotypical decomposition* of W. Moreover, for every G-equivariant linear operator $A : W \to W$, we have that $A(W_k) \subseteq W_k$ for all $k = 0, 1, 2, \ldots$.

Using the Zorn lemma one can easily establish

PROPOSITION 2.12. *Given* (19) *and* (20), *for any finite subset $X \subset W_\infty$, the subspace* $\operatorname{span} G(X)$ *spanned by the orbits of points from X, is finite-dimensional and G-invariant.*

2.3. *G-vector bundles and G-manifolds*

Let E and B be two topological spaces, V a Banach space, and $p\colon E \to B$ a continuous map which satisfies the following conditions:

(1) For each $x \in B$, the set $p^{-1}(x)$ has a structure of a real Banach space (with topology induced from E);

(2) For each $x \in B$, there exists an open neighborhood U and a homeomorphism $\varphi_U\colon p^{-1}(U) \to U \times V$ such that the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi_U"] \arrow[dr, "p"'] & & U \times V \arrow[dl, "\pi_1"', "\pi_1(u,v)=u"] \\ & U & \end{tikzcd}$$

and, in addition,

(3) For each $x \in U$ the map $\varphi_{U,x} := (\varphi_U)_{|p^{-1}(x)}\colon p^{-1}(x) \to \{x\} \times V =: V$ is an isomorphism of Banach spaces.

Then, the triple $\xi = (p, E, B)$ is called a *Banach vector bundle* modeled on the Banach space V, E – *total space*, B – *base space*, $p^{-1}(x)$ – *fiber over* x, p – *vector bundle projection* and the pair (U, φ_U) – *local trivialization*.

A Banach vector bundle $\xi = (p, E, B)$, where E, B are two G-spaces, is called a *G-vector bundle* if

(a) p is an equivariant map;

(b) For every $g \in G$, the map $g\colon p^{-1}(x) \to p^{-1}(gx)$, defined by $g(v) = gv$, $v \in p^{-1}(x)$, is an isomorphism of Banach spaces.

A G-morphism of two G-vector bundles $\xi = (p, E, B)$ and $\xi' = (p', E', B')$ is a pair of G-equivariant maps (φ, ψ), $\varphi\colon E \to E'$ and $\psi\colon B \to B'$, such that the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "p"] & E' \arrow[d, "p'"'] \\ B \arrow[r, "\psi"] & B' \end{tikzcd}$$

and $\varphi_x := \varphi_{|p^{-1}(x)}\colon p^{-1}(x) \to p'^{-1}(\psi(x))$ is a linear (bounded) operator. If φ and ψ are two G-equivariant homeomorphisms, we say that the G-vector bundles (p, E, B) and (p', E', B') are G-isomorphic.

If M is a G-space with a structure of a finite-dimensional smooth manifold such that the action $\varphi\colon G \times M \to M$ of G on M is a smooth map, then M is called a *G-manifold*. Given

an (orthogonal) G-representation V, any open invariant subset (equipped with the action induced from V) provides an example of a G-manifold.

In the case of a G-vector bundle (p, E, B) such that E and B are G-manifolds and $p : E \to B$ is a smooth mapping admitting smooth local trivializations, we say that (p, E, B) is a *smooth G-vector bundle*. If M is a G-manifold, then it is easy to see that the *tangent vector bundle* $\tau(M)$ of M is a smooth G-vector bundle. Let M be a Riemannian G-manifold. By using the Haar integral, it is easy to construct a G-invariant Riemannian metric $\langle\cdot,\cdot\rangle : \tau(M) \times \tau(M) \to \mathbb{R}$. Suppose that M is a G-submanifold of a Riemannian G-manifold W (equipped with an invariant metric). Then, the *normal vector bundle* $\nu(M)$ of M in W is a smooth G-vector bundle (in what follows, we will use the symbol $\nu_x(M)$ to denote the normal slice at $x \in M$).

Using the standard G-vector bundle techniques one can easily establish

PROPOSITION 2.13. *Assume that N is a G-submanifold of a Riemannian G-manifold M. Then, for every G-invariant compact set $K \subset N$, there exists an invariant neighborhood U of K, called invariant tubular neighborhood of N over K, satisfying the properties*:

(a) *$U_o := U \cap N$ is an invariant neighborhood of K in N*;

(b) *there exists $\varepsilon > 0$ such that the exponential map $\exp : \nu(N, \varepsilon)_{|U_o} \to M$ is a G-diffeomorphism onto U, where $\nu(N, \varepsilon)_{|U_o}$ denotes the ε-disc bundle of the normal bundle $\nu(N)$ restricted to U_o. In particular, every point $x \in U$ can be represented by a pair $(u, v) \in \nu(N, \varepsilon)_{|U_o}$, where $u \in U_o$ and v is a vector of norm less than ε orthogonal to N at u.*

Notice that if $M = V$, where V is an orthogonal G-representation, the map $\exp : \nu(N, \varepsilon)_{|U_o} \to V$ is simply defined by $\exp(u, v) = u + v$.

We complete this subsection with the following important

THEOREM 2.14. (Cf. [84,26].) *Let M be a G-manifold and H a subgroup of G. Then*:

(i) *$M_{(H)}$ is a G-invariant submanifold of M*;

(ii) *$M_{(H)}/G$ is a manifold. If $M_{(H)}$ is connected, then $M_{(H)}/G$ is also connected*;

(iii) *If (H) is a maximal orbit type in M, then $M_{(H)}$ is closed in M*;

(iv) *If (H) is a minimal orbit type in M and M/G is connected, then $M_{(H)}/G$ is a connected, open and dense subset of M/G*;

(v) *M_H is a $W(H)$-manifold with free $W(H)$-action, and $M_H/W(H)$ is a smooth manifold,*

where the minimal and maximal orbit types are taken with respect to the partial order relation (10).

2.4. *Fredholm operators*

Let $\mathbb{E}$ and $\mathbb{F}$ be two Banach spaces. Consider on $\mathbb{E} \oplus \mathbb{F}$ the norm defined by $\|x\| + \|y\|$ for $x \in \mathbb{E}$ and $y \in \mathbb{F}$. An (unbounded in general) linear operator $L : \mathrm{Dom}(L) \to \mathbb{F}$ (where $\mathrm{Dom}(L) \subset \mathbb{E}$ is a dense subspace), is called a *Fredholm operator* if

(i) L is a *closed* operator, i.e. the graph $\mathrm{Gr}(L)$ of L given by

$$\mathrm{Gr}(L) := \big\{(v, Lv):\ v \in \mathrm{Dom}(L)\big\},$$

is a closed subspace in $\mathbb{E} \oplus \mathbb{F}$,

(ii) $\mathrm{Im}(L) := \{Lv:\ v \in \mathrm{Dom}(L)\}$ is a closed subspace of $\mathbb{F}$,

(iii) $\dim\ker L$ and $\operatorname{codim}\operatorname{Im} L := \dim \mathbb{F}/\operatorname{Im} L$ are finite.

For a given Fredholm operator L, the number $i(L) = \dim\ker L - \dim\mathbb{F}/\operatorname{Im}(L)$ is called the *index* of L.

Let $L : \mathrm{Dom}(L) \subset \mathbb{E} \to \mathbb{F}$ be an unbounded Fredholm operator and denote by $\pi_1 : \mathbb{E} \oplus \mathbb{F} \to \mathbb{E}$ (resp. $\pi_2 : \mathbb{E} \oplus \mathbb{F} \to \mathbb{F}$) the natural projection on $\mathbb{E}$ (resp. $\mathbb{F}$). Since $\pi_1 : \mathrm{Gr}(L) \to \mathrm{Dom}(L)$ is a bounded linear operator, which is one-to-one and onto, the subspace $\mathrm{Dom}(L)$ can be equipped with a new norm $\|\cdot\|_L$ defined by

$$\|v\|_L := \|v\|_{\mathbb{E}} + \|Lv\|_{\mathbb{F}}, \tag{21}$$

where $\|\cdot\|_{\mathbb{E}}$ (resp. $\|\cdot\|_{\mathbb{F}}$) denotes the norm in $\mathbb{E}$ (resp. in $\mathbb{F}$), i.e. the norm $\|v\|_L$ is simply the norm of $\pi_1^{-1}(v)$ in $\mathbb{E} \oplus \mathbb{F}$. Since π_1 is one-to-one, we immediately obtain that $\mathbb{E}_L := (\mathrm{Dom}(L), \|\cdot\|_L)$ is a Banach space (equipped with so-called "*graph norm*") and the operator $L : \mathbb{E}_L \to \mathbb{F}$ is a bounded operator. Since $\ker L$ and $\operatorname{Im} L$ are still the same, the operator $L : \mathbb{E}_L \to \mathbb{F}$ is a bounded Fredholm operator of the same index as the original unbounded operator $L : \mathrm{Dom}(L) \subset \mathbb{E} \to \mathbb{F}$.

Consider the set $\mathrm{Sub}(\mathbb{E} \oplus \mathbb{F})$ of all the closed subspaces in $\mathbb{E} \oplus \mathbb{F}$ and denote by $\mathrm{Op}(\mathbb{E}, \mathbb{F})$ the set of all (in general unbounded) densely defined closed linear operators $L : \mathrm{Dom}(L) \subset \mathbb{E} \to \mathbb{F}$. It is convenient to identify any such operator L with its graph $\mathrm{Gr}(L)$, thus $\mathrm{Op}(\mathbb{E}, \mathbb{F})$ can be considered as a subspace of $\mathrm{Sub}(\mathbb{E} \oplus \mathbb{F})$.

The set $\mathrm{Sub}(\mathbb{E} \oplus \mathbb{F})$ and consequently, $\mathrm{Op}(\mathbb{E}, \mathbb{F})$ can be equipped with the *Hausdorff metric*. More precisely, consider a Banach space $\mathbb{V}$ and the set $\mathfrak{B}(\mathbb{V})$ consisting of all bounded closed subsets of $\mathbb{V}$. Then, for two sets $X, Y \in \mathfrak{B}(\mathbb{V})$, define

$$D(X, Y) := \inf\big\{r > 0:\ Y \subset X + B_r(0)\big\}, \quad B_r(0) = \big\{v \in \mathbb{V}:\ \|v\|_{\mathbb{V}} < r\big\}.$$

The *Hausdorff metric* d_H on $\mathfrak{B}(\mathbb{V})$ is given by

$$d_H(X, Y) := \max\big\{D(X, Y), D(Y, X)\big\}, \quad X, Y \in \mathfrak{B}(\mathbb{V}). \tag{22}$$

Using the Hausdorff metric, define the metric on $\mathrm{Sub}(\mathbb{E} \oplus \mathbb{F})$ by

$$d(V, W) := d_H\big(S(V), S(W)\big), \quad V, W \in \mathrm{Sub}(\mathbb{E} \oplus \mathbb{F}), \tag{23}$$

where $S(V)$ (resp. $S(W)$) denotes the unit sphere in V (resp. in W).

THEOREM 2.15. *The set* $\mathfrak{F}_0(\mathbb{E}, \mathbb{F})$ (*of Fredholm operators of index zero*) *is open in* $\mathrm{Op}(\mathbb{E}, \mathbb{F})$.

Theorem 2.15 (belonging to a functional analysis folklore) provides a theoretical basis for many of the arguments justifying the functional settings used in what follows. In particular,

COROLLARY 2.16.

(a) *Let* $L \in \mathfrak{F}_0(\mathbb{E}, \mathbb{E})$ *and assume that* $0 \in \sigma(L)$, *where* $\sigma(L)$ *denotes the spectrum of* L. *Then, for all sufficiently small reals* $t > 0$, $L - t\,\mathrm{Id}$ *is a one-to-one* (*bijective onto* $\mathbb{E}$) *Fredholm operator of index zero* (*in particular,* 0 *is an eigenvalue of* L *of finite multiplicity and it is an isolated point in* $\sigma(L)$).

(b) *Let* $L \in \mathfrak{F}_0(\mathbb{E}, \mathbb{E})$ *and let* $j : \mathbb{E}_L \to \mathbb{E}$ *be a compact operator. Then, for all* $t \in \mathbb{R}$, $L - t\,\mathrm{Id} \in \mathfrak{F}_0(\mathbb{E}, \mathbb{E})$ (*in particular,* $\sigma(L)$ *is a discrete in* $\mathbb{R}$ *set composed of eigenvalues of* L, *all of them of finite multiplicity*).

Observe that Theorem 2.15 admits an equivariant version. Namely, let $\mathbb{E}$ and $\mathbb{F}$ be real isometric Banach G-representations. Denote by $\mathrm{Op}^G := \mathrm{Op}^G(\mathbb{E} \oplus \mathbb{F})$ the set of all closed G-equivariant linear operators from $\mathbb{E}$ to $\mathbb{F}$. Clearly, for $L \in \mathrm{Op}^G$, the graph $\mathrm{Gr}(L)$ is a closed invariant subspace of $\mathbb{E} \oplus \mathbb{F}$, where we assume that G acts diagonally on $\mathbb{E} \oplus \mathbb{F}$.

The space $\mathbb{E}_L$ (cf. (21)) is a Banach G-representation. It is clear that $L : \mathbb{E}_L \to \mathbb{F}$ is a continuous equivariant operator.

Equip Op^G with the metric

$$\mathrm{dist}(L_1, L_2) = d\big(\mathrm{Gr}(L_1), \mathrm{Gr}(L_2)\big), \quad L_1, L_2 \in \mathrm{Op}^G,$$

where $d(\cdot,\cdot)$ is the metric on $\mathrm{Sub}(\mathbb{E} \oplus \mathbb{F})$ defined by (23). Let $\mathfrak{F}_0^G$ be the set of all closed G-equivariant Fredholm operators of index zero from $\mathbb{E}$ to $\mathbb{F}$. Since $\mathfrak{F}_0^G = \mathfrak{F}_0 \cap \mathrm{Op}^G$, by Theorem 2.15, $\mathfrak{F}_0^G$ is an open subset of Op^G.

We complete this subsection with the following important

REMARK AND NOTATION 2.17. Consider a Fredholm operator of index zero $L : \mathrm{Dom}(L) \subset \mathbb{E} \to \mathbb{F}$. A finite-dimensional linear operator $K : \mathbb{E} \to \mathbb{F}$ is called a (finite-dimensional) *resolvent* of L if the map $L + K : \mathrm{Dom}(L) \to \mathbb{F}$ is one-to-one. Since the operator $L : \mathbb{E}_L \to \mathbb{F}$ is continuous, $L + K : \mathbb{E}_L \to \mathbb{F}$ is also continuous and one-to-one. Hence, using open mapping theorem and the fact that a compact linear perturbation of a (bounded) Fredholm operator does not change its index, we obtain that $L + K$ is surjective and $(L + K)^{-1} : \mathbb{F} \to \mathbb{E}_L$ is bounded. Consequently, by applying the natural inclusion $j : \mathbb{E}_L \hookrightarrow \mathbb{E}$, the inverse $(L + K)^{-1} : \mathbb{F} \to \mathbb{E}$ is a bounded operator. Moreover, if the natural inclusion j is compact, the inverse $(L + K)^{-1}$ is also compact.

2.5. *Bibliographical remarks*

There is a lot of excellent graduate texts and monographs devoted to Lie group theory (see, for instance, [144,29,143,44]). For the background on transformation groups and G-spaces we refer to [26,84,43]. The material related to compact Lie group representations (both finite- and infinite dimensional) can be found in [23,27,58,142,151,152].

For the concepts and constructions related to smooth equivariant topology we refer to [26,37,84] (see also [54,72] for the fiber bundle background). The proof of Theorem 2.15 can be found, for instance, in [19].

3. Twisted equivariant degree: Construction and basic properties

3.1. *Topology behind the construction: Equivariant extensions and fundamental domains*

Induction over Orbit Types As it was mentioned in the Introduction, the equivariant degree "measures" homotopy obstructions for an equivariant map to have equivariant extensions without zeros on a set composed of several orbit types. Therefore, in this subsection we briefly discuss the following problem:

$$(P_A)\quad \begin{cases} \textit{Assume that } V \textit{ is a finite-dimensional } G\textit{-representation, } Y := V \setminus \{0\}, \\ X \textit{ is a } G\textit{-space and } A \subset X \textit{ is a closed invariant subset in } X. \textit{ Let} \\ f: A \to Y \textit{ be an equivariant map. Under which conditions does there} \\ \textit{exist an equivariant extension of } f \textit{ over } X \textit{ (resp. over a } G\textit{-invariant} \\ \textit{neighborhood of } A \textit{ in } X)? \end{cases}$$

Recall the principle of constructing equivariant maps via *induction over orbit types* (see [43]). Suppose G acts on X with finitely many orbit types $\mathcal{J}(X) = \{(H_1), (H_2), \ldots, (H_k)\}$, where $\mathcal{J}(X)$ is equipped with the partial order such that $(H_i) \leqslant (H_l)$ implies $i \geqslant l$ (cf. (10)). Assume that $Y^{H_l} \neq \emptyset$ for $l = 1, \ldots, k$, which is a necessary condition for the existence of equivariant maps from X to Y. Define a filtration of X by closed (in X) G-invariant subsets

$$A = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k = X,$$

where, for $l \geqslant 1$,

$$X_l := \big\{x \in X\colon (G_x) = (H_j) \text{ for some } j \leqslant l\big\} \cup A.$$

Suppose that $f_{l-1}: X_{l-1} \to Y$ is an equivariant map for some $l \geqslant 1$. We are interested in the existence of an equivariant extension $f_l : X_l \to Y$ of the map f_{l-1}. It is well-known (cf. [43]) that if the map $f_{l-1}^{H_l} : X_{l-1}^{H_l} \to Y^{H_l}$ admits a $W(H_l)$-equivariant extension $s : X_l^{H_l} \to Y^{H_l}$, then there exists a unique G-equivariant extension $f_l : X_l \to Y$ such that $f_{l}{}_{|X_l^{H_l}} = s$. Therefore, we arrive at the following question: *When does there exist a $W(H_l)$-equivariant extension s?* Notice that $W(H_l)$ acts freely on $X_l^{H_l} \setminus X_{l-1}^{H_l}$, thus the general problem (P_A) can be reduced to the following one:

$$(P_B)\quad \begin{cases} \textit{Let } X, A, Y \textit{ and } f \textit{ be as above and assume that } G \textit{ acts freely on } X \setminus A. \\ \textit{Find a } G\textit{-equivariant extension of } f \textit{ over } X \textit{ (resp. over a } G\textit{-invariant} \\ \textit{neighborhood of } A \textit{ in } X). \end{cases}$$

Fundamental domains The key to the extension results we are looking for is the following notion.

DEFINITION 3.1. Let a topological group Q act on a finite-dimensional metric space X. Let $D_o \subset X$ be open in its closure D. Then, D is said to be a *fundamental domain* if the following conditions are satisfied:

(i) $Q(D) = X$;
(ii) $g(D_o) \cap h(D_o) = \emptyset$ for $g, h \in Q$, $g \neq h$;
(iii) $X \setminus Q(D_o) = Q(D \setminus D_o)$;
(iv) $\dim D = \dim X/Q$, $\dim(D \setminus D_o) < \dim D$, $\dim Q(D \setminus D_o) < \dim X$, where dim stands for the covering dimension.

It turns out that

THEOREM 3.2. (Cf. [99].) *Let G be a compact Lie group and X be a finite-dimensional metric G-space on which G acts freely. Then, a fundamental domain $D \subset X$ always exists.*

IDEA OF THE PROOF. Since G is a compact Lie group acting freely,

(a) the orbit map $p : X \to X/G$ is the fiber bundle projection (cf. Gleason lemma);
(b) X/G is metrizable (cf., for instance, [99]);
(c) $\dim X/G = \dim X - \dim G$ (cf. [120]).

Using (a)–(c), one can construct $T_o \subset X/G$ such that:

(α) T_o is dense in X/G;
(β) T_o is open in X/G;
(γ) $\dim((X/G) \setminus T_o) < \dim X/G$;
(δ) the fiber bundle p is trivial over T_o.

Fix a trivialization $\psi : p^{-1}(T_o) \to G \times T_o$ and put $D_o := \psi^{-1}(\{1\} \times T_o)$. Then, $D = \overline{D_o}$ is the fundamental domain. □

Equivariant Kuratowski–Dugundji theorem Let us return to problem (P_B) (recall that we assume $X \setminus A$ is a free G-subspace). By Theorem 3.2, there exists a fundamental domain $D^{(0)} \subset L^{(0)} := X \setminus A$. Let $D_o^{(0)}$ be the corresponding open subset of $D^{(0)}$ satisfying the conditions (ii)–(iv) of Definition 3.1, and let $X^{(1)} := A \cup G(D^{(0)} \setminus D_o^{(0)})$, $L^{(1)} := X^{(1)} \setminus A$. Now, by applying Theorem 3.2 to $X^{(1)} \setminus A$, we obtain $X^{(2)}$ and $L^{(2)}$, etc. Consequently, by following the same steps, we obtain a closed finite G-invariant filtration

$$X = X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset \cdots \supset X^{(r)} = A.$$

PROPOSITION 3.3. (Cf. [99].) *Under the above assumptions, any G-equivariant map $A \to Y$ extends over X if for all $i \geqslant 1$, any equivariant map $X^{(i)} \to Y$ has a* (*non-equivariant*) *extension over $X^{(i)} \cup D^{(i-1)}$.*

It should be pointed out that Proposition 3.3 together with the induction over orbit types reduces the equivariant extension problem to the non-equivariant one. For example, combining Proposition 3.3 with the classical Kuratowski–Dugundji theorem (see, for example, [99]), one can easily establish the corresponding version of this result for equivariant maps.

THEOREM 3.4. (Cf. [99].) *Let V be a finite-dimensional G-representation, $Y := V \setminus \{0\}$, X a G-space and $A \subset X$ a closed invariant subset in X with $\mathcal{J}(X \setminus A) = \{(H_1), \ldots, (H_k)\}$. Let $f : A \to Y$ be a G-equivariant map. Then*:

(i) *There always exists a G-equivariant extension of f over a G-invariant neighborhood of A in X;*

(ii) *Assume that* $\dim V^{H_l} = n_l$ *for $l = 1, \ldots, k$ and* $\dim(X_{(H_l)} \setminus A)/G < n_l$ *for all $l = 1, \ldots, k$. Then f extends equivariantly over X.*

Under the same assumptions on X, A and V as in Problem (P_A), we say that a G-equivariant map $\mathfrak{h} : [0, 1] \times X \to V$ (where G will always be assumed to act trivially on $[0, 1]$) is a *G-homotopy* on X between $f_0 := \mathfrak{h}(0, \cdot)$ and $f_1 := \mathfrak{h}(1, \cdot)$.

In applications one is often interested in constructing G-equivariant extensions of homotopies without zeros. We have the following immediate consequence of Theorem 3.4.

COROLLARY 3.5. *Under the same assumptions on X, A and V as in Theorem* 3.4, *if* $\dim(X_{(H_l)} \setminus A)/G < n_l - 1$ *for all $l = 1, \ldots, k$, then every G-equivariant homotopy $\mathfrak{h} : [0, 1] \times A \to Y$ has an equivariant extension over* $[0, 1] \times X$.

Regular fundamental domains To make the equivariant extension techniques based on the use of fundamental domains compatible with an appropriate equivariant degree theory, a more careful analysis of a geometry of a fundamental domain is needed.

DEFINITION 3.6. Under the notations of Definition 3.1, assume there exists an open *contractible* subset $T_0 \subset X/Q$ such that the natural projection $p : X \to X/Q$ induces the homeomorphism

$$(p_{|D_o})^{-1} : T_o \to D_o. \tag{24}$$

Then, D (resp. (24)) is called a *regular fundamental domain* (resp. *lifting homeomorphism*).

THEOREM 3.7. *Let X be a G-manifold on which a compact Lie group G acts freely. Let X/G be connected. Then, there always exists a regular fundamental domain.*

The proof of Theorem 3.7 uses the same ideas as Theorem 3.2 combined with the following statement applied to X/G.

THEOREM 3.8. *Let M be a smooth connected n-dimensional manifold* (*in general non-compact*). *Take a smooth countable triangulation of M. Then, there always exists a subset T_o of M satisfying the following conditions*:

(i) *T_o is open in M;*

(ii) *T_o is dense in M;*

(iii) *T_o is contractible;*

(iv) *$M \setminus T_o$ is contained in the $n - 1$-dimensional skeleton.*

Equivariant Dugundji theorem For several reasons we will also use the following equivariant analogue of the Dugundji theorem.

PROPOSITION 3.9. (Cf. [3,4].) *Let X be a metric G-space, $A \subset X$ a closed invariant subspace, E an isometric Banach G-representation, $C \subset E$ a convex closed invariant subset and $f : A \to C$ an equivariant map. Then, there exists an equivariant extension $F : X \to C$ of f.*

In particular, given two closed disjoint and invariant subsets A and B of a metric G-space X, by applying Proposition 3.9, one can construct an invariant Urysohn function from X to $[0, 1]$ (with the trivial G-action on $[0, 1]$), which in turn allows to obtain small invariant (disjoint) neighborhoods around A and B.

3.2. *Analysis behind the construction: Regular normal approximations*

Throughout this subsection, G stands for a compact Lie group, V for an orthogonal G-representation, $\Omega \subset \mathbb{R} \oplus V$ is an open bounded G-invariant subset and $f : \mathbb{R} \oplus V \to V$ an equivariant map.

DEFINITION 3.10. Given an orthogonal G-representation V and a bounded open invariant set $\Omega \subset \mathbb{R} \oplus V$, an equivariant map $f : \mathbb{R} \oplus V \to V$ is called Ω-*admissible* if $f(x) \neq 0$ for all $x \in \partial\Omega$ (in such a case we will call (f, Ω) an *admissible pair*). Similarly, one can define an Ω-*admissible homotopy*.

The "differential viewpoint" that we follow to assign the twisted equivariant degree to an Ω-admissible map, amounts to choosing "nice" representatives of (equivariant) homotopy classes satisfying reasonable regularity/transversality condition. However, such a choice should be determined by the following two observations:

(i) Since we are dealing with group actions admitting *several* orbit types, usual transversality conditions are not reasonable. Moreover, even in the case of one orbit type, the equivariance may be in "conflict" with regularity due to the restriction requirements on the dimensions of the orbits of zeros.

(ii) Any concept of "nice" representatives should provide a way to separate the sets of zeros belonging to different orbit types (in particular, to be compatible with the principle of constructing equivariant maps via induction over orbit types), and, in addition, to be "stable" with respect to the suspension procedure (in particular, to allow a passage to infinite dimensions).

The observation (ii) leads to a concept of *normal* maps, while (i) leads to a special *regularity* requirement restricted to separate orbit types.

Normality Recall that for every (closed) subgroup $H \subset G$, the set $\Omega_{(H)}$ is a G-submanifold of Ω. In particular, one can speak about the normal G-vector bundle around $\Omega_{(H)}$.

DEFINITION 3.11. (Cf. [60,96,99].) An Ω-admissible map $f:\mathbb{R}\oplus V\to V$ is called *normal* in Ω if for every $(H)\in\mathcal{J}(\Omega)$ and every

$$x\in f^{-1}(0)\cap\Omega_H,$$

the following (H)-*normality* condition at x is satisfied: *There exists* $\delta_x>0$ *such that for all* $w\in\nu_x(\Omega_{(H)})$ *with* $\|w\|<\delta_x$,

$$f(x+w)=f(x)+w=w$$

(here $\nu_x(\cdot)$ stands for the normal slice).

Similarly, an Ω-admissible G-homotopy $h:[0,1]\times(\mathbb{R}\oplus V)\to V$ is called a *normal homotopy* in Ω, if for every $(H)\in\mathcal{J}(\Omega)$ and for every $(t,x)\in h^{-1}(0)\cap([0,1]\times\Omega_H)$, the following (H)-*normality* condition at (t,x) is satisfied: There exists $\delta_{(t,x)}>0$ such that for all $w\in\nu_{(t,x)}([0,1]\times\Omega_{(H)})$ with $\|w\|<\delta_{(t,x)}$,

$$h(t,x+w)=h(t,x)+w=w. \tag{25}$$

From Definition 3.11, it follows immediately (cf. observation (ii) above)

PROPOSITION 3.12. *If* $f:\mathbb{R}\oplus V\to V$ *is normal in* Ω*, then*

$$\overline{\left(f^{-1}(0)\cap\Omega_{(K)}\right)}\cap\overline{\left(f^{-1}(0)\cap\Omega_{(H)}\right)}=\emptyset\quad \textit{for all } (K),(H)\in J(\Omega), \tag{26}$$

provided $(K)\neq(H)$.

Regular normality Introduce the following

DEFINITION 3.13. (Cf. [60,96,99].) Let $f:\mathbb{R}\oplus V\to V$ be and Ω-admissible G-equivariant map. We say that f is a *regular normal* map in Ω if:

(i) f is of class C^1;
(ii) f is normal in Ω;
(iii) for every $(H)\in\mathcal{J}(f^{-1}(0)\cap\Omega)$, zero is a regular value of $f_H:=f|_{\Omega_H}:\Omega_H\to V^H$.

Similarly, one can define the notion of *regular normal* homotopy.

Using induction over orbit types, Proposition 2.13 and Theorem 2.14, one can establish

PROPOSITION 3.14. (See [60,96,99].) *Each* Ω*-admissible map* $f:\mathbb{R}\oplus V\to V$ *admits an arbitrarily close regular normal approximation.*

3.3. *Algebra behind the construction: Twisted groups and Burnside modules*

Hereafter, we use the following notations: Γ stands for a compact Lie group and $G:=\Gamma\times S^1$, where S^1 is the unit circle in $\mathbb{C}$.

The twisted equivariant degree we are going to construct takes its values in a certain module (denoted by $A_1^t(G)$) over the Burnside ring $A(\Gamma)$. In this subsection, we briefly discuss generators and multiplication structure of $A_1^t(G)$.

Numbers $n(L,H)$ We start our exposition of algebraic ideas underlying the construction of twisted equivariant degree with introducing certain integers intimately related to the module $A_1^t(G)$. Moreover, as we will see later on, these numbers play a substantial role in the computational formulae for the twisted degree.

Given two closed subgroups $L \subset H$ of a compact Lie group Q, define the set (cf. [74,99])

$$N(L,H) = \{g \in Q\colon\ gLg^{-1} \subset H\} \tag{27}$$

and put

$$n(L,H) := \left|\frac{N(L,H)}{N(H)}\right|, \tag{28}$$

where the symbol $|X|$ stands for the cardinality of the set X.

It turns out that

PROPOSITION 3.15. *Let $L \subset H$ be two closed subgroups of a compact Lie group such that* $\dim W(L) = \dim W(H)$. *Then, the number $n(L,H)$ is finite.*

The above statement can be easily proved using the following well-known

PROPOSITION 3.16. (Cf. [26], Corollary 5.7.) *Let $L \subset H$ be two closed subgroups of the compact Lie group Q. Then, the orbit space*

$$\frac{(Q/H)^L}{W(L)} \quad ((Q/H)^L \text{ is considered as the left } W(L)\text{-space})$$

is finite.

Observe that the number $n(L,H)$ (where $L \subset H$ are two closed subgroups of Q with $\dim W(H) = \dim W(L)$) has a very simple geometric interpretation provided by the following

LEMMA 3.17. *Let L and H be two subgroups of a compact Lie group Q such that $L \subset H$ and* $\dim W(H) = \dim W(L)$. *Then, $n(L,H)$ represents the number of different subgroups $\widetilde{H}$ in the conjugacy class (H) such that $L \subset \widetilde{H}$. In particular, if V is an orthogonal Q-representation such that (L), $(H) \in \mathcal{J}(V)$, then $V^L \cap V_{(H)}$ is a disjoint union of exactly $m = n(L,H)$ sets V_{H_j}, $j = 1, 2, \ldots, m$, satisfying $(H_j) = (H)$.*

NOTATION. In what follows, in the case of two orbit types (L) and (H) such that $(L) \leqslant (H)$ (with the order relation defined by (10)), we will assume that the number $n(L,H)$ corresponds to representatives L and H such that $L \subset H$. In the case the orbit types (L) and (H) are not comparable with respect to the order relation (10), we will simply put $n(L,H) = 0$.

REMARK 3.18. It should be pointed out that the numbers $n(L,H)$ are essentially of non-abelian nature: if Q is an abelian compact Lie group and $L \subset H \subset Q$ are (closed) subgroups, then $n(L,H) \equiv 1$. In a certain sense, the numbers $n(L,H)$ demonstrate a principal difference between the (twisted) equivariant degree theory for non-abelian groups from that for abelian ones.

Burnside ring $A(\Gamma)$ Recall some basic facts related to the Burnside Ring $A(\Gamma)$ of the compact Lie group Γ.

Denote by $\Phi_0(\Gamma)$ the set of conjugacy classes (H) for $H \subset \Gamma$, such that $W(H) = N(H)/H$ is finite, i.e. $\dim W(H) = 0$. Let $A(\Gamma)$ be the free abelian group generated by $(H) \in \Phi_0(\Gamma)$. There is a *multiplication* operation on $A(\Gamma)$ which induces a structure of a ring with identity on $A(\Gamma)$. In order to define the multiplication operation, observe that

$$\begin{aligned}
(\Gamma/H \times \Gamma/K)_{(L)}/\,\Gamma &\cong (\Gamma/H \times \Gamma/K)_L/N(L) \\
&\subset (\Gamma/H \times \Gamma/K)^L/N(L) \\
&= (\Gamma/H^L \times \Gamma/K^L)/W(L) \\
&= \left(\frac{N(L,H)}{H} \times \frac{N(L,K)}{K}\right)\Big/ W(L).
\end{aligned}$$

Since, by Proposition 3.16, the spaces Γ/H^L and Γ/K^L consist of finitely many $W(L)$-orbits and, by assumption, $W(L)$ is finite, one obtains that Γ/H^L and Γ/K^L are finite. Consequently the set $(\Gamma/H \times \Gamma/K)_{(L)}/\,\Gamma$ is finite.

The multiplication table of the generators (H) is given by the relation

$$(H) \cdot (K) = \sum_{(L)\in\Phi_0(\Gamma)} n_L(H,K)(L) \tag{29}$$

where $n_L(H,K)$ denotes the number of elements in the set $(\Gamma/H \times \Gamma/K)_{(L)}/\,\Gamma$, i.e.

$$n_L(H,K) := \left|(\Gamma/H \times \Gamma/K)_{(L)}/\,\Gamma\right|,$$

where we denote by $|X|$ the number of elements in the set X. In other words, the number $n_L(H,K)$ represents the number of the orbits of the orbit type (L) contained in the space $\Gamma/H \times \Gamma/K$.

The ring $A(\Gamma)$ is called the *Burnside Ring* of Γ. We refer to [93] for more details and proofs related to the above definition of the Burnside Ring.

The computations of the multiplication table for $A(\Gamma)$ can be effectively conducted using a simple recurrence formula:

$$n_L(H, K) = \frac{n(L, H)|W(H)|n(L, K)|W(K)| - \sum_{(\tilde{L})>(L)} n(L, \tilde{L})n_{\tilde{L}}|W(\tilde{L})|}{|W(L)|}, \tag{30}$$

where $|X|$ denotes the number of elements in the set X.

Twisted Subgroups of $G = \Gamma \times S^1$ Here, given a compact Lie group Γ, we briefly discuss generators of the $\mathbb{Z}$-module $A_1^t(G)$, where $G := \Gamma \times S^1$. To this end, we need to classify subgroups H of G.

Consider the following diagram

$$\begin{tikzcd} & H \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \Gamma & & S^1 \end{tikzcd}$$

where π_1 and π_2 are projections (homomorphisms) on Γ and S^1 respectively. Let $K := \pi_1(H)$ and consider $\ker \pi_1 = H \cap \{e\} \times S^1$, where e denotes the neutral element of Γ. If $\ker \pi_1 = \{e\} \times S^1$, then simply $H = K \times S^1$, i.e. H is a “product” subgroup. Otherwise $\ker \pi_1 = \{e\} \times \mathbb{Z}_l$, for some $l \geqslant 1$, and in this case, we are dealing with “twisted” subgroups. In the second case, we still have that $H/\ker \pi_1 \subset \Gamma \times S^1$, thus we can consider the diagram below.

$$\begin{tikzcd} & H \arrow[d, "\mu_l"] & \\ & H/\ker \pi_1 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ K \arrow[rr, dotted, "\varphi"] & & S^1 \end{tikzcd}$$

Since $\pi_1 : H/\ker \pi_1 \to K$ is one-to-one and onto, we can define the homomorphism $\varphi := \pi_2 \circ \pi_1^{-1} : K \to S^1$, and consequently the subgroup $H/\ker \pi_1$ is the graph of φ, i.e.

$$H/\ker \pi_1 := \left\{(k, z) \in K \times S^1 \colon \varphi(k) = z\right\},$$

and since the subgroup H is the inverse image $\mu_l^{-1}(H/\ker \pi_1)$, we obtain:

$$H = \left\{(k, z) \in K \times S^1 \colon \varphi(k) = z^l\right\}.$$

In this case, we call the subgroup H a *twisted* (by the homomorphism $\varphi : K \to S^1$) *l-folded* subgroup which is denoted by $K^{\varphi,l}$ (for $l = 1$ we simply write K^{φ}). It is easy to see that each (closed) subgroup $K \subset \Gamma \cong \Gamma \times \{1\} \subset G$, where 1 is the unit in S^1, is (trivially) twisted by the homomorphism $\varphi : K \to S^1$ defined as $\varphi(x) \equiv 1$.

Consider a twisted l-folded subgroup $H = K^{\varphi,l}$ and the normalizer $N(H)$ of the group H. Notice that

$$
\begin{aligned}
N(H) = N(K^{\varphi,l}) &= \{(\gamma, w) \in \Gamma \times S^1\colon (\gamma, w)K^{\varphi,l}(\gamma, w)^{-1} = K^{\varphi,l}\} \\
&= \{(\gamma, w) \in \Gamma \times S^1\colon \gamma k\gamma^{-1} \in K,\ \varphi(\gamma k\gamma^{-1}) = \varphi(k)\ \forall k \in K\} \\
&= \{\gamma \in N(K)\colon \varphi(\gamma k\gamma^{-1}) = \varphi(k)\ \forall k \in K\} \times S^1 =: N_o \times S^1.
\end{aligned}
$$

Consequently,

$$\dim W(H) = \dim N_o/K + 1 \leqslant \dim W_\Gamma(K) + 1,$$

where $W_\Gamma(\cdot)$ stands for the Weyl group in Γ. In the case $(K) \in \Phi_0(\Gamma)$, i.e. $\dim W_\Gamma(K) = 0$, we have that $\dim W(H) = 1$.

REMARK 3.19. Notice that if $H = K^{\varphi,l}$ is a twisted subgroup and $(\widetilde{H}) \leqslant (H)$, then $\widetilde{H}$ is also a twisted subgroup. In particular, every subgroup $H' \in (H)$ is twisted. Consequently, it makes sense to talk about the lattice of the conjugacy classes of twisted subgroups in G. Moreover, if $\dim W_\Gamma(K) = 0$ and $L^{\psi,l'}$ is a twisted subgroup such that $(L^{\psi,l'}) \geqslant (K^{\varphi,l})$, then it is easy to see that $\dim W_\Gamma(L) = 0$.

With these preliminaries on hands, introduce the following

DEFINITION 3.20. Denote by $\Phi_1^t(G)$ the set of all conjugacy classes of the φ-twisted l-folded subgroups $H = K^{\varphi,l}$, $l = 1, 2, \ldots$, such that the Weyl group $W_\Gamma(K)$ of K in Γ is finite (i.e. $\dim W_\Gamma(K) = 0$). We call any element from $\Phi_1^t(G)$ a *twisted conjugacy class*.

In what follows, twisted conjugacy classes will perform as generators of the module $A_1^t(G)$. Clearly, the construction of twisted subgroups provides (algebraic) features allowing to classify symmetric properties of periodic solutions to dynamical systems. What is, probably, less obvious, twisted subgroups are intimately connected to *orientability* properties of the corresponding to them Weyl groups, which, in turn is extremely important for the construction of the (twisted) equivariant degree.

To be more specific, let Q be a compact Lie group and M a Q-manifold. Take $x \in M$ with the isotropy $Q_x = H$. By passing to the $W(H)$-manifold M^H, we arrive at the following question: *does the orbit $N := W(H)(x)$ admit a natural $W(H)$-invariant orientation?* Since $W(H)$ is an orientable manifold, one can choose an orientation $\mathfrak{o}_{W(H)}$ on $W(H)$ and use the action of $W(H)$ on M^H to induce an orientation $\mathfrak{o}$ on N via the diffeomorphism $\mu_{x_o}\colon W(H) \to N$ given by $\mu_{x_o}(h) := hx_o$, $h \in W(H)$, for a certain distinguished point $x_o \in N$. Clearly, in order to this construction to be natural, the orientation of N obtained in this way should not depend on a choice of the point x_o. However, this property is satisfied only if the orientation $\mathfrak{o}_{W(H)}$ is invariant with respect to *right* translations on $W(H)$. On the other hand, the constructed orientation $\mathfrak{o}$ of the orbit N is $W(H)$-invariant if and only if $\mathfrak{o}_{W(H)}$ is invariant with respect to *left* translations on $W(H)$. Consequently, we obtain that an orbit N admits a natural $W(H)$-invariant orientation if and only if $W(H)$ admits an

orientation which is invariant with respect to *both* left and right translations (in this case, $W(H)$ is called *bi-orientable*).

Clearly, any reasonable equivariant degree theory allowing an "algebraic count" of orbits of periodic solutions should deal with the above mentioned *natural* $W(H)$-invariant orientation on orbits in question (or, equivalently, with bi-orientable groups $W(H)$). It is easy to see that if $W(H)$ is finite, Abelian or contains an odd number of connected components, then it is bi-orientable. On the other hand, the group O(2) presents an example of the non-bi-orientable group. Our choice of the module $A_1^t(G)$ for the range of values of the (twisted) equivariant degree is essentially based on the following

PROPOSITION 3.21. *Let Γ be a compact Lie group, $G := \Gamma \times S^1$ and $H = K^{\varphi,l}$ a twisted subgroup of G such that $(H) \in \Phi_1^t(G)$. Then*:

(i) *the connected component of $1 \in W(H)$ can be canonically identified with S^1*;

(ii) *the Weyl group $W(H)$ of H in G is bi-orientable and can be equipped with the natural orientation induced from S^1.*

$A(\Gamma)$-module $A_1^t(G)$ Take the set $\Phi_1^t(G)$ introduced in Definition 3.20 and consider the free $\mathbb{Z}$-module $A_1^t(G) := \mathbb{Z}[\Phi_1^t(G)]$. Using Proposition 3.16 and the (finiteness) argument similar to that for the Burnside ring (cf. formula (29)), one can easily establish the following

THEOREM 3.22. *There is a "multiplication" function $\cdot : A(\Gamma) \times A_1^t(G) \to A_1^t(G)$ defined on the generators $(K) \in \Phi_0(\Gamma)$ and $(H^{\varphi,l}) \in \Phi_1^t(G)$ as follows*:

$$(K) \cdot (H^{\varphi,l}) = \sum_{(L)} n_L (L^{\varphi,l}), \tag{31}$$

where the summation is taken over all subgroups L such that $W(L)$ is finite, $L = \gamma^{-1} K \gamma \cap H$ for some $\gamma \in \Gamma$ and

$$n_L = \left| \left(\frac{G}{K \times S^1} \times \frac{G}{H^{\varphi,l}} \right)_{(L^{\varphi,l})} \Big/ G \right|.$$

The $\mathbb{Z}$-module $A_1^t(G)$ equipped with the multiplication "$\cdot$" becomes an $A(\Gamma)$-module.

Below we describe the process one can follow to establish the multiplication tables for the $A(\Gamma)$-module $A_1^t(G)$, based on the same idea of a recurrence formula (30).

Let $(K) \in \Phi_0(\Gamma)$ be a generator of $A(\Gamma)$ and $(H^{\varphi,l}) \in \Phi_1^t(G)$ a generator of $A_1^t(G)$. In order to compute the coefficients n_L given in formula (31), apply the following recurrence formula

$$n_L = \Bigg[n(L, K) \big| W(K) \big| n(L^{\varphi,l}, H^{\varphi,l}) \left| \frac{W(H^{\varphi,l})}{S^1} \right|$$

$$- \sum_{(\tilde{L})>(L)} n(L^{\varphi,l}, \tilde{L}^{\varphi,l}) n_{\tilde{L}} \left| \frac{W(\tilde{L}^{\varphi,l})}{S^1} \right| \Big] \Big/ \left| \frac{W(L^{\varphi,l})}{S^1} \right| \tag{32}$$

where for a set Y, $|Y|$ denotes the number of elements in Y.

REMARK 3.23. Notice that the multiplication formulae for conjugacy classes of trivially twisted subgroups $(H) \in \Phi_1^t(G)$ (i.e. $H = H^{\varphi,1}$ with $\varphi \equiv 1$), where $(H) \in \Phi_0(\Gamma)$, coincide with the multiplication formulae for the Burnside ring $A(\Gamma)$. That means, the multiplication formulae for the Burnside ring are contained in the $A(\Gamma)$-module multiplication formulae for $A_1^t(G)$.

REMARK 3.24. Let us point out that the formula (32) allows us to implement the multiplication tables for the $A(\Gamma)$-module $A_1^t(G)$ into a computer software, which can be effectively used for the computations of the twisted degree.

3.4. *Construction*

We are now in a position to present the construction of twisted equivariant degree.

We start with the regular normal case, i.e. take an orthogonal G-representation V, open bounded G-invariant subset $\Omega \subset \mathbb{R} \oplus V$ and an Ω-admissible *regular normal* map $f : \mathbb{R} \oplus V \to V$. Fix $(H) \in \Phi_1^t(G, \Omega) := \{(H) \in \mathcal{J}(\Omega)\colon\ (H) \in \Phi_1^t(G)\}$. By Theorem 2.14(v), Ω_H is a free $W(H)$-manifold. Assuming, without loss of generality, that $\Omega_H/W(H)$ is connected, and using Theorem 3.7, one can find a regular fundamental domain D on Ω_H with the lifting homeomorphism

$$\xi := (p_{|D_o})^{-1} : T_o \to D_o$$

(cf. Definitions 3.1 and 3.6). Moreover, since f is normal, it satisfies the condition (26). Hence, one can choose D in such a way that

$$f_H^{-1}(0) \cap (D \setminus D_o) = \emptyset, \tag{33}$$

where $f_H := f|_{\Omega_H}$. Since f is regular normal, the set

$$p\big(f_H^{-1}(0) \cap D_o\big)$$

is finite, therefore, one can construct T_o providing

$$p\big(f_H^{-1}(0)\big) \subset T_o. \tag{34}$$

Since $\Omega_H/W(H)$ is a (smooth) manifold (see Theorem 2.14(v)) and T_o is open in $\Omega_H/W(H)$, T_o is a manifold as well. Since T_o is *contractible*, it is *orientable*. By assumption, $(H) \in \Phi_1^t(G)$, therefore, fix the natural orientation on $W(H)$ which is invariant

with respect to both left and right translations (cf. Proposition 3.21). Also, fix an orientation on V^H inducing an orientation on $\mathbb{R} \oplus V^H$. By bi-orientability of $W(H)$, for every $x \in T_o$, the orbit $N := W(H)(\xi x)$ has a natural $W(H)$-invariant orientation. Therefore, we can assume that the orientation on T_o is chosen in such a way that for *all* $x \in T_o$, we have that the orientation on $\mathbb{R} \oplus V^H$ is obtained from the orientation on D_o followed by the orientation on N, and the lifting homeomorphism $\xi : T_o \to D_o$ preserves the orientation.

Put

$$n_H = n_H(f) := \deg(f_H \circ \xi, T_o) \tag{35}$$

(here "deg" denotes the (local) Brouwer degree with respect to zero (this degree is correctly defined according to (33) and (34))).

Furthermore, take $A_1^t(G)$ and define the twisted equivariant degree $G\text{-Deg}^t(f, \Omega)$ by:

$$G\text{-Deg}^t(f, \Omega) = \sum_{(H) \in \Phi_1^t(G,\Omega)} n_H(H), \tag{36}$$

where n_H is defined by (35).

If now $f : \mathbb{R} \oplus V \to V$ is an *arbitrary* Ω-admissible map (in general, neither normal nor regular), choose a regular normal Ω-admissible map $g : \mathbb{R} \oplus V \to V$ close to f (see Proposition 3.14) and put

$$G\text{-Deg}^t(f, \Omega) := G\text{-Deg}^t(g, \Omega). \tag{37}$$

REMARK 3.25. Some comments related to the definition of twisted degree are in order.

(i) To some extent, the bi-orientability of $W(H)$ is used to define an "index of an orbit" ("local twisted degree"), while the contractibility of T_o allows a correct definition on the whole T_o ("global twisted degree").

(ii) If we choose an orientation on D_o in such a way that ξ preserves it, then $\deg(f_H, D_o)$ is correctly defined and coincides with $\deg(f_H \circ \xi, T_o)$. In this sense, one can think of $n_H(f)$ as a "degree of f_H on a fundamental domain D".

(iii) It is possible to show that the definition of twisted degree given by (35)–(37) is independent of a choice of fundamental domains as well as regular normal approximations.

3.5. *Axiomatic approach to twisted degree*

Basic properties The twisted degree defined above satisfies all the standard properties required from any reasonable "degree theory". To see that, we need the following

DEFINITION 3.26. Let V be an orthogonal G-representation with $G = \Gamma \times S^1$ and $f : \mathbb{R} \oplus V \to V$ a regular normal map such that $f(x_o) = 0$ with $G_{x_o} = H$ and $(H) \in \Phi_1^t(G, V)$. Let

$U_{G(x_o)}$ be a G-invariant tubular neighborhood around $G(x_o)$ such that $f^{-1}(0) \cap U_{G(x_o)} = G(x_o)$ (cf. Proposition 2.13). Then, f is called a *tubular map*.

Next, fix the natural orientation on $W(H)$ which is invariant with respect to both left and right translations. Also, fix an orientation on V^H inducing an orientation on $\mathbb{R} \oplus V^H$. By bi-orientability of $W(H)$, choose the natural orientation on the orbit $W(H)(x_o)$ and assume that the orientation on the normal slice $\nu_{x_o} := \nu_{x_o}(W(H)(x_o))$ in $\mathbb{R} \oplus V^H$ is oriented in such a way that the orientation on $\mathbb{R} \oplus V^H$ is obtained from the orientation on ν_{x_o} followed by the orientation on $W(H)(x_o)$. Then, we call $n_{x_o} = \operatorname{sign}\det Df^H(x_o)|_{\nu_{x_o}}$ the *local index* of f at x_o in $U_{G(x_o)}$ (here $f^H := f|_{\Omega^H}$ and "D" stands for the differential).

PROPOSITION 3.27. *Let V be an orthogonal G-representation with $G = \Gamma \times S^1$, $\Omega \subset \mathbb{R} \oplus V$ an open bounded invariant subset and $f : \mathbb{R} \oplus V \to V$ an equivariant Ω-admissible map. Then, the twisted equivariant degree defined by* (35)–(37) *satisfies the following properties*:

(P1) (Existence) *If $G\text{-}\mathrm{Deg}^t(f, \Omega) = \sum_{(H)} n_H(H)$ is such that $n_{H_o} \neq 0$ for some $(H_o) \in \Phi_1^t(G, \Omega)$, then there exists $x \in \Omega$ with $f(x) = 0$ and $G_x \supset H_o$.*

(P2) (Additivity) *Assume that Ω_1 and Ω_2 are two G-invariant open disjoint subsets of Ω such that $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then,*

$$G\text{-}\mathrm{Deg}^t(f, \Omega) = G\text{-}\mathrm{Deg}^t(f, \Omega_1) + G\text{-}\mathrm{Deg}^t(f, \Omega_2).$$

(P3) (Homotopy) *Suppose $h : [0, 1] \times \mathbb{R} \oplus V \to V$ is an Ω-admissible G-equivariant homotopy. Then,*

$$G\text{-}\mathrm{Deg}^t(h_t, \Omega) = \mathrm{const}$$

(*here $h_t := h(t, \cdot, \cdot)$, $t \in [0, 1]$*).

(P4) (Suspension) *Suppose that W is another orthogonal G-representation and let U be an open bounded G-invariant neighborhood of* 0 *in W. Then,*

$$G\text{-}\mathrm{Deg}^t(f \times \mathrm{Id}, \Omega \times U) = G\text{-}\mathrm{Deg}^t(f, \Omega).$$

(P5) (Normalization) *Suppose f is a tubular map around $G(x_o)$ with $H := G_{x_o}$ and the local index n_{x_o} of f at x_o in a tubular neighborhood $U_{G(x_o)}$. Then,*

$$G\text{-}\mathrm{Deg}^t(f, U_{G(x_o)}) = n_{x_o}(H).$$

(P6) (Elimination) *Suppose f is normal in Ω and $\Omega_H \cap f^{-1}(0) = \emptyset$ for every $(H) \in \Phi_1^t(G, \Omega)$. Then,*

$$G\text{-}\mathrm{Deg}^t(f, \Omega) = 0.$$

(P7) (Excision) *If $f^{-1}(0) \cap \Omega \subset \Omega_0$, where $\Omega_0 \subset \Omega$ is an open invariant subset, then*

$$G\text{-}\mathrm{Deg}^t(f, \Omega) = G\text{-}\mathrm{Deg}^t(f, \Omega_0).$$

(P8) (Hopf property) *Suppose that* $\Omega_H/W(H)$ *is connected for all* $(H) \in \Phi_1^t(G,\Omega)$ *and* $\Omega_K = \emptyset$ *for all* $(K) \notin \Phi_1^t(G,\Omega)$. *Let* $f, g : \mathbb{R} \oplus V \to V$ *be two* Ω*-admissible* G*-equivariant maps such that*

$$G\text{-Deg}^t(f,\Omega) = G\text{-Deg}^t(g,\Omega).$$

Then, f *and* g *are* G*-equivariantly homotopic by an* Ω*-admissible homotopy.*

PROOF. To establish properties (P1)–(P4), (P7) in the regular normal case, one should combine Proposition 3.12 and formulae (35), (36) with appropriate properties of the (local) Brouwer degree. In the general case, take regular normal approximations sufficiently close to f (resp. h) and use the standard compactness argument.

Property (P5) (resp. (P6)) follows immediately from the regular value definition of the Brouwer degree (resp. definition of the twisted degree).

To prove property (P8), assume (without loss of generality) that f and g are regular normal. Combining Proposition 3.12 with the classical Hopf Theorem (see, for instance, [44]) applied to appropriate fundamental domains, construct "local" homotopies between f and g around the corresponding zeros and extend the obtained homotopies by G-equivariance. Next, using Theorem 3.4 (see also Corollary 3.5), extend the previous "partial homotopy" over $\Omega \times [0,1]$. The latter homotopy may have "new" zeros which can be eliminated by using Proposition 3.9. □

REMARK 3.28. Following the same lines as above, one can associate with an open Γ-invariant bounded set $U \subset W$ (W is an orthogonal Γ-representation) and a continuous Γ-equivariant map $g : (U, \partial U) \to (W, W \setminus \{0\})$ the Γ-equivariant degree $\Gamma\text{-Deg}(g,U)$ taking its values in the Burnside ring $A(\Gamma)$ and satisfying the properties similar to (P1)–(P8) with obvious modifications (see [93] and [99] for more details).

Axiomatic approach The following result provides an axiomatic approach to the twisted degree theory.

THEOREM 3.29. *There exists a unique* $A_1^t(G)$*-valued function on the set of* Ω*-admissible* G*-equivariant maps and homotopies satisfying conditions* (P1)–(P6).

PROOF. The *existence* part of Theorem 3.29 is provided by Proposition 3.27. To prove the *uniqueness* part, assume $\overline{G\text{-Deg}^t}$ is another 'degree' satisfying (P1)–(P6). Take an arbitrary admissible pair (f,Ω). By homotopy property, f can be assumed to be regular normal. By additivity (i.e. excision) and elimination properties, we can assume that $\Omega \cap f^{-1}(0)$ contains only points of the orbit types $(H) \in \Phi_1^t(G)$. Since f is regular normal, the set $\Omega \cap f^{-1}(0)$ is composed of a finite number of G-orbits. Take tubular neighborhoods isolating the above orbits (this is doable, since we have finitely many zero orbits). By additivity property, $\overline{G\text{-Deg}^t}(f,\Omega)$ is equal to the sum of degrees of restrictions of f to the tubular neighborhoods. Finally, by normalization property, the orbits in question lead to "local indices", which implies

$$\overline{G\text{-Deg}^t}(f,\Omega) = G\text{-Deg}^t(f,\Omega).$$ □

Theorem 3.29 provides the application of the twisted degree without going into topological, analytical and algebraic roots underlying its construction – this is an obvious advantage of the axiomatic approach. In addition, this theorem can be extended to equivariant maps $\mathbb{R}^n \oplus V \to V$ with G replaced by $\Gamma \times T^n$, where T^n stands for the n-dimensional torus (such a setting naturally appears in multiparameter bifurcation problems). However, Theorem 3.29 amounts to the *normality* property (see (P5)–(P6)) which (being of great theoretical importance) is easy to formulate but difficult to achieve in practice.

Fortunately, in the setting relevant to our discussion, there is an effective way “to go around the normality problem”, namely:

(i) to define the twisted degree for $G = S^1$ by a list of axioms (of course, equivalent to those presented in Proposition 3.27) with the normality property not being addressed whatsoever;

(ii) for an arbitrary $G = \Gamma \times S^1$, to obtain a formula canonically reducing the computations of G-degree to those for the S^1-degree and numbers $n(L, H)$.

3.6. S^1*-degree*

Assume $\Gamma = \{e\}$, i.e. $G = \{e\} \times S^1 \cong S^1$. Recall that any Abelian compact Lie group is bi-orientable. Therefore, $A_1^t(S^1) := A_1(S^1)$ is the free $\mathbb{Z}$-module generated by $(\mathbb{Z}_k)$, $k = 1, 2, 3 \ldots.$

Take an S^1-representation V, an open S^1-invariant bounded set $\Omega \subset \mathbb{R} \oplus V$, and an Ω-admissible S^1-equivariant map $f : \mathbb{R} \oplus V \to V$. Then,

$$S^1\text{-Deg}(f, \Omega) = n_{k_1}(\mathbb{Z}_{k_1}) + \cdots + n_{k_r}(\mathbb{Z}_{k_r}), \tag{38}$$

where $n_{k_i} \in \mathbb{Z}$. To provide an axiomatic approach to the degree given by (38), we need two auxiliary constructions.

Two constructions

(i) *Basic maps*. Denote by $\mathcal{V}_k$, $k = 1, 2, 3, \ldots$, the (non-trivial) k-th real irreducible S^1-representation, i.e. $\mathcal{V}_k$ is the space $\mathbb{R}^2 = \mathbb{C}$ with the S^1-action given by $\gamma z := \gamma^k \cdot z$, $\gamma \in S^1$, $z \in \mathbb{C}$, and define the set

$${}^k\Omega := \left\{(t, z) \in \mathbb{R} \oplus \mathcal{V}_k\colon\ |t| < 1,\ \frac{1}{2} < |z| < 2\right\} \tag{39}$$

and $b : \mathbb{R} \oplus \mathcal{V}_k \to \mathcal{V}_k$ by

$$b(t, z) := \big(1 - |z| + \mathrm{i}t\big) \cdot z, \quad (t, z) \in \mathbb{R} \oplus \mathcal{V}_k, \tag{40}$$

where “$\cdot$” denotes the complex multiplication in $\mathcal{V}_k = \mathbb{C}$. Clearly, b is S^1-equivariant and ${}^k\Omega$-admissible. In what follows, b is called the k-th *basic map*.

(ii) *l-folding*. For every integer $l = 1, 2, 3, \ldots$, define the homomorphism $\theta_l : S^1 \to S^1$ (called *l-folding*), by $\theta_l(\gamma) = \gamma^l$, $\gamma \in S^1$, and define the induced by θ_l homomorphism $\Theta_l : A_1(S^1) \to A_1(S^1)$, by

$$\Theta_l(\mathbb{Z}_k) := (\mathbb{Z}_{kl}), \quad k = 1, 2, 3, \ldots,$$

i.e. $\Theta_l(\mathbb{Z}_k) = (\theta_l^{-1}(\mathbb{Z}_k))$, where $(\mathbb{Z}_k)$ are the free generators of $A_1(S^1)$.

If $f : \mathbb{R} \oplus V \to V$ is an Ω-admissible S^1-equivariant map for a certain open bounded S^1-invariant subset $\Omega \subset \mathbb{R} \oplus V$, then for every integer $l = 1, 2, 3, \ldots$, define the *associated l-folded* S^1-representation $^l(V)$, which is the same vector space V with the S^1-action '$\cdot$' given by

$$\gamma \cdot v := \theta_l(\gamma)v = \gamma^l v, \quad \gamma \in S^1,\ v \in V.$$

Next, the map f considered from $\mathbb{R} \oplus {}^l(V)$ to $^l(V)$, is S^1-equivariant as well. The set Ω considered as an S^1-subset of $\mathbb{R} \oplus {}^l(V)$ is denoted by $^l(\Omega)$. In what follows, we will say that the pair $(f, {}^l(\Omega))$ is the *l-folded* admissible pair *associated* with (f, Ω).

Axiomatic approach to S^1-degree

THEOREM 3.30. *There exists a unique $A_1(S^1)$-valued function* S^1-Deg *defined on Ω-admissible S^1-equivariant maps and homotopies and satisfying properties* (P1)–(P4) (*with $G = S^1$*) *as well as the following ones*:

(P5′) (Normalization) *For the* 1*-st basic map $b : \mathbb{R} \oplus \mathcal{V}_1 \to \mathcal{V}_1$,*

$$S^1\text{-Deg}(b, {}^1\Omega) = (\mathbb{Z}_1).$$

(P6′) (Elimination) *If V is a trivial S^1-representation, then*

$$S^1\text{-Deg}(f, \Omega) = 0.$$

(F) (Folding) *Let $^l(V)$ be the l-folded representation associated with V, and $(f, {}^l(\Omega))$ the l-folded admissible pair associated with (f, Ω). Then,*

$$S^1\text{-Deg}(f, {}^l(\Omega)) = \Theta_l\big[S^1\text{-Deg}(f, \Omega)\big].$$

PROOF. The *existence* part of Theorem 3.30 is obvious. The *uniqueness* easily follows from

LEMMA 3.31. *Under the conditions and notations of Definition* 3.26, *assume $G = S^1$ and $G_{x_o} = \mathbb{Z}_{k_o}$. Assume that* S^1-Deg *is a function provided by Theorem* 3.30. *Then,*

$$S^1\text{-Deg}(f, \Omega) = n_{x_o}(\mathbb{Z}_{k_o}),$$

where n_{x_o} is the local index of f at x_o. □

3.7. *Computational techniques for twisted degree*

Statement of the problem The goal of this subsection is to show how the axiomatic approach described in the previous two sections, being combined with some additional techniques, allows to obtain twisted degree results for an important class of equivariant maps which naturally appear in the symmetric Hopf bifurcation problems.

We start with a simple observation that every S^1-representation admits the so-called *natural complex structure*, which turns out to be a convenient setting for the discussion of Hopf bifurcation problems and a natural way of describing the S^1-action to carry out certain computations. To be more specific, let V be an (orthogonal) S^1-representation with $V^{S^1} = \{0\}$. Then, one can define on V a complex structure sensitive to the S^1-action as follows. Assume, for a moment, that $V = \mathcal{V}_k$. Then, for $z \in \mathbb{C}$, put $z = \gamma|z|$, where $\gamma = \mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in [0, 2\pi)$. The complex multiplication of $v \in \mathcal{V}_k$ by the number z is defined by

$$z \cdot v := |z|\mathrm{e}^{\frac{\mathrm{i}\theta}{k}} v. \tag{41}$$

Suppose, further, that V is (in general) reducible, and we have the following S^1-isotypical decomposition (cf. (13) and (14)):

$$V = V_{k_1} \oplus V_{k_2} \oplus \cdots \oplus V_{k_s}, \tag{42}$$

where V_{k_j} is modeled on the irreducible S^1-representation $\mathcal{V}_{k_j}$, $j = 1, 2, \ldots, s$. Since for every j, $\mathcal{V}_{k_j}$ can be equipped with the complex structure according to (41), every isotypical component from (42) also admits such a structure. In this way, we obtain on V a complex structure which we call the *natural complex structure*.

Let Γ be a compact Lie group. As we will see later on, the problem of studying Γ-symmetric Hopf bifurcations in many cases can be reduced to the following one.

PROBLEM 3.32. Let $G = \Gamma \times S^1$ and let V be an orthogonal G-representation with $V^{S^1} = \{0\}$ (here S^1 is identified with $\{e\} \times S^1$). Suppose V (considered as the S^1-representation) is equipped with the natural complex structure and put

$$\mathcal{O} := \left\{(\lambda, v) \in \mathbb{C} \oplus V\colon\ \|v\| < 2,\ \frac{1}{2} < |\lambda| < 4\right\}. \tag{43}$$

Assume G acts trivially on $\mathbb{C}$ and $\mathbb{R}$ and take a continuous map $a : S^1 \to GL^G(V)$, where $GL^G(V)$ stands for the set of all G-equivariant linear invertible maps in V. Define a G-equivariant map $f : \overline{\mathcal{O}} \to \mathbb{R} \oplus V$ by

$$f_a(\lambda, v) = \left(|\lambda|\big(\|v\| - 1\big) + \|v\| + 1, a\left(\frac{\lambda}{|\lambda|}\right)v\right), \quad (\lambda, v) \in \overline{\mathcal{O}}. \tag{44}$$

How can one compute the twisted degree $G\text{-}\mathrm{Deg}^t(f_a, \mathcal{O})$?

Our approach to attack the above problem contains the following four ingredients:

(i) *Recurrence Formula* (see Theorem 3.41) allowing a reduction of the general problem to the computation of the corresponding S^1-degree;

(ii) *Splitting Lemma* (cf. Lemma 3.37) allowing a reduction to subrepresentations;

(iii) *Homotopy Factorization* (cf. Corollaries 3.36 and 3.38) allowing a factorization of a given map through canonical representatives of the elements of $\pi_1(GL^G(k,\mathbb{C}))$ and next deformations to the so-called $\mathbb{C}$-complementing maps being natural “complex counterparts” for the k-th basic maps (cf. Definition 3.33);

(iv) *Suspension Procedure* allowing to reduce the computation of the S^1-degree of $\mathbb{C}$-complementting maps to the one of k-th basic maps (cf. Proposition 3.34).

The last three techniques come together at the end of this subsection (see Theorem 3.39 where the S^1-degree for (44) is given). Observe that the Splitting Lemma is presented in a form much more general than is needed to establish Theorem 3.39.

$\mathbb{C}$-complementing maps and suspension procedure We start with the following

DEFINITION 3.33. Let $b:\mathbb{R}\oplus\mathcal{V}_k\to\mathcal{V}_k$ (resp. $b^-:\mathbb{R}\oplus\mathcal{V}_k\to\mathcal{V}_k$) be the k-th basic map defined by (40) (resp. the map defined by $b^-(t,v)=(1-\|v\|-it)v$, $t\in\mathbb{R}$, $v\in\mathcal{V}_k$) and let ${}^k\Omega$ be defined by (39). Assume that $\mathcal{V}_k$ is equipped with the natural complex structure and $\mathcal{O}$ is given by (43) with $V:=\mathcal{V}_k$. Suppose, finally, that $f:\mathbb{C}\oplus\mathcal{V}_k\to\mathbb{R}\oplus\mathcal{V}_k$ (resp. $f^-:\mathbb{C}\oplus\mathcal{V}_k\to\mathbb{R}\oplus\mathcal{V}_k$) is defined by $f(\lambda,v)=(|\lambda|(\|v\|-1)+\|v\|+1,\lambda\cdot v)$ (resp. $f^-(\lambda,v)=(|\lambda|(\|v\|-1)+\|v\|+1,\bar{\lambda}\cdot v)$), where $\lambda\in\mathbb{C}$, $v\in\mathcal{V}_k$. Then, the pair $(f,\mathcal{O})$ (resp. $(f^-,\mathcal{O})$) is called a $\mathbb{C}$-*complementing pair* to $(b,{}^k\Omega)$ (resp. denoted by $(b^-,{}^k\Omega)$).

It is clear that $(f,\mathcal{O})$, $(f^-,\mathcal{O})$, $(b,{}^k\Omega)$ and $(b^-,{}^k\Omega)$ are admissible pairs. The following statement (basically, its proof is a careful usage of properties (P2)–(P4)) justifies the above definition.

PROPOSITION 3.34. *Let $(f,\mathcal{O})$ (resp. $(f^-,\mathcal{O})$) be a $\mathbb{C}$-complementing pair to $(b,{}^k\Omega)$ (resp. $(b^-,{}^k\Omega)$). Then f (resp. f^-) is S^1-homotopic (by an $\mathcal{O}$-admissible homotopy) to a map $\overline{f_1}$ (resp. $\overline{f_1^-}$), which is a suspension of b (resp. b^-) on an open subset containing zeros of $\overline{f_1}$ (resp. $\overline{f_1^-}$). In particular,*

$$S^1\text{-Deg}(f,\mathcal{O})=S^1\text{-Deg}(b,{}^k\Omega)=(\mathbb{Z}_k),\tag{45}$$

$$S^1\text{-Deg}(f^-,\mathcal{O})=S^1\text{-Deg}(b^-,{}^k\Omega)=-(\mathbb{Z}_k).\tag{46}$$

Homotopy factorization: Properties of $GL^G(V)$ Consider a continuous map $\varphi:S^1\to\mathbb{C}\setminus\{0\}$. Denote by $\tilde{\varphi}$ a continuous extension of φ to the space $\mathbb{C}$. Put $B:=\{z\in\mathbb{C}:\ |z|<1\}$. Clearly, the Brouwer degree $\deg(\tilde{\varphi},B)$ depends only on the map φ. Moreover, by classical Hopf theorem (see [44]), for two continuous maps $\varphi,\psi:S^1\to\mathbb{C}\setminus\{0\}$, φ is homotopic to ψ if and only if $\deg(\tilde{\varphi},B)=\deg(\tilde{\psi},B)$.

Denote by $GL(m,\mathbb{C})$ the group of all complex invertible $m\times m$-matrices. The following fact is well-known.

PROPOSITION 3.35. (*See, for instance, [93].*)

(i) *Two continuous maps* $\Phi, \Psi : S^1 \to GL(m, \mathbb{C})$, $m \geqslant 1$, *are homotopic if and only if the maps* $\varphi := \det_{\mathbb{C}} \circ \Phi$ *and* $\psi := \det_{\mathbb{C}} \circ \Psi$ *are homotopic, i.e.*

$$\deg(\tilde{\varphi}, B) = \deg(\tilde{\psi}, B).$$

(ii) *For every map* $\Phi : S^1 \to GL(m, \mathbb{C})$, *there exists* $l \in \mathbb{Z}$ *such that* Φ *is homotopic to* Φ_l *given by*

$$\Phi_l(\gamma) := \begin{bmatrix} \gamma^l & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}, \quad \gamma \in S^1. \tag{47}$$

In particular, for $\varphi_l := \det_{\mathbb{C}} \circ \Phi_l$, *one has* $\deg(\tilde{\varphi}_l, B) = l$.

Combining Proposition 3.35 with Proposition 2.9 yields

COROLLARY 3.36. *Let* V *be an orthogonal* G*-representation and* V_{k_o} *an isotypical component of* V *modeled on an irreducible* G*-representation* $\mathcal{V}_{k_o}$ *of complex type. Assume* $m = \dim V_{k_o} / \dim \mathcal{V}_{k_o}$. *Then*:

(i) $GL^G(V_{k_o}) \simeq GL(m, \mathbb{C})$;

(ii) *for each* $a \in \pi_1(GL^G(V_{k_o}))$, *there exists a representative* $\varphi_a : S^1 \to GL(m, \mathbb{C})$, *such that*

$$\varphi_a(\lambda) = \begin{bmatrix} \lambda^l & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}, \quad \lambda \in S^1,$$

for some $l \in \mathbb{Z}$. *In particular, we have an isomorphism* $\mu_{k_o} : \pi_1(GL^G(V_{k_o})) \to \mathbb{Z}$, *where* $\mu_{k_o}(a) = l$.

Splitting lemma Combining properties (P1)–(P4) of the twisted degree with standard techniques from complex analytic function theory, one can establish

LEMMA 3.37 (Splitting Lemma). *Let* V_1 *and* V_2 *be orthogonal* G*-representations and* $V = V_1 \oplus V_2$. *Assume that the* G*-isotypical decomposition of* V *contains only components modeled on irreducible* G*-representations of complex type. Suppose that* $a_j : S^1 \to GL^G(V_j)$, $j = 1, 2$, *are two continuous maps and* $a : S^1 \to GL^G(V)$ *is given by*

$$a(\lambda) = a_1(\lambda) \oplus a_2(\lambda), \quad \lambda \in S^1.$$

Assume that $\mathcal{O}$ *and* f_a *are defined by* (43) *and* (44), *respectively. Put*

$$\mathcal{O}_j := \left\{ (\lambda, v_j) \in \mathbb{C} \oplus V_j : \ \|v_j\| < 2, \ \frac{1}{2} < |\lambda| < 4 \right\},$$

$$f_{a_j}(\lambda, v_j) := \left(|\lambda|\big(\|v_j\| - 1\big) + \|v_j\| + 1, a_j\left(\frac{\lambda}{|\lambda|}\right)v_j \right),$$

where $j = 1, 2$ *and* $v_j \in V_j$. *Then,*

$$G\text{-Deg}^t(f_a, \mathcal{O}) = G\text{-Deg}^t(f_{a_1}, \mathcal{O}_1) + G\text{-Deg}^t(f_{a_2}, \mathcal{O}_2). \tag{48}$$

S^1-degree formulae Here, we combine the above results to compute the S^1-degree of (44). We start with the following

COROLLARY 3.38. *Let* $V := \mathcal{V}_k$ *be the* k*-th irreducible* S^1*-representation* $(k > 0)$ *equipped with the natural complex structure,* $l \in \mathbb{Z}$ *and*

$$\tilde{f}(\lambda, v) = \left(|\lambda|\big(\|v\| - 1\big) + \|v\| + 1, \left(\frac{\lambda}{|\lambda|}\right)^l v \right), \quad (\lambda, v) \in \overline{\mathcal{O}},$$

where $\mathcal{O}$ *is given by* (43). *Then,*

$$S^1\text{-Deg}(\tilde{f}, \mathcal{O}) = l(\mathbb{Z}_k).$$

PROOF. For the sake of definiteness, assume that $l > 0$ (the case $l \leqslant 0$ can be treated using similar arguments), and consider the map

$$\tilde{f} \times \mathrm{Id} : \overline{\mathcal{O}} \times \overline{B_{l-1}} \to \mathbb{R} \oplus \mathcal{V}_k \oplus \big[\underbrace{\mathcal{V}_k \oplus \cdots \oplus \mathcal{V}_k}_{l-1}\big],$$

where $B_{l-1} = \underbrace{B(\mathcal{V}_k) \times \cdots \times B(\mathcal{V}_k)}_{l-1}$ and $B(\mathcal{V}_k)$ denotes the unit ball in $\mathcal{V}_k$. Then, by suspension property,

$$S^1\text{-Deg}(\tilde{f}, \mathcal{O}) = S^1\text{-Deg}(\tilde{f} \times \mathrm{Id}, \mathcal{O} \times B_{l-1}).$$

Obviously, $\tilde{f} \times \mathrm{Id}$ is equivariantly homotopic, by an $\mathcal{O} \times B_{l-1}$-admissible homotopy, to f_a given by (44) where $v \in V = \underbrace{\mathcal{V}_k \oplus \cdots \oplus \mathcal{V}_k}_{l}$ and $a : S^1 \to GL^{S^1}(V)$ is defined by

$$a(\gamma) = \begin{bmatrix} \gamma^l & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}, \quad \gamma \in S^1.$$

Using Proposition 3.35, it is easy to see that f_a is equivariantly homotopic (by an $\mathcal{O} \times B_{l-1}$-admissible homotopy) to f_b given by

$$f_b(\lambda, v) = \left(|\lambda|\|\big(\|v\| - 1\big) + \|v\| + 1, b\left(\frac{\lambda}{|\lambda|}\right)v \right),$$

with $b: S^1 \to GL^{S^1}(V)$ defined by

$$b(\gamma) = \begin{bmatrix} \gamma & 0 & \dots & 0 \\ 0 & \gamma & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma \end{bmatrix}, \quad \gamma \in S^1.$$

Since $S^1\text{-Deg}(\tilde{f}, \mathcal{O}) = S^1\text{-Deg}(f_b, \mathcal{O} \times B_{l-1})$, by Splitting Lemma (cf. Lemma 3.37) and Proposition 3.34,

$$S^1\text{-Deg}(\tilde{f}, \mathcal{O}) = \underbrace{(\mathbb{Z}_k) + \cdots + (\mathbb{Z}_k)}_{l} = l(\mathbb{Z}_k).$$

□

By combining Proposition 3.35, Corollary 3.36, Splitting Lemma (cf. Lemma 3.37), and Corollary 3.38, we immediately obtain

THEOREM 3.39. *Let V be an orthogonal S^1-representation with $V^{S^1} = \{0\}$, admitting the isotypical decomposition* (42) *and equipped with the natural complex structure. Let $\mathcal{O}$* (*resp. f_a*) *be defined by* (43) (*resp.* (44)). *Then,*

$$S^1\text{-Deg}(f, \mathcal{O}) = \sum_{j=1}^{s} l_j (\mathbb{Z}_{k_j}), \tag{49}$$

where $l_j := \deg(\det_{\mathbb{C}} \circ a_j, B)$, $a_j(\lambda) := a(\lambda)|_{V_{k_j}} : V_{k_j} \to V_{k_j}$, *for* $j = 1, 2, \dots, s$ *and* $B := \{z \in \mathbb{C}\colon\ |z| < 1\}$.

As an immediate consequence of Theorem 3.39, one has

COROLLARY 3.40. *Let V and $\mathcal{O}$ be as in Theorem* 3.39. *Let $l_j \in \mathbb{Z}$, $j = 1, 2, \dots, s$, be given integers and assume that* $\dim_{\mathbb{C}} V_{k_j} = m_j$. *Define $f : \overline{\mathcal{O}} \to \mathbb{R} \oplus V$ by*

$$f(\lambda, v_1, \dots, v_n) = \big(|\lambda|\big(\|v\| - 1\big) + \|v\| + 1, \lambda^{l_1} v_1, \dots, \lambda^{l_s} v_s\big),$$

where $\lambda \in \mathbb{C} \setminus \{0\}$, $v_j \in V_{k_j}$. *Then,*

$$S^1\text{-Deg}(f, \mathcal{O}) = \sum_{j=1}^{n} m_j l_j (\mathbb{Z}_{k_j}). \tag{50}$$

Recurrence formula We complete this subsection with the so-called Recurrence Formula for the twisted G-equivariant degree, which, to some extent, can be counted as the Borsuk–Ulam theorem in the case of one free parameter. This formula allows a reduction of computations of G-equivariant twisted degree to the S^1-degree.

Let G, V, Ω and f be as in Proposition 3.27. Take $(H) \in \Phi_1^t(G)$ and put $f^H := f_{|\mathbb{R}\oplus V^H}$. Clearly (cf. Proposition 3.21(i)), f^H is S^1-equivariant and Ω^H-admissible, thus

$$S^1\text{-Deg}(f^H, \Omega^H) = n_{k_1}(\mathbb{Z}_{k_1}) + \cdots + n_{k_r}(\mathbb{Z}_{k_r}) \tag{51}$$

is correctly defined (cf. (38)). Put

$$\deg_k(f^H, \Omega^H) := n_k. \tag{52}$$

By combining Propositions 3.14, 3.15 and Lemma 3.17, one can prove

THEOREM 3.41 (Recurrence Formula). *Let G, V, $\Omega \subset \mathbb{R} \oplus V$ and f be as in Proposition* 3.27. *Assume, further, that*

$$G\text{-Deg}^t(f, \Omega) = \sum_{(H)\in\Phi_1^t(G,\Omega)} n_H \cdot (H) \tag{53}$$

is defined according to (35)–(37). *Then,*

$$n_H = \left[\sum_k \deg_k(f^H, \Omega^H) - \sum_{(K)>(H)} n_K n(H,K)\left|W(K)/S^1\right|\right] \Big/ \left|W(H)/S^1\right| \tag{54}$$

(*see* (52) (*resp.* (28)) *for the definition of* $\deg_k(f^H, \Omega^H)$ (*resp.* $n(H,K)$)).

3.8. *General concept of basic maps*

The case of one free parameter The concept of the *basic maps*, which was introduced in Subsection 3.6 for S^1-equivariant maps (see (39) and (40)), can be generalized to G-equivariant maps, where $G = \Gamma \times S^1$. Following the same lines as in Subsection 3.6, we also define the counter part of a basic map, called a $\mathbb{C}$-*complementing* map.

DEFINITION 3.42.

(a) Under the notations explained in Remark and Notation 2.6, let $\mathcal{O} \subset \mathbb{R} \oplus \mathcal{V}_{j,l}$ be defined by

$$\mathcal{O} = \left\{(t,v) \colon \frac{1}{2} < \|v\| < 2,\ -1 < t < 1\right\}, \tag{55}$$

and $b \colon \overline{\mathcal{O}} \to \mathcal{V}_{j,l}$ be defined by

$$b(t,v) = \big(1 - \|v\| + \mathrm{i}t\big) \cdot v, \quad (t,v) \in \overline{\mathcal{O}}. \tag{56}$$

Then, the map b is called a *basic map* on $\mathcal{O}$, and the pair $(b, \mathcal{O})$ is called a *basic pair* for the irreducible G-representation $\mathcal{V}_{j,l}$.

(b) Let

$$\Omega = \left\{(\lambda, v) \in \mathbb{C} \oplus \mathcal{V}_{j,l}\colon \|v\| < 1,\ \frac{1}{2} < |\lambda| < 4\right\} \tag{57}$$

and $\mathfrak{f}\colon \overline{\Omega} \to \mathbb{R} \oplus \mathcal{V}_{j,l}$ be defined as

$$\mathfrak{f}(\lambda, v) = \big(|\lambda|\big(\|v\| - 1\big) + \|v\| + 1, \lambda \cdot v\big), \quad (\lambda, v) \in \overline{\Omega}, \tag{58}$$

where $\lambda \cdot v$ denotes the usual complex multiplication of v by λ. Then, the map $\mathfrak{f}$ is called a $\mathbb{C}$-*complementing map* on Ω, and the pair $(\mathfrak{f}, \Omega)$ is called a $\mathbb{C}$-*complementing pair* for the irreducible G-representation $\mathcal{V}_{j,l}$.

Obviously, both (56) and (58) are G-equivariant. Moreover, b is $\mathcal{O}$-admissible and $\mathfrak{f}$ is Ω-admissible. Therefore, $G\text{-Deg}^t(b, \mathcal{O})$ and $G\text{-Deg}^t(\mathfrak{f}, \Omega)$ are correctly defined. Although b and $\mathfrak{f}$ are rather different in nature (b seems to be the simplest one having a non-trivial twisted equivariant degree, while $\mathfrak{f}$ is motivated by the applications to the bifurcation theory, where it appears naturally), we have the following statement parallel to Proposition 3.34.

PROPOSITION 3.43. *Let $\mathcal{U}_j$ be a complex irreducible Γ-representation and $\mathcal{V}_{j,l} := {}^l\mathcal{U}_j$ the associated irreducible representation of $G = \Gamma \times S^1$ (cf. Remark and Notation* 2.6). *Then,*

$$G\text{-Deg}^t(b, \mathcal{O}) = G\text{-Deg}^t(\mathfrak{f}, \Omega), \tag{59}$$

where $\mathcal{O}$, b, Ω and $\mathfrak{f}$ are given by (55)–(57) *and* (58), *respectively.*

For the sequel, it is convenient to introduce

DEFINITION 3.44. Let $(b, \mathcal{O})$ be a basic pair for the irreducible G-representation $\mathcal{V}_{j,l}$. Put

$$\deg_{\mathcal{V}_{j,l}} := G\text{-Deg}^t(b, \mathcal{O}), \tag{60}$$

and call $\deg_{\mathcal{V}_{j,l}}$ the *basic degree* for $\mathcal{V}_{j,l}$.

Using the same ideas as in the proof of Corollary 3.38 in compliance with Corollary 3.36, one can easily establish a formula allowing in certain important cases a reduction to basic degrees.

PROPOSITION 3.45. *Under the notations and assumptions of Problem* 3.32 *suppose*, *in addition*, *that*:

(i) *V has a single G-isotypical component modeled on an irreducible G-representation $\mathcal{V}_{j,l}$ and $m = \dim V / \dim \mathcal{V}_{j,l}$;*
(ii) *the map a from formula* (44)*, identified with a map $S^1 \to GL(m, \mathbb{C})$ (cf. Corollary* 3.36(ii)) *admits a representation* $\mathrm{diag}\{(\lambda^k, 1, 1, \dots, 1\}$.

Then,

$$G\text{-Deg}^t(f_a, \mathcal{O}) = k \cdot \deg_{\mathcal{V}_{j,l}}. \tag{61}$$

Observe that basic maps play an essential role in practical computations of the twisted degree. A typical degree-theoretical approach to a concrete problem is usually devised on the so-called *linearization techniques*. These techniques are based on local or global linear approximations, which can be considered as a way of transforming the original map to the one for which the degree can be easily evaluated. In the case of the equivariant degree with one free parameter, the linearization techniques are also used. However, due to the nature of the equivariant degree, it is inappropriate to expect the same type of computations for all kinds of groups. That means, in order to establish a standard method for the computations of the (twisted) equivariant degree, one needs to identify the key elements in this process. It turns out that the *basic degrees* are essential for the computations of the twisted degrees of arbitrary maps. By providing for a considered group $\Gamma \times S^1$ the tables of basic degrees, one is able to remove essential technical obstructions from the computation process of the twisted degree.

It should be pointed out that the computational formulae presented in Subsection 3.7 are enough to completely evaluate $\deg_{\mathcal{V}_{j,l}}$. Indeed, consider the representation $\mathcal{V}_{j,1}$ (the case of arbitrary $l > 1$ can be analyzed similarly). The Recurrence Formula (see Theorem 3.41) reduces the computations of $\deg_{\mathcal{V}_{j,1}}$ to $\deg_1(f^H, \Omega^H)$ and numbers $n(H, K)$, where $(H), (K) \in \Phi_1^t(G)$. Next, by Corollary 3.40, one obtains

$$\deg_1(f^H, \Omega^H) = \frac{1}{2} \dim \mathcal{V}_{j,1}^H. \tag{62}$$

Thus, to compute $\deg_{\mathcal{V}_{j,1}}$ one needs to know: (i) twisted orbit type lattice in $\mathcal{V}_{j,1}$, (ii) dimensions of H-fixed-point subspaces in $\mathcal{V}_{j,1}$ ($(H) \in \Phi_1^t(G)$), and (iii) numbers $n(H, K)$. Therefore, the computations of basic degrees can be completely computerized.

The case without free parameter Similarly to the situation of one free parameter, one could look for an appropriate concept of *basic maps* in the case without free parameters (cf. Remark 3.28). However, in the latter case, this concept turns out to be almost trivial. To be more specific, consider an irreducible orthogonal representation $\mathcal{V}_i$ of the compact Lie group Γ, and let $\mathcal{D}$ be the unit ball in $\mathcal{V}_i$. It is clear that every linear Γ-equivariant isomorphism A from $\mathcal{V}_i$ onto $\mathcal{V}_i$ is a $\mathcal{D}$-admissible Γ-equivariant map. Therefore, we can consider this type of Γ-equivariant maps as the most elementary, for which the equivariant degree $\Gamma\text{-Deg}(A, \mathcal{D})$ is defined. On the other hand, the representation $\mathcal{V}_i$ can be of one of three types: real, complex or quaternionic (cf. Proposition 2.9(ii)). It is clear that in the case $\mathcal{V}_i$ is of complex or quaternionic type, the operator A can be connected by a path in $GL^\Gamma(\mathcal{V}_i) := \{B \in L^G(\mathcal{V}_i) \colon B \text{ is an isomorphism}\}$ with the identity operator Id. Thus,

Γ-Deg$(A, \mathcal{D}) = (\Gamma)$, which can be considered as a trivial case. In the case the representation $\mathcal{V}_i$ is of real type, the group $GL^{\Gamma}(\mathcal{V}_i)$ has two connected components, so if A does not belong to the same connected component as Id, it is homotopic to $-\mathrm{Id}$, and consequently,

$$\Gamma\text{-Deg}(A, \mathcal{D}) = \Gamma\text{-Deg}(-\mathrm{Id}, \mathcal{D}).$$

Therefore, we call the map $-\mathrm{Id} : \mathcal{V}_i \to \mathcal{V}_i$ a *basic map* for the irreducible representation $\mathcal{V}_i$ (see [93] for more information about this type of basic maps) and put

$$\deg_{\mathcal{V}_i} := \Gamma\text{-Deg}(-\mathrm{Id}, \mathcal{D}). \tag{63}$$

It turns out that $\deg_{\mathcal{V}_i}$ can be completely evaluated using a recurrence formula similar to (53)–(54). More precisely, consider the set $\Phi_0(\Gamma, \mathcal{V}_i)$ of all the orbit types (L_k) in $\mathcal{V}_i$ such that $(L_k) \in \Phi_0(\Gamma)$. Then (see [93]),

$$\deg_{\mathcal{V}_i} = \sum_k n_{L_k} \cdot (L_k), \tag{64}$$

where

$$n_{L_{k_o}} = \frac{(-1)^{n_{k_o}} - \sum_{k=1}^{k_o-1} n(L_{k_o}, L_k) \cdot n_{L_k} \cdot |W(L_k)|}{|W(L_{k_o})|} \tag{65}$$

and $n_k = \dim \mathcal{V}_i^{L_k}$.

3.9. *Multiplicativity property*

In this subsection, we connect the multiplication structures on $A(\Gamma)$ (resp. $A_1^t(G)$) to the Γ-equivariant degree (resp. twisted degree) of Γ-equivariant maps without free parameter (resp. G-equivariant maps with one free parameter). The well-known multiplicativity property of the classical Brouwer degree, which asserts that the degree of a direct product of maps is equal to the product of their degrees (in the ring $\mathbb{Z}$), can be thought of as the prototype of the theorems following below (their proofs use a regular normality argument combined with the definitions of the multiplications in $A(\Gamma)$ (resp. $A_1^t(G)$) and a careful count of orbits of zeros in the product maps).

THEOREM 3.46. *Let Γ be a compact Lie group, U and W two orthogonal Γ-representations, $\Omega_1 \subset U$, $\Omega_2 \subset W$ two open bounded Γ-invariant sets. Assume that $s : V \to V$ (resp. $g : W \to W$) is an Ω_1-admissible (resp. Ω_2-admissible) equivariant map. Then,*

$$\Gamma\text{-Deg}(s \times g, \Omega_1 \times \Omega_2) = \Gamma\text{-Deg}(s, \Omega_1) \cdot \Gamma\text{-Deg}(g, \Omega_2), \tag{66}$$

where '$\cdot$' stands for the multiplication in the Burnside ring $A(\Gamma)$ (see also Remark 3.28).

REMARK 3.47. Clearly, under the notations of Theorem 3.46, one immediately obtains the multiplicativity property for the usual Brouwer degree by putting $\Gamma = \{e\}$.

THEOREM 3.48 (Multiplicativity property). *Let Γ be a compact Lie group, $G = \Gamma \times S^1$, V (resp. W) an orthogonal G-representation (resp. Γ-representation), $\Omega_1 \subset \mathbb{R} \oplus V$ (resp. $\Omega_2 \subset W$) an invariant open bounded set and $f_1 : \mathbb{R} \oplus V \to V$ (resp. $f_2 : W \to W$) an Ω_1-admissible (resp. Ω_2-admissible) equivariant map. Then,*

$$G\text{-Deg}^t(f_1 \times f_2, \Omega_1 \times \Omega_2) = \Gamma\text{-Deg}(f_2, \Omega_2) \cdot G\text{-Deg}^t(f_1, \Omega_1),$$

where the multiplication is taken in the $A(\Gamma)$-module $A_1^t(G)$ (see also Remark 3.28).

REMARK 3.49. Obviously, in the case $\Gamma = \{e\}$, one has to take for Γ-Deg(f_2, Ω_2) the usual Brouwer degree and for "$\cdot$" the usual multiplication by an integer.

3.10. *Infinite dimensional twisted degree*

In order to apply the twisted degree theory developed above to symmetric Hopf bifurcation problems, one should extend it to reasonable classes of equivariant vector fields on *infinite dimensional* Banach G-representations. To this end, it is necessary to adopt the standard infinite dimensional techniques in the equivariant setting. Below, we briefly discuss the corresponding framework for two classes of vector fields relevant to the applications we are dealing with in this paper: (i) compact fields and (ii) condensing fields. Observe, however, that extensions to many other important classes can be easily done as well.

Throughout this subsection, W is a real isometric Banach G-representation ($G = \Gamma \times S^1$), $E := \mathbb{R} \oplus W$ is equipped with the norm $\|(\lambda, v)\| = \max\{|\lambda|, \|v\|\}$ (G acts trivially on $\mathbb{R}$) and $\pi : E \to W$ stands for the natural projection on W. Given a G-invariant subset $M \subset E$ and a G-equivariant map $F : M \to W$, define $f : M \to W$ by

$$f := \pi - F, \tag{67}$$

and call f a *G-equivariant field* on M.

Leray–Schauder twisted degree Recall the following

DEFINITION 3.50. Let $X \subset E$ be a bounded subset. Then, a continuous map $F : X \to W$ is called *compact*, if $\overline{F(X)}$ is compact in W.

THEOREM 3.51 (Equivariant Schauder approximation theorem). *Let $X \subset E$ be a G-invariant bounded subset and $F : X \to W$ a G-equivariant compact map. Then, for every $\varepsilon > 0$, there exists a G-equivariant finite-dimensional map $F_\varepsilon : X \to W$ (i.e. the image $F_\varepsilon(X)$ is contained in a finite-dimensional subrepresentation of W) such that*

$$\|F_\varepsilon(x) - F(x)\| < \varepsilon \quad \textit{for all } x \in X. \tag{68}$$

PROOF. Take W_∞ defined by (17)–(19). By (20), there exists a finite set $N = \{v_1, \dots, v_n\} \subset W_\infty$ such that $F(X) \subset N_\varepsilon := N + B_\varepsilon(0)$, where $B_\varepsilon(0) = \{v \in W\colon \|v\| < \varepsilon\}$. Define the functions $\nu_k : N_\varepsilon \to \mathbb{R}$, $k = 1, \dots, n$, by $\nu_k(v) := \max\{0, \varepsilon - \|v - v_k\|\colon k = 1, 2, \dots, n\}$, $v \in N_\varepsilon$, and put $P_\varepsilon(v) := \frac{1}{\sum_{k=1}^n \nu_k(v)} \sum_{k=1}^n \nu_k(v) v_k \in \operatorname{conv}\{v_1, \dots, v_n\}$, $v \in N_\varepsilon$, where "conv" stands for the convex hull.

Next, define the map $\widetilde{F}_\varepsilon : X \to W$ by $\widetilde{F}_\varepsilon(x) := P_\varepsilon(F(x))$ for $x \in X$. Clearly, $\widetilde{F}_\varepsilon$ satisfies (68) and $\widetilde{F}_\varepsilon(X) \subset \operatorname{span}\{G(v_1), \dots, G(v_n)\} =: A$. By Proposition 2.12, A is G-invariant and finite-dimensional.

To complete the proof, put $F_\varepsilon(x) := \int_G g \widetilde{F}_\varepsilon(g^{-1}x)\, \mathrm{d}\mu(g)$, $x \in X$ (here μ denotes the Haar measure on G). $\square$

The construction of the Leray–Schauder equivariant degree is standard.

DEFINITION 3.52. Assume that $\Omega \subset E$ is an open bounded G-invariant subset, $F : \overline{\Omega} \to W$ is a compact equivariant map.

(a) A G-equivariant field f defined by (67) is called *compact*. Such a field f is called Ω-*admissible* if $f(x) \neq 0$ for all $x \in \partial\Omega$.

(b) Two G-equivariant Ω-admissible compact fields $f_0 = \pi - F_0$ and $f_1 = \pi - F_1$ are said to be *G-equivariantly homotopic*, if there exists a G-equivariant compact map $H : [0, 1] \times \overline{\Omega} \to W$ with $h := \pi - H$ satisfying the following conditions:

 (i) $h(0, \cdot) = f_0$, $h(1, \cdot) = f_1$;

 (ii) for each $\lambda \in [0, 1]$, $h(\lambda, \cdot)$ is Ω-admissible.

 The map h will be referred to as a *G-equivariant homotopy of compact fields* joining f_0 and f_1.

In order to define the twisted degree $G\text{-}\mathrm{Deg}^t(f, \Omega)$ of an Ω-admissible G-equivariant compact field f, take an equivariant finite-dimensional map $F_\varepsilon : \overline{\Omega} \to W$ such that $F_\varepsilon(\overline{\Omega}) \subset W_*$, $\dim W_* < \infty$ and

$$\big\| F(x) - F_\varepsilon(x) \big\| < \inf\big\{ \big\| f(y) \big\|\colon\ y \in \partial\Omega \big\},$$

and next, define the ε-approximation f_ε of f as $f_\varepsilon(\lambda, v) = v - F_\varepsilon(\lambda, v)$, $(\lambda, v) \in \overline{\Omega}$ (cf. Theorem 3.51). Then, put

$$G\text{-}\mathrm{Deg}^t(f, \Omega) \overset{\text{def}}{=} G\text{-}\mathrm{Deg}^t(f_\varepsilon|_{\overline{\Omega}_*}, \Omega_*) \in A_1^t(G), \tag{69}$$

where $\Omega_* := \Omega \cap (\mathbb{R} \oplus W_*)$, and call $G\text{-}\mathrm{Deg}^t(f, \Omega)$ the *Leray–Schauder twisted degree*.

By applying the standard arguments, one can verify that the definition, given by formula (69), neither depends on a choice of the equivariant approximation f_ε, nor on an invariant finite-dimensional subspace W_* (see [93] for more details).

Using Definition 3.52 and following the standard lines (cf. [81] and [93]), one can formulate and prove an analogue of Proposition 3.27 for compact fields and homotopies.

Nussbaum–Sadovskii twisted degree Denote by $\mathcal{M}$ the class of all bounded subsets of E.

DEFINITION 3.53. A function $\mu : \mathcal{M} \to [0, \infty)$ is called a *measure of noncompactness* on E if for all $A, B \in \mathcal{M}$, the following conditions are satisfied:

(a) $\mu(A) = 0 \Leftrightarrow \bar{A}$ is compact;
(b) $\mu(A) = \mu(\bar{A})$;
(c) $\mu(\mathrm{conv}(A)) = \mu(A)$;
(d) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$;
(e) $\mu(rA) = |r| \cdot \mu(A)$, $r \in \mathbb{R}$;
(f) $\mu(A + B) \leqslant \mu(A) + \mu(B)$.

We refer to [1] and [93], where many important examples of measures of noncompactness are considered as well as their properties are studied.

DEFINITION 3.54. Let μ be a measure of noncompactness defined on $\mathcal{M}$. Take $X \subset E$ and $Y \subset W$. Consider a continuous map $F : X \to Y$ taking bounded subsets of X to bounded subsets of Y. We say that F is:

(i) a *μ-Lipschitzian map* with a constant $L \geqslant 0$, if $\mu(F(A)) \leqslant L\mu(A)$ for all bounded subsets A of X;
(ii) a *completely continuous map*, if it is μ-Lipschitzian with a constant $L = 0$;
(iii) a *condensing map*, if it is μ-Lipschitzian with a constant $L = 1$ and $\mu(F(A)) < \mu(A)$ for every bounded subset $A \subset X$ such that $\mu(A) > 0$.

REMARK 3.55.

(i) Clearly, any compact map is also completely continuous. The opposite statement is true under the assumption that the domain of the map is bounded. Also, any completely continuous map is, of course, condensing with respect to any measure of noncompactness.
(ii) Obviously, if $F : E \to W$ is a contraction map (i.e. there exists $0 \leqslant q < 1$ such that $\|F(x) - F(y)\| \leqslant q\|x - y\|$ *for all* $x, y \in E$), then F is condensing with respect to any measure of noncompactness on $\mathcal{M}$ (the applications to neutral FDEs we will be dealing with appeal exactly to this situation).

The idea of extending the equivariant Leray–Schauder degree to the class of equivariant condensing fields, is based on the same standard construction that is used in order to make such an extension in the non-equivariant case (see for example [1] and [93], Chapter 4). For the sake of completeness of our presentation, we discuss shortly some of the main steps of this construction.

DEFINITION 3.56. Let $\Omega \subset E$ be an open bounded G-invariant subset and $F : \overline{\Omega} \to W$ a condensing equivariant map with respect to some (fixed) measure of noncompactness μ on E.

(a) The map f defined by (67) is called a *condensing field*.
(b) Two G-equivariant Ω-admissible condensing fields $f_0 = \pi - F_0$ and $f_1 = \pi - F_1$ are called *G-equivariantly homotopic*, if there exists a G-equivariant condensing map $H : [0, 1] \times \overline{\Omega} \to W$ with $h := \pi - H$ satisfying the following conditions:

(i) $h(0,\cdot) = f_0$, $h(1,\cdot) = f_1$;

(ii) for each $\lambda \in [0,1]$, $h(\lambda,\cdot)$ is Ω-admissible.

The map h will be referred to as a *G-equivariant homotopy of condensing fields* joining f_0 and f_1. The map H is called a *condensing G-equivariant homotopy* between F_0 and F_1.

The following notion (see [1] for the non-equivariant setting) plays the central role in our considerations.

DEFINITION 3.57. Let $M \subset E$ be an invariant subset and $F : M \to W$ an equivariant map. A subset $Q \subset W$ is called *G-fundamental* for F, if it satisfies the following conditions:

(a) Q is non-empty, compact, convex and G-invariant;

(b) $F(M \cap Q) \subset Q$;

(c) if $x_o \in M \setminus Q$, then $x_o \notin \mathrm{conv}(F(x_o) \cup Q)$.

Observe that a G-fundamental set Q for F contains all the fixed points of F.

Similarly, one can define a notion of a G-fundamental set for a G-equivariant deformation $H : [0,1] \times M \to W$, namely, this set should be G-fundamental for each $H(\lambda,\cdot)$.

REMARK 3.58. Obviously, if $X \subset W$ is a (non-empty) compact G-invariant subset, then so is $\overline{\mathrm{conv}(X)}$. Therefore, for any G-equivariant compact map $F : M \to W$, where $M \subset E$, the set $\overline{\mathrm{conv}(F(M))}$ is G-fundamental.

It turns out (cf. [1])

LEMMA 3.59. *Let $\Omega \subset E$ be an open bounded G-invariant subset, $H : [0,1] \times \overline{\Omega} \to W$ a G-equivariant Ω-admissible condensing homotopy, $K \subset W$ an arbitrary G-invariant compact subset. Then, there exists a G-fundamental set for H containing K.*

Assume that $\Omega \subset E$ is an open bounded G-invariant subset, $\pi - F : \overline{\Omega} \to W$ a G-equivariant condensing Ω-admissible field. Let $Q \subset W$ be a G-fundamental set for F provided by Lemma 3.59. By the equivariant Dugundji theorem (see Proposition 3.9), there exists a G-equivariant extension $\widetilde{F} : E \to Q$ of the map $F|_{\overline{\Omega} \cap Q}$. Put $\overline{F} := \widetilde{F}|_{\overline{\Omega}}$. Take the compact G-equivariant field $\pi - \overline{F}$ and put

$$G\text{-}\mathrm{Deg}^t(\pi - F, \Omega) := G\text{-}\mathrm{Deg}^t(\pi - \overline{F}, \Omega). \tag{70}$$

We call $G\text{-}\mathrm{Deg}^t(\pi - F, \Omega)$ the *Nussbaum–Sadovskii equivariant degree*. Using Lemma 3.59, it is easy to show that the definition given by formula (70), neither depends on a choice of a G-fundamental set, nor on an equivariant extension $\overline{F}$ (see [1,93] for more details).

Using Definition 3.56 and following the standard lines (cf. [93] and [1]), one can formulate and prove an analogue of Theorem 3.27 for condensing maps.

3.11. *Bibliographical remarks*

Equivariant extensions via fundamental domains is one of the crucial ideas underlying our approach to the equivariant degree. In the context relevant to our discussion, a simplicial fundamental domain for an action of a cyclic group of prime order was used (probably, for the first time) in [47]. The case of an arbitrary cyclic group action on a sphere was considered by M. Krasnosel'skii [88] (see also the paper of P. Zabreiko [149], where a slightly more general case was considered). The case of an arbitrary finite group action on a subset of $\mathbb{R}^n$ was studied by Z. Balanov and S. Brodsky in [9]. In full generality, Theorems 3.2 and 3.4 were proved by Z. Balanov and A. Kushkuley in [20,21,98,99]. The concept of regular fundamental domain (as well as the proof of Theorem 3.7) was suggested in [14] (see also [19]). The detailed study of the case of linear actions of Abelian groups can be found in [80,81]. For the equivariant extension problem in a more general setting, we refer to [3,5,7].

The concept of normal/regular normal approximations (being, together with fundamental domains, another important ingredient of our approach to studying (twisted) equivariant degree) was discussed, on the one hand, by K. Gęba, W. Krawcewicz and J. Wu in [60], and on the other hand, by A. Kushkuley and Z. Balanov in [99] (cf. [96]). For other "equivariant general position" results (including those for different actions on the domain and image) closed in spirit to regular normal approximations, we refer to [99] (cf. [115,150]) and [81].

The numbers $n(L, H)$ were defined by E. Ihrig and M. Golubitsky in [74] (see also [99]). Proposition 3.15 and Lemma 3.17 were suggested in [14]. For a detailed discussion on the concept of Burnside ring (as well as for computations of several multiplication tables), we refer to [43] and [93]. Twisted subgroups of $\Gamma \times S^1$ for certain groups Γ of practical meaning, were studied in [10,11,15,17,18,14,67,65,66,91,92,145] (some additional information (including the numbers $n(L, H)$), can be found in [27,29,32,67,99]). The notion of bi-orientability was introduced by G. Peschke in [126] (see also [60]). Some important multiplication tables for $A_1^t(G)$ can be found in [10,18,92]. In fact, the $A(\Gamma)$-module structure on $A_1^t(\Gamma \times S^1)$ coincides with the appropriate restrictions of the multiplication in the Euler ring $U(\Gamma \times S^1)$ (cf. [43,59,134]), where $A(\Gamma)$ is identified with $\mathbb{Z}[\Phi(\Gamma \times S^1)] \subset U(\Gamma \times S^1)$. In the case Γ is a finite group, we have $U(\Gamma \times S^1) \simeq A(\Gamma) \oplus A_1^t(\Gamma \times S^1)$ (cf. [43]).

The concept of twisted degree was developed in [11,14,19]. Among the important "predecessors" of (twisted) S^1-equivariant degree, one should mention the rational-valued homotopy invariants introduced and studied in [57,39,40,42,41]. Actually, the approach suggested in Subsection 3.4 to construct the twisted degree, can be easily extended to a more general setting: G is an arbitrary compact Lie group (not necessarily of the form $G = \Gamma \times S^1$) and $f : \mathbb{R}^n \oplus V \to V$ is an equivariant map with $n \geqslant 1$ free parameters. In this case, the corresponding invariant (known as the *primary equivariant degree*) takes its values in the $\mathbb{Z}$-module $\mathbb{Z}[\Phi_n^+(G)]$, where $\Phi_n^+(G)$ consists of all "primary" conjugacy classes (H), i.e. $\dim W(H) = n$ and $W(H)$ is bi-orientable. It was considered by K. Gęba, W. Krawcewicz and J. Wu in [60] (following the differential topology framework), G. Peschke in [126] (using the algebraic topology ideas) and W. Krawcewicz and H. Xia in [96] (using an analytical approach). The axiomatic approach to the primary degree based on the use of (regular) fundamental domains was suggested by Z. Balanov,

W. Krawcewicz and H. Ruan in [14] (see also [19]). In turn, the primary equivariant degree is a part of the construction of equivariant degree defined by J. Ize, I. Massabó and A. Vignoli in [78] which is an equivariant homotopy class of a certain auxiliary equivariant map of G-representation spheres (a detailed exposition of this concept for *Abelian* group representations (including many non-trivial and elegant examples) can be found in [81]). In particular, this degree (as a complete equivariant homotopy invariant) takes into account the so-called "secondary" orbit types, i.e. those with $\dim W(H) < n$. Observe, however, that (i) the primary degree was defined in [60] (see also [45,46] for $G = S^1$) independently of the work [78] , and (ii) the general construction of the equivariant degree given in [78] (see also [81]), being of great theoretical importance, does not give practical hints for its computation.

The importance of S^1-basic and $\mathbb{C}$-complementing maps was indicated in [14] (see also [49,93,18]). The Splitting Lemma was proved in [14] based on ideas from [91]. The computational formulae for the S^1-degree were established in [14] (see also [46] and [93]). The Recurrence Formula (in a slightly different version) was suggested in [91] (see also [14] and [19]).

Twisted degrees of basic maps (being another important ingredient of our approach), for various irreducible representations, were considered in [10,11,18,14,91,92], where the importance of the numbers $n(L, H)$ for the computations of the basic degrees was first indicated. The multiplicativity property of the equivariant degree, which is crucial from the applications viewpoint, was studied in [17,18,91] (see also [81] for the Abelian group actions). The case without free parameter was considered in [93,141]. Finally, a systematic exposition of the equivariant degree theory for many important classes of non-Abelian group actions can be found in [19].

4. Hopf bifurcation problem for ODEs without symmetries

In this section, we start the (twisted) equivariant degree treatment of the Hopf bifurcation phenomena in (symmetric) dynamical systems. To clarify the main ingredients of our approach, we begin with the simplest situation – a parameterized by $\alpha \in \mathbb{R}$ system of ODEs $\dot{x} = f(\alpha, x)$, $x \in \mathbb{R}^N$, *without* spatial symmetries. It turns out that it is possible:

(i) to reformulate the original problem as the S^1-fixed point problem for a compact map $\mathcal{F} : \Omega \subset \mathbb{R} \oplus W \to W$, where W is an appropriate space of periodic $\mathbb{R}^N$-valued functions with the S^1-action induced by the shift of time variable;

(ii) to associate to the compact field $\mathfrak{F} := \pi - \mathcal{F}$ the S^1-degree whose nontriviality indicates the occurrence of a Hopf bifurcation (here, $\pi : \mathbb{R} \oplus W \to W$ is the natural projection).

In the next section, assuming the system to satisfy some Γ-symmetry conditions, we associate to $\mathfrak{F}$ the $\Gamma \times S^1$-equivariant twisted degree allowing to classify the bifurcating solutions according to their symmetries. The main advantage of this method rests on the fact that it can be equivalently applied (with certain cosmetic modifications) to other classes of equations (for instance, functional differential equations, neutral functional differential equations or partial functional differential equations).

4.1. *Statement of the problem*

Put $V := \mathbb{R}^N$. We begin with the following system of ODEs:

$$\begin{cases} \dot{x} = f(\alpha, x), & x \in V,\ \alpha \in \mathbb{R}, \\ x(0) = x(p), & \text{for a certain } p > 0, \end{cases} \tag{71}$$

where the function $f : \mathbb{R} \times V \to V$ is of class C^1 and satisfies the property

$$f(\alpha, 0) = 0 \quad \text{for all } \alpha \in \mathbb{R}. \tag{72}$$

The stationary point, at which we are analyzing the occurrence of small amplitude periodic solutions for the system (71), is (by the assumption (72)) the origin $0 \in V$. We say that for $\alpha = \alpha_o$, the system (71) has a *Hopf bifurcation* occurring at $(\alpha_o, 0)$ corresponding to the "limit period" $\frac{2\pi}{\beta_o}$, if there exists a family of p_s-periodic non-constant solutions $\{(\alpha_s, x_s(t))\}_{s\in\Lambda}$ (for a proper index set Λ) of (71) satisfying the conditions:

(1) The set $K := \overline{\bigcup_{s\in\Lambda}\{(\alpha_s, x_s(t)) \colon t \in \mathbb{R}\}}$ contains a compact connected (infinite) set C such that $(\alpha_o, 0) \in C$,

(2) $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that

$$\forall \big(\alpha_s, x_s(t)\big) \in C \quad \sup_t \big\| x_s(t) \big\| < \delta \Rightarrow \|\alpha_o - \alpha_s\| < \varepsilon \text{ and } \left\| p_s - \frac{2\pi}{\beta_o} \right\| < \varepsilon.$$

With the system (71), associate the linearized at $x = x_o$ system

$$\dot{x} = D_x f(\alpha, x_o)x, \quad x \in V. \tag{73}$$

Taking the complexification of (73) and substituting the function $x(t) = \mathrm{e}^{\lambda t} \cdot v$ with $\lambda \in \mathbb{C}$ and $v \in V$, leads to the following *characteristic operator*

$$\Delta_{(\alpha, x_o)}(\lambda) := \lambda\, \mathrm{Id} - D_x f(\alpha, x_o) : V^c \to V^c, \tag{74}$$

where $V^c = \mathbb{C}^N$ denotes the complexification of V, $D_x f(\alpha, x_o)(z \otimes x) = z \otimes D_x f(\alpha, x_o)x$ for $z \otimes x \in V^c$, and the *characteristic equation*

$$\det\nolimits_{\mathbb{C}}\big[\Delta_{(\alpha, x_o)}(\lambda)\big] = 0, \quad \lambda \in \mathbb{C}. \tag{75}$$

A solution to (75) is called a *characteristic root* at the stationary point (α, x_o). It is obvious that the Hopf bifurcation occurs for the linearized system (73) with $\alpha = \alpha_o$ if and only if the characteristic equation (75) (for $\alpha = \alpha_o$) has a (non-zero) purely imaginary root $\lambda = \mathrm{i}\beta_o$. It will turn out that the appearance of a purely imaginary *characteristic root* can provide appropriate information to predict the occurrence of Hopf bifurcation for the non-linear system (71). In what follows, put $\Delta_\alpha(\cdot) := \Delta_{(\alpha,0)}(\cdot)$.

4.2. S^1*-equivariant reformulation of the problem*

Normalization of the period Introduce the unknown period p as an additional parameter to the considered system (71) of ODEs. For this purpose, substitute $u(t) := x(\frac{p}{2\pi}t)$, which leads to

$$\begin{cases} \dot{u} = \frac{p}{2\pi} f(\alpha, u), \quad u \in V, \\ u(0) = u(2\pi), \end{cases} \tag{76}$$

where the unknown function $u(t)$ is considered to be of the fixed 2π-period. Next, reparametrize the system (76) by introducing $\beta := \frac{2\pi}{p}$, i.e. we obtain

$$\begin{cases} \dot{u} = \frac{1}{\beta} f(\alpha, u), \quad u \in V, \\ u(0) = u(2\pi). \end{cases} \tag{77}$$

Setting in functional spaces and S^1*-isotypical decomposition* The system (77) can be reformulated as an equation in the appropriate functional spaces. Denote by $W := H^1(S^1; V)$ the first Sobolev space of V-valued functions defined on S^1, which can be identified with the corresponding space of 2π-periodic V-valued functions via the identification $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$. Formula

$$(\zeta u)(t) := u(t + \tau) \quad (t \in \mathbb{R},\ u \in W,\ \zeta = e^{i\tau} \in S^1) \tag{78}$$

determines on W the structure of isometric Hilbert S^1-representation with respect to the standard inner product in W given by

$$\langle u, v \rangle_{H_1} := \int_0^{2\pi} \big(u(t) \bullet v(t) + u'(t) \bullet v'(t)\big)\, dt.$$

Identify the subspace of W consisting of all constant functions with V (it can be thought of as the “0-mode” of the isotypical decomposition of W). To describe other S^1-isotypical components W_l, $l = 1, 2, 3, \dots$, of the space W, observe that each component W_l is generated (as a linear space) by the functions $u(t) = \cos lt \cdot a_l + \sin lt \cdot b_l$, for $t \in [0, 2\pi]$, $a_l, b_l \in V$. Consequently, the function $u(t)$ can be viewed as

$$u(t) = e^{ilt} \cdot (x_l + iy_l), \quad x_l, y_l \in V. \tag{79}$$

Using (79), we can describe the action of $e^{i\tau} \in S^1$ on $u(t)$ as the ‘complex multiplication’ by $e^{il\tau}$. Moreover, each of the subspaces W_l, equipped with this S^1-action, is S^1-isomorphic to the complexification V^c of V, considered with the S^1-action defined by the l-folding, i.e. $W_l \simeq {}^l V^c$ (cf. Subsection 3.6 and Remark and Notation 2.6). Also, we can

write the elements $u(t) \in W_l$ as $u(t) = \mathrm{e}^{\mathrm{i}lt} \cdot z$, $z \in \mathbb{C}^N$. Therefore, we have the following S^1-isotypical decomposition of W:

$$W = \overline{V \oplus \bigoplus_{l=1}^{\infty} W_l}. \tag{80}$$

S^1-equivariant fixed-point problem Define the following operators:

$$L : W \to L^2(S^1; V), \quad L(u) = \dot{u}, \tag{81}$$

$$j : W \to C(S^1; V), \quad j(u) = \tilde{u}, \tag{82}$$

where $u \in W$ and $C(S^1; V)$ denotes the space of 2π-periodic, V-valued continuous functions on $\mathbb{R}$ equipped with the usual sup-norm and $\tilde{u} = u$ a.e. (cf. [137]). For a function $u(t) = \mathrm{e}^{\mathrm{i}lt} \cdot z \in W_l$, we have $L(u) = \mathrm{i}lu$, i.e. $L|_{W_l} = \mathrm{i}l \,\mathrm{Id}|_{W_l}$. Put $\mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}_+$, $\mathbb{R}_+ := \{r \in \mathbb{R}\colon r > 0\}$, and let $\pi\colon \mathbb{R}^2 \oplus W \to W$ be the natural projection. Denote by N_f the Nemytsky operator associated with the function f, i.e. for every $\alpha \in \mathbb{R}$, put

$$\big(N_f(\alpha, v)\big)(t) := f\big(\alpha, v(t)\big), \quad v \in C(S^1; V). \tag{83}$$

Notice that the system (71) is equivalent to the following operator equation

$$Lu = \frac{1}{\beta} N_f\big(\alpha, j(u)\big), \quad (\alpha, \beta) \in \mathbb{R}^2_+,\ u \in W. \tag{84}$$

Equation (84) can be transformed to an S^1-equivariant fixed-point problem in $\mathbb{R}^2 \oplus W$. Define the operator $K : W \to L^2(S^1; V)$ by

$$K(u) := \frac{1}{2\pi} \int_0^{2\pi} u(t)\,\mathrm{d}t, \quad u \in W, \tag{85}$$

which is simply a projection on the subspace V of constant functions (cf. Remark and Notation 2.17). Then, it can be easily verified that the operator $L + K : W \to L^2(S^1; V)$ is an isomorphism. Put

$$\begin{aligned} &\mathcal{F}(\alpha, \beta, u) := (L + K)^{-1}\left[\frac{1}{\beta} N_f\big(\alpha, j(u)\big) + K(u)\right], \\ &\mathfrak{F}(\alpha, \beta, u) := u - \mathcal{F}(\alpha, \beta, u). \end{aligned} \tag{86}$$

In this way, the following equation is equivalent to (84):

$$\mathfrak{F}(\alpha, \beta, u) = 0, \tag{87}$$

where $(\alpha, \beta) \in \mathbb{R}^2_+$, $u \in W$ (cf. Subsection 3.6 and Remark and Notation 2.6).

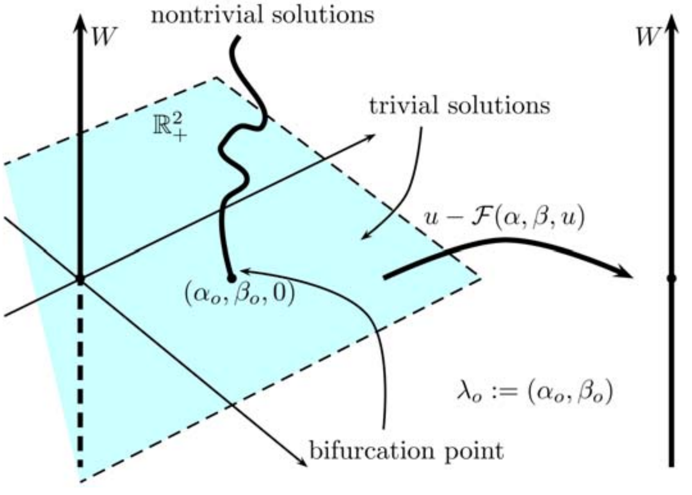

Fig. 1. Hopf bifurcation as a two-parameter fixed-point problem in functional spaces.

4.3. *S^1-degree method for Hopf bifurcation problem*

It is clear that every point $(\alpha, \beta, 0) \in \mathbb{R}^2_+ \times W$ is a solution to Eq. (87) (called a *trivial solution*). Denote the set $\mathbb{R}^2_+ \times \{0\} \subset \mathbb{R}^2_+ \times W$ by M. We would like to identify the so-called *bifurcation points* to (87), i.e. the trivial solutions $(\alpha_o, \beta_o, 0) \in M$ such that in every neighborhood of $(\alpha_o, \beta_o, 0)$ there is a non-trivial solution $(\alpha, \beta, u) \in \mathbb{R}^2_+ \times W$. Actually, in the case $(\alpha_o, \beta_o, 0) \in M$ is a bifurcation point, there is a "branch" in $\mathbb{R}^2_+ \times W$ of non-trivial solutions bifurcating from the trivial solution $(\alpha_o, \beta_o, 0) \in M$. The bifurcation problem for Eq. (87) is illustrated on Fig. 1.

Necessary condition for the occurrence of Hopf bifurcation By the implicit function theorem, the *necessary condition* for the point $(\alpha_o, \beta_o, 0)$ to be a bifurcation point, is that the linear operator

$$\mathrm{Id} - D_u\mathcal{F}(\alpha_o, \beta_o, 0) : W \to W$$

is not an isomorphism. In such a case, we say that $(\alpha_o, \beta_o, 0) \in M$ is a *W-singular point*. It follows directly from the definition of the map $\mathcal{F}$ that

$$a(\alpha, \beta) := \mathrm{Id} - D_u\mathcal{F}(\alpha, \beta, 0) = \mathrm{Id} - (L + K)^{-1}\left[\frac{1}{\beta} N_{D_x f}(\alpha, 0) + K\right], \tag{88}$$

where the operator $N_{D_x f}(\alpha, 0)$ is the Nemytsky operator of $D_x f(\alpha, 0)$, defined by

$$\big(N_{D_x f}(\alpha, 0) j(u)\big)(t) := D_x f(\alpha, 0) j(u)(t), \quad u \in W.$$

Clearly, $a(\alpha,\beta)(V)\subset V$ and $a(\alpha,\beta)(W_l)\subset W_l$, for all $l=1,2,3,\dots$. Put

$$a_0(\alpha,\beta):=a(\alpha,\beta)|_V,\quad a_l(\alpha,\beta):=a(\alpha,\beta)|_{W_l},\quad l=1,2,3,\dots. \tag{89}$$

It is clear that for a constant function $\bar u\in V\subset W$,

$$a_0(\alpha,\beta)\bar u=-\frac{1}{\beta}D_x f(\alpha,0)\bar u, \tag{90}$$

and for $e^{ilt}\cdot z\in W_l$, where $z=x+iy$, $x,y\in V$, we have (see (74) and (79))

$$a_l(\alpha,\beta)=\frac{1}{il\beta}\Delta_\alpha(il\beta). \tag{91}$$

Consequently, if $(\alpha_o,\beta_o,0)$ is a W-singular point, then at least one of the linear operators $a_l(\alpha_o,\beta_o)$, $l=0,1,2,\dots$, cannot be an isomorphism. In order to exclude the possibility for the occurrence of the so-called steady-state bifurcation (i.e. a bifurcation of constant solutions), we assume that the operator (90) is an isomorphism, i.e.

$$\det D_x f(\alpha_o,0)\neq 0. \tag{92}$$

Using (89)–(92) one can easily establish

PROPOSITION 4.1 (Necessary condition for Hopf bifurcation). *Assume the operator* $a(\alpha_o,\beta_o): W\to W$ (*see* (88)) *is not an isomorphism for some* $(\alpha_o,\beta_o)\in\mathbb{R}\times\mathbb{R}_+$, *and suppose condition* (92) *is satisfied. Then, the characteristic equation* (75) *has a purely imaginary root* $il\beta_o$ *for* $\alpha=\alpha_o$ *and some* $l\in\mathbb{N}$.

DEFINITION 4.2.

(i) A point $(\alpha_o,x_o)\in\mathbb{R}\oplus\mathbb{R}^N$ is called a *center* for the system (71), if its characteristic equation (75) has a purely imaginary characteristic value $i\beta_o$ ($\beta_o>0$) for $\alpha=\alpha_o$.

(ii) A center (α_o,x_o) for the system (71) is called *isolated* if it is the only center for (71) in a certain neighborhood of (α_o,x_o) in $\mathbb{R}\oplus V$.

Sufficient conditions for the occurrence of Hopf bifurcation In what follows, it is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$, and instead of writing $(\alpha,\beta)\in\mathbb{R}^2$, we use the notation $\lambda=\alpha+i\beta$. In particular, we write $\lambda_o=\alpha_o+i\beta_o$.

Assume $(\alpha_o,0)$ is an isolated center for the system (71) with $\lambda_o=\alpha_o+i\beta_o$ (see Definition 4.2). Then, the point $(\lambda_o,0)\in\mathbb{R}^2_+\times W$ is an isolated W-singular point of $\mathfrak{F}$ (cf. (86)), i.e. for a sufficiently small $\varepsilon>0$, the linear operator $a(\lambda)=\mathrm{Id}-D_u\mathcal{F}(\lambda,0): W\to W$, where $\lambda=\alpha+i\beta$ and $|\lambda-\lambda_o|<\varepsilon$, is not an isomorphism only if $\lambda=\lambda_o$. Consequently, by the implicit function theorem, there exists $r>0$ such that for all points $(\lambda,u)\in\mathbb{R}^2_+\oplus W$ with $|\lambda-\lambda_o|=\varepsilon$ and $0<\|u\|\leqslant r$, we have $u-\mathcal{F}(\lambda,u)\neq 0$. Define the set $\Omega\subset\mathbb{R}^2\oplus W$ by

$$\Omega=\{(\lambda,u)\in\mathbb{R}^2\oplus W:\ |\lambda-\lambda_o|<\varepsilon \text{ and } \|u\|<r\}. \tag{93}$$

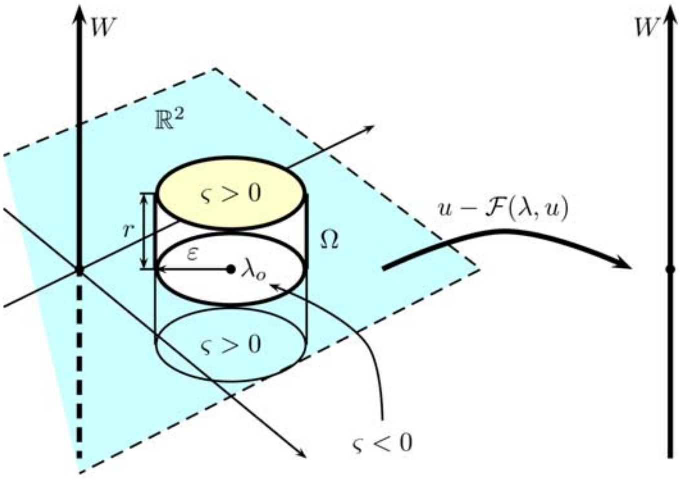

Fig. 2. Auxiliary function for Hopf bifurcation.

Put

$$\partial_0 := \overline{\Omega} \cap \big(\mathbb{R}^2 \times \{0\}\big) \quad \text{and} \quad \partial_r := \big\{(\lambda, u) \in \overline{\Omega}\colon\ \|u\| = r\big\}.$$

DEFINITION 4.3. An S^1-invariant continuous function $\varsigma : \overline{\Omega} \to \mathbb{R}$ is called an *auxiliary* function for $\mathfrak{F}$ in Ω, if

$$\begin{cases} \varsigma(\lambda, u) > 0 & \text{for } (\lambda, u) \in \partial_r, \\ \varsigma(\lambda, u) < 0 & \text{for } (\lambda, u) \in \partial_0 \end{cases}$$

(see Fig. 2).

The existence of an auxiliary function is provided by Proposition 3.9.

Take an auxiliary function ς and define the map $\mathfrak{F}_\varsigma : \overline{\Omega} \to \mathbb{R} \oplus W$ by

$$\mathfrak{F}_\varsigma(\lambda, u) := \big(\varsigma(\lambda, u), u - \mathcal{F}(\lambda, u)\big), \quad (\lambda, u) \in \overline{\Omega}. \tag{94}$$

Since $\mathfrak{F}_\varsigma(\lambda, u) \neq 0$ for $(\lambda, u) \in \partial\Omega$, it follows that $\mathfrak{F}_\varsigma$ is an Ω-admissible S^1-equivariant compact field, so the Leray–Schauder S^1-equivariant degree $S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega)$ is well-defined.

THEOREM 4.4. *Given the system* (71), *assume conditions* (72) *and* (92) *are satisfied*. *Suppose*, *further*, $(\alpha_o, 0)$ *is an isolated center for* (71) (*cf*. *Definition* 4.2). *Take* $\mathfrak{F}$ *defined by* (86) *and construct* Ω *according to* (93). *Let* $\varsigma : \overline{\Omega} \to \mathbb{R}$ *be an auxiliary function* (*see Definition* 4.3) *and let* $\mathfrak{F}_\varsigma$ *be defined by* (94). *Assume*, *finally*, $S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega) \neq 0$.

Then, there exists a branch of nontrivial solutions bifurcating from the point $(\lambda_o, 0) \in \Omega$. *More precisely, the closure of the set composed of all non-trivial solutions* $(\lambda, u) \in \Omega$ *to* (87), *i.e.*

$$\overline{\{(\lambda, u) \in \Omega\colon\ u - \mathcal{F}(\lambda, u) = 0,\ u \neq 0\}},$$

contains a compact connected subset C *such that*

$$(\lambda_o, 0) \in C \quad \textit{and} \quad C \cap \partial_r \neq \emptyset.$$

The main ingredients of the proof of Theorem 4.4 are: invariant Urysohn function techniques, Zorn lemma and the following statement, which is an immediate consequence of Theorem 3 from [97], p. 170.

LEMMA 4.5. *Let* X *be a metric space,* $A, B \subset X$ *two disjoint closed sets in* X, *and* K *a compact set in* X *such that* $K \cap A \neq \emptyset \neq K \cap B$. *If the set* K *does not contain a connected component* K_o *such that* $K_o \cap A \neq \emptyset \neq K_o \cap B$, *then there exist two open disjoint sets* V_1 *and* V_2 *such that* $A \cup B \cup K \subset V_1 \cup V_2$, $A \subset V_1$ *and* $B \subset V_2$.

4.4. *Deformation of the map* $\mathfrak{F}_\varsigma$*: Reduction to a product map*

Theorem 4.4 reduces the study of the Hopf bifurcation for (71) to computing S^1-Deg$(\mathfrak{F}_\varsigma, \Omega)$. In this subsection, we deform $\mathfrak{F}_\varsigma$ to a field with “more computable” S^1-degree.

Complementing function and linearization Replace the set Ω (resp. the auxiliary function ς) with the set Ω_1 (resp. the so-called *complementing function* $\tilde{\varsigma}: \overline{\Omega}_1 \to \mathbb{R}$) defined as

$$\Omega_1 := \left\{(\lambda, u) \in \mathbb{C} \oplus W\colon\ \frac{\varepsilon}{4} < |\lambda - \lambda_o| < \varepsilon,\ \|u\| < r\right\}. \tag{95}$$

Respectively,

$$\tilde{\varsigma} := |\lambda - \lambda_o|\big(\|u\| - r\big) + \|u\| + \varepsilon\frac{r}{2}. \tag{96}$$

For each $(\lambda, u) \in \Omega_1$, put

$$\eta(\lambda) := \lambda_o + \frac{\varepsilon(\lambda - \lambda_o)}{2|\lambda - \lambda_o|} \tag{97}$$

and define the map $\tilde{\mathfrak{F}}: \overline{\Omega}_1 \to \mathbb{R} \oplus W$ by

$$\tilde{\mathfrak{F}}(\lambda, u) := \big(\tilde{\varsigma}(\lambda, u), a\big(\eta(\lambda)\big)u\big), \tag{98}$$

where $a(\cdot)$ is given by (88). Obviously, the map (98) is Ω_1-admissible S^1-equivariant and (by excision and homotopy property of twisted degree)

$$S^1\text{-Deg}(\tilde{\mathfrak{F}}, \Omega_1) = S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega). \tag{99}$$

Finite-dimensional normalization Obviously, one may consider $a(\eta(\cdot))$ from (98) as a continuous map

$$\left\{\lambda \in \mathbb{C}\colon\ \frac{\varepsilon}{4} < |\lambda - \lambda_o| < \varepsilon\right\} \xrightarrow{a(\eta(\cdot))} GL_c^{S^1}(W), \tag{100}$$

where $GL_c^{S^1}(W)$ stands for the group of S^1-equivariant linear completely continuous fields in W. Put

$$\Sigma := \left\{\lambda \in \mathbb{C}\colon\ |\lambda - \lambda_o| = \frac{\varepsilon}{2}\right\}. \tag{101}$$

By (97), the map (100) is completely determined by its restriction to $\Sigma \simeq S^1$. Denote by $\bar{a}(\cdot)$ the restriction of (100) to (101). Also, given $m \in \mathbb{N}$, define $W^m := V \oplus \bigoplus_{l=1}^m W_l \subset W$. Using the standard compactness argument, one can easily show

PROPOSITION 4.6. *There exists $m \in \mathbb{N}$ such that*:

(i) $W = W^m \oplus (W^m)^\perp$;

(ii) *for any $\lambda \in \Sigma$, the map $\tilde{a}(\lambda)$, defined by $\tilde{a}(\lambda) := \bar{a}(\lambda)|_{W^m} + \mathrm{Id}\,|_{(W^m)^\perp}$, belongs to $GL_c^{S^1}(W)$*;

(iii) *the map $\mathfrak{F}_2 : \overline{\Omega}_1 \to \mathbb{R} \oplus W$, defined by $\mathfrak{F}_2(\lambda, u) := (\tilde{\varsigma}(\lambda, u), \tilde{a}(\eta(\lambda))u)$, is the compact equivariant Ω_1-admissible field* (*cf.* (95));

(iv) S^1-Deg$(\tilde{\mathfrak{F}}, \Omega_1) = S^1$-Deg$(\mathfrak{F}_2, \Omega_1)$ (*cf.* (99)).

Isotypical deformations and finite-dimensional reduction Consider the map $\tilde{a}$ defined in Proposition 4.6(ii) more intently. Given $\lambda \in \Sigma$, put

$$\tilde{a}_0(\lambda) := \tilde{a}(\lambda)|_V; \qquad \tilde{a}_l(\lambda) := \tilde{a}(\lambda)|_{W_l}, \quad l > 0.$$

It follows immediately from (88)–(90), (92), that the map $\tilde{a}_0 : \Sigma \to GL(V) = GL(N, \mathbb{R})$ is homotopic to a constant map

$$a_0(\lambda_o) := -\frac{1}{\beta_o} D_x f(\alpha_o, 0), \tag{102}$$

where $\lambda_o = \alpha_o + \mathrm{i}\beta_o$.

Further, $\tilde{a}_l(\lambda) : W_l \to W_l$ is S^1-equivariant, therefore, $\tilde{a}_l(\lambda) \in GL^{S^1}(W_l) \simeq GL(N, \mathbb{C})$ (cf. Proposition 2.9). Thus, the map $\tilde{a}_l : \Sigma \to GL(N, \mathbb{C})$ determines an element $\mu \in$

$\pi_1(GL(N,\mathbb{C}))$. By Proposition 3.35(ii), μ contains a representative $b_l : \Sigma \to GL(N,\mathbb{C})$ of the form

$$b_l(\lambda) = \begin{bmatrix} (\lambda-\lambda_o)^{\mathfrak{k}_l} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} : \mathbb{C}^N \to \mathbb{C}^N, \tag{103}$$

where $\mathfrak{k}_l$ is an integer.

Using Proposition 4.6(i), (ii), (102) and (103), define the map $\hat{a} : \Sigma \to GL_c^{S^1}(W)$ by

$$\hat{a}(\lambda) := a_0(\lambda_o) + \bigoplus_{l=1}^{m} b_l(\lambda) + \mathrm{Id}\,|_{(W^m)^\perp}, \tag{104}$$

and the compact S^1-equivariant field $\hat{\mathfrak{F}} : \overline{\Omega}_1 \to W$ by $\hat{\mathfrak{F}}(\lambda, u) := (\tilde{\varsigma}(\lambda, u), \hat{a}(\eta(\lambda))u)$. Put $\Omega_o := \Omega_1 \cap (\mathbb{C} \oplus V \oplus \bigoplus_{l=1}^m W_l)$ and consider the map $\mathfrak{F}_o := \hat{\mathfrak{F}}|_{\overline{\Omega_o}}$. By construction of the Leray–Schauder equivariant degree and the suspension property,

$$S^1\text{-Deg}(\mathfrak{F}_2, \Omega_1) = S^1\text{-Deg}(\mathfrak{F}_o, \Omega_o). \tag{105}$$

Product formula for S^1-Deg$(\mathfrak{F}_\varsigma, \Omega)$ Take $\Omega_* := B \times \Omega_{o*}$, where $\Omega_{o*} := \Omega_o \cap (\mathbb{C} \oplus \bigoplus_{l=1}^m W_l)$ and B is the unit ball in V, and define the map $\mathfrak{F}_{o*} : \overline{\Omega_{o*}} \to \mathbb{R} \oplus \bigoplus_{l=1}^m W_l$ by

$$\mathfrak{F}_{o*}(\lambda, v) := \left(\tilde{\varsigma}(\lambda, v), \bigoplus_{l=1}^{m} b_l\big(\eta(\lambda)\big)v_l\right)$$

(here v_l is an orthogonal projection of $v \in \bigoplus_{l=1}^m W_l$ to W_l (cf. (103))). Next, take the product map

$$\mathfrak{F}_* := a_0(\lambda_o) \times \mathfrak{F}_{o*} : \overline{B \times \Omega_{o*}} \to V \oplus \left(\mathbb{R} \oplus \bigoplus_{l=1}^{m} W_l\right)$$

(see (102)). Using Theorem 3.48 and Remark 3.49, one obtains

$$S^1\text{-Deg}(\mathfrak{F}_o, \Omega_o) = S^1\text{-Deg}(\mathfrak{F}_*, \Omega_*) = \deg\big(a_0(\lambda_o), B\big) \cdot S^1\text{-Deg}(\mathfrak{F}_{o*}, \Omega_{o*}),$$

where "deg" stands for the usual Brouwer degree. Observe, $\deg(a_0(\lambda_o), B) = \operatorname{sign}\det a_0(\lambda_o)$. Therefore, by applying Proposition 3.34, Lemma 3.37 and Theorem 3.39, we obtain

PROPOSITION 4.7. *Under the assumptions and notations of Theorem* 4.4, *one has*:

$$S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega) = \operatorname{sign}\det a_0(\lambda_o) \cdot \sum_{l=1}^{m} \mathfrak{k}_l(\mathbb{Z}_l), \tag{106}$$

where a_0 *is given by* (90) *and the numbers* $\mathfrak{k}_l$ *are defined by*

$$\mathfrak{k}_l = \deg\bigl(\det_{\mathbb{C}}(a_l), \widetilde{D}\bigr) \tag{107}$$

for a_l *given by* (91) *and*

$$\widetilde{D} := \bigl\{\lambda \in \mathbb{C}\colon |\lambda - \lambda_o| < \varepsilon\bigr\}. \tag{108}$$

4.5. *Crossing numbers*

Under the assumption that one is able to effectively compute the numbers $\mathfrak{k}_l$ in formula (107), Proposition 4.7 together with Theorem 4.4 provide an effective way to study the Hopf bifurcation for the system (71). Fortunately, this task can be accomplished by using the information related to the S^1-equivariant spectral decomposition of $D_x f(\alpha, 0)$.

It is convenient to replace the set $\widetilde{D}$ in Proposition 4.7 with a set D defined by

$$D := \bigl\{\lambda = \alpha + \mathrm{i}\beta \in \mathbb{C}\colon \max\bigl\{|\alpha - \alpha_o|, |\beta - \beta_o|\bigr\} < \varepsilon\bigr\} \tag{109}$$

with, possibly, smaller ε. By excision and homotopy properties of Brouwer degree,

$$\mathfrak{k}_l = \deg\bigl(\det_{\mathbb{C}}(a_l), D\bigr).$$

Clearly, if the number $\mathrm{i}l\beta_o$ is not a root of the characteristic equation, then $\mathfrak{k}_l = 0$. Suppose, therefore, that $\mathrm{i}l\beta_o$ is a root of the characteristic equation (75) for $\alpha = \alpha_o$. Since $(\alpha_o, 0)$ is an isolated center, we can assume, without loss of generality, that the number ε in (109) is so small that for every α satisfying $0 < |\alpha - \alpha_o| < \varepsilon$, there is no purely imaginary root $\mathrm{i}\beta'$ of (75) with $|\beta' - l\beta_o| \leqslant \varepsilon$. Also, there exists $\delta > 0$ such that there are no characteristic roots $\tau + \mathrm{i}\beta$ of Eq. (75) for $|\alpha - \alpha_o| \leqslant \varepsilon$ belonging to the boundary of the set

$$S := \bigl\{\tau + \mathrm{i}\beta\colon 0 < \tau < \delta, |\beta - l\beta_o| < \varepsilon\bigr\}, \tag{110}$$

except for the root $\mathrm{i}l\beta$ corresponding to $\alpha = \alpha_o$ (see Fig. 3).

Using a simple Brouwer degree argument, one can show that

$$\mathfrak{k}_l = \deg(\det_{\mathbb{C}} \Delta_{\alpha_-}, S) - \deg(\det_{\mathbb{C}} \Delta_{\alpha_+}, S) \quad (\alpha_\pm := \alpha_o \pm \varepsilon). \tag{111}$$

DEFINITION 4.8. Define the following integers

$$\mathfrak{t}_l^{\pm}(\alpha_o, \beta_o) := \deg(\det_{\mathbb{C}} \Delta_{\alpha_\pm}, S),$$

and

$$\mathfrak{t}_l(\alpha_o, \beta_o) := \mathfrak{t}_l^{-}(\alpha_o, \beta_o) - \mathfrak{t}_l^{+}(\alpha_o, \beta_o). \tag{112}$$

The integer $\mathfrak{t}_l := \mathfrak{t}_l(\alpha_o, \beta_o)$ is called the *l-th isotypical crossing number* at (α_o, β_o).

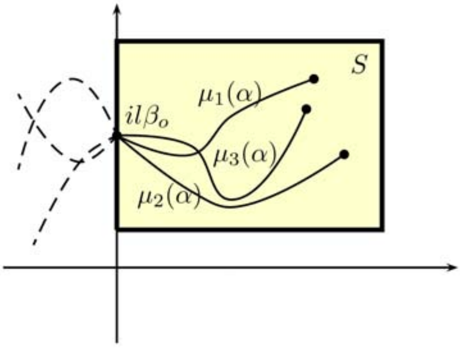

Fig. 3. Characteristic values $\mu_k(\alpha)$ near $\mathrm{i}l\beta_o$.

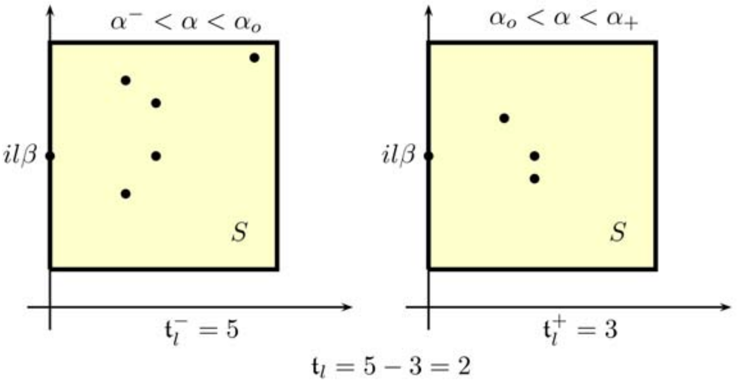

Fig. 4. The l-th isotypical crossing number.

The crossing number $\mathfrak{t}_l = \mathfrak{t}_l^- - \mathfrak{t}_l^+$ has a very simple interpretation. In the case $\mathrm{i}l\beta_o$ is an eigenvalue of the matrix $D_x f(\alpha_o, 0)$, the number $\mathfrak{t}_l^-$ counts the number of all the eigenvalues (with multiplicity) in the set S before α crosses the value α_o, and the number $\mathfrak{t}_l^+$ counts the eigenvalues in S after α crosses α_o. The difference, which is exactly the number $\mathfrak{t}_l$, represents the net number of the eigenvalues which 'escaped' (if $\mathfrak{t}_l$ is positive) or 'entered' (if $\mathfrak{t}_l$ is negative) the set S when α was crossing α_o. This situation is illustrated on Fig. 4.

4.6. *Conclusions*

Return to the system (71). Take an isolated center $(\alpha_o, 0)$ corresponding to the purely imaginary characteristic root $\mathrm{i}\beta_o$, $\beta_0 > 0$. Then, Theorem 4.4 and Proposition 4.7 show that the equivariant degree S^1-Deg$(\mathfrak{F}_\varsigma, \Omega)$ is a *local Hopf bifurcation invariant* describing a possible occurrence of the bifurcation. Namely, the non-zero components of the equivariant

degree provide the information about the existence of branches of non-constant periodic solutions to (71) bifurcating from the point $(\alpha_o, 0)$ (with the limit frequency β_o).

Furthermore, formulae (110)–(112), in compliance with Theorem 4.4 and Proposition 4.7, allow us to formulate an effective sufficient condition for the occurrence of the Hopf bifurcation in terms of the right-hand side of the system (71) only. Namely,

THEOREM 4.9. *Let* $V := \mathbb{R}^N$ *and let* $f : \mathbb{R} \oplus V \to V$ *and* $\alpha_o \in \mathbb{R}$ *satisfy the following conditions*:

(i) f *is of class* C^1;
(ii) $f(\alpha, 0) = 0$ *for all* $\alpha \in \mathbb{R}$;
(iii) $\det D_x f(\alpha_o, 0) \neq 0$, *where* $D_x f$ *denotes the derivative of* f *with respect to* $x \in V$;
(iv) $(\alpha_o, 0)$ *is an isolated center for the system* (71) *with* $\mathrm{i}\beta_o$ $(\beta_o > 0)$ *being the corresponding characteristic root* (*cf. Definition* 4.2 *and formulae* (74), (75));
(v) $\mathfrak{t}_1(\alpha_o, \beta_o) = \mathfrak{t}_1^-(\alpha_o, \beta_o) - \mathfrak{t}_1^+(\alpha_o, \beta_o) \neq 0$ (*see formulae* (110)–(112)).

Then, *there exists a sequence* (*in fact*, *it is a connected branch*) *of non-constant* $\frac{2\pi}{\beta_n}$-*periodic*[1] *solutions* $x_{\alpha_n}(t)$ *of the system* (71) (*with* $\alpha = \alpha_n$) *such that*

$$\begin{cases} \alpha_n \to \alpha_o, \\ x_{\alpha_n}(t) \to 0, \end{cases} \quad \text{as } n \to \infty.$$

PROOF. By conditions (i), (ii) and (iv), one can associate to (α_0, β_o) the degree $S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega)$ (cf. formulae (93) and (94)). By Proposition 4.7 and formulae (110)–(112),

$$S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega) = \operatorname{sign}\det a_0 \sum_{l=1}^{m} \mathfrak{k}_l(\mathbb{Z}_l) = \operatorname{sign}\det a_0 \sum_{l=1}^{m} \mathfrak{t}_l(\alpha_o, \beta_o)(\mathbb{Z}_l)$$

for some $m \geqslant 1$. By condition (v), $\mathfrak{t}_1(\alpha_o, \beta_o) \neq 0$, and the conclusion follows from Theorem 4.4 (condition (iii) excludes possible steady-state bifurcation). □

4.7. *Bibliographical remarks*

The standard approach to attack the Hopf bifurcation problem occurring in ODEs is based on the use of the Central Manifold theorem (allowing a reduction to the two-dimensional case) and further studying the properties of the Poincaré map associated with the induced system (an excellent exposition of this concept is presented in [113], see also [8,68,75]). Other techniques for studying Hopf bifurcation problems, which are based on Lyapunov–Schmidt reduction, normal forms and applications of the singularity theory can be found in [33,34,64,111]. The idea of a crossing number $\mathfrak{t}$ can be traced back to [2], where the framed bordism method was used to show that, if $\mathfrak{t}$ is odd, then a Hopf bifurcation occurs. The general result stating that, if $\mathfrak{t} \neq 0$ then a Hopf bifurcation takes place, was proved with the use of the Fuller index in [35]. One should point out that the framed bordism invariants

[1] We do not assume here that $\frac{2\pi}{\beta_n}$ is a minimal period.

and Fuller index can be expressed in terms of the equivariant degree (cf. [81,19]). The S^1-equivariant degree approach to the above problem described in this section was discussed in [61,93,19] (see also [46,49,81,95]). On the other hand, for Hamiltonian systems (resp. systems with first integral), important results were obtained by N.E. Dancer in [39] (resp. by E.N. Dancer and J.F. Toland in [40,42,41]).

5. Hopf bifurcation problem for ODEs with symmetries

5.1. *Symmetric Hopf bifurcation and local bifurcation invariant*

Statement of the problem In this section, we extend the setting and methods developed in Section 4 to discuss the Hopf bifurcation phenomenon for the system (71) admitting a certain compact Lie group of symmetries. More precisely, suppose $V := \mathbb{R}^N$ is an orthogonal representation of a compact Lie group Γ and assume that

(A0) $f : \mathbb{R} \oplus V \to V$ is Γ-equivariant.

Also, assume that f and $\alpha_o \in \mathbb{R}$ satisfy the hypotheses (i)–(iv) from Theorem 4.9.

We are interested in describing possible symmetries of non-constant periodic solutions to the system (71) bifurcating from the origin.

$\Gamma \times S^1$-Equivariant reformulation of the problem Following the same scheme as in the non-symmetric case (see Subsection 4.2), we start with the normalization of the period (see (76) and (77)) and setting in functional spaces (see (84)). Observe that the space $W = H^1(S^1; V)$ is a natural Hilbert isometric $\Gamma \times S^1$-representation with the action given by $((\gamma, \zeta)u)(t) = \gamma u(t + \tau)$, $t \in \mathbb{R}$, $\gamma \in \Gamma$, $\zeta = e^{i\tau} \in S^1$, $u \in W$. In the same way, the space $\mathcal{C} = C(S^1; V)$ is a Banach isometric G-representation. Further, consider $\mathbb{R}^2_+$ (identified with the subspace of $\mathbb{C}$) as a trivial G-space. Then, all the operators L, j, N_f, and K (cf. (81)–(83) and (85)) turn out to be G-equivariant. Consequently, the system (71) translates as the G-equivariant fixed-point problem (87).

G-isotypical decomposition of W The S^1-isotypical decomposition (80) of W is clearly G-invariant, therefore we can use it to construct the G-isotypical decomposition of W. Consider the real (resp. complex) Γ-isotypical decomposition of V (resp. V^c):

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_r, \qquad V^c = U_0 \oplus U_1 \oplus \cdots \oplus U_s, \tag{113}$$

where the isotypical components V_i (resp. U_j) are modeled on the real (resp. complex) Γ-irreducible representations $\mathcal{V}_i$ (resp. $\mathcal{U}_j$). Notice that it is possible that there is a different number of isotypical components in the isotypical decompositions of V and V^c. The isotypical components of W_l, $l \geqslant 1$, are easily obtained (see Remark and Notation 2.6):

$$W_l = V_{0,l} \oplus V_{1,l} \oplus \cdots \oplus V_{s,l}, \tag{114}$$

where the components $V_{j,l} := {}^{l}U_j$, $j = 0, 1, \dots, s$, are modeled on the irreducible G-representation $\mathcal{V}_{j,l}$. In this way, we obtain the following G-isotypical decomposition of W:

$$W = [V_0 \oplus V_1 \oplus \cdots \oplus V_r] \oplus \overline{\bigoplus_{l=1}^{\infty} (V_{0,l} \oplus V_{1,l} \oplus \cdots \oplus V_{s,l})}. \tag{115}$$

Dominating orbit types The following notion will be used to estimate the *minimal number* of different periodic solutions (as well as their symmetries) to the system (71) (see Theorem 5.4).

DEFINITION 5.1. An orbit type (H) in W is called *dominating*, if (H) is maximal (with respect to the usual order relation (10)) in the class of all φ-twisted *one-folded* orbit types in W (i.e. $H = K^{\varphi}$, where $\varphi : K \to S^1$ is a homomorphism, and $\dim W(K) = 0$ (see Section 3.3 for more details)).

REMARK 5.2. Let (H) be a dominating orbit type in W. Using the maximality property of (H), it is easy to see that there exists an irreducible subrepresentation $\mathcal{V} \subset W$ and a non-zero vector $u \in \mathcal{V}$ such that $G_u = H$. Consequently, the dominating orbit types in W can be easily recognized from the isotypical decomposition of W and lattices of orbit types of the corresponding to this decomposition irreducible G-representations.

REMARK 5.3. Assume there is a solution $u_o \in W$ to (77) (for $\alpha = \alpha_o$ and some $\beta > 0$), for which one has $G_{u_o} \supset H_o$. If (H_o) is a dominating orbit type in W with $H_o = K^{\varphi}$ for some $K \subset \Gamma$ and $\varphi : K \to S^1$, then, by maximality condition, $(G_{u_o}) = (K^{\varphi,l})$ with $l \geqslant 1$, and the corresponding orbit $G(u_o)$ is composed of exactly $|G/G_{u_o}|_{S^1}$ different periodic functions (where $|Y|_{S^1}$ denotes the number of S^1-orbits in Y). It is easy to check that the number of S^1-orbits in G/G_{u_o} is $|\Gamma/K|$ (where $|X|$ stands for the number of elements in X).

On the other hand, suppose that x_o is, say, a p-periodic solution to (71) canonically corresponding to the above u_o. It follows from the definition of l-folding and $\Gamma \times S^1$-action on W that x_o is also a $\frac{p}{l}$-periodic solution to (71). The pair $(x_o, \frac{p}{l})$ canonically determines an element $u_o' \in W$ being a solution to (77) (for $\alpha = \alpha_o$ and some β') satisfying the condition $G_{u_o'} = H_o$. In this way, we obtain that (71) has at least $|\Gamma/K|$ different periodic solutions with the orbit type exactly (H_o).

Sufficient conditions for symmetric Hopf bifurcation Similarly to the non-equivariant case, define the set Ω by (93). Since W is an isometric G-representation, Ω is clearly G-invariant. Combining the standard Urysohn function techniques (cf. Proposition 3.9) with Definition 4.3, construct a G-invariant auxiliary function $\varsigma : \overline{\Omega} \to \mathbb{R}$. Then, define the map $\mathfrak{F}_{\varsigma} : \overline{\Omega} \to \mathbb{R} \oplus W$ by (94), which is a G-equivariant Ω-admissible compact field. Therefore, one can assign to $\mathfrak{F}_{\varsigma}$ the twisted degree $G\text{-}\mathrm{Deg}^t(\mathfrak{F}_{\varsigma}, \Omega)$.

Combining the properties of the twisted degree with Remark 5.3 and Lemma 4.5 (cf. Theorem 4.4), one can easily establish

THEOREM 5.4. *Given the system* (71), *assume conditions* (i)–(iv) *from Theorem* 4.9 *and condition* (A0) *to be satisfied. Take* $\mathfrak{F}$ *defined by* (86) *and construct* Ω *according to* (93). *Let* $\varsigma : \overline{\Omega} \to \mathbb{R}$ *be a* G*-invariant auxiliary function* (*see Definition* 4.3) *and let* $\mathfrak{F}_\varsigma$ *be defined by* (94).

(i) *Assume* $G\text{-Deg}^t(\mathfrak{F}_\varsigma, \Omega) \neq 0$, *i.e.*

$$G\text{-Deg}^t(\mathfrak{F}_\varsigma, \Omega) = \sum_{(H)} n_H(H) \quad \textit{and} \quad n_{H_o} \neq 0 \tag{116}$$

for some $(H_o) \in \Phi_1^t(G)$. *Then, there exists a branch of non-trivial solutions to* (71) *bifurcating from the point* $(\alpha_o, 0)$ (*with the limit frequency* $l\beta_o$ *for some* $l \in \mathbb{N}$). *More precisely, the closure of the set composed of all non-trivial solutions* $(\lambda, u) \in \Omega$ *to* (87), *i.e.*

$$\overline{\{(\lambda, u) \in \Omega\colon\ \mathfrak{F}(\lambda, u) = 0,\ u \neq 0\}}$$

contains a compact connected subset C *such that*

$$(\lambda_o, 0) \in C \quad \textit{and} \quad C \cap \partial_r \neq \emptyset, \quad C \subset \mathbb{R}^2_+ \times W^{H_o},$$

($\lambda_o = \alpha_o + \mathrm{i}\beta_o$) *which, in particular, implies that for every* $(\alpha, \beta, u) \in C$, *we have* $G_u \supset H_o$.

(ii) *If, in addition,* (H_o) *is a dominating orbit type in* W, *then there exist at least* $|G/H_o|_{S^1}$ *different branches of periodic solutions to Eq.* (71) *bifurcating from* $(\alpha_o, 0)$ (*with the limit frequency* $l\beta_o$ *for some* $l \in \mathbb{N}$). *Moreover, for each* (α, β, u) *belonging to these branches of* (*nontrivial*) *solutions one has* $(G_u) = (H_o)$ (*considered in the space* W).

REMARK 5.5.

(i) Since the order (10) on the set $\Phi_1^t(G)$ is partial, it is usually the case that there is more than one dominating orbit type in W contributing to the lower estimate of all bifurcating branches of solutions.

(ii) In addition, if there is also a coefficient $n_K \neq 0$ such that (K) is not a dominating orbit type, but $n_H = 0$ for all dominating orbit types (H) such that $(K) < (H)$, then we can also predict the existence of multiple branches by analyzing all the dominating orbit types (H) larger than (K). However, the exact orbit type of these branches (as well as the corresponding estimate) cannot be determined precisely.

Local bifurcation invariant Similarly to the non-equivariant case, the non-zero coefficient n_{H_o} in (116) indicates that in the subspace $\mathbb{R}^2_+ \oplus W^{H_o}$, there exists a branch of solutions to (87), which is "invariant" (i.e. does not disappear) under small *equivariant* perturbations of the map $\mathfrak{F}_\varsigma$. This observation justifies the following

DEFINITION 5.6. Put

$$\omega(\lambda_o) = \omega(\alpha_o, \beta_o) := G\text{-Deg}^t(\mathfrak{F}_\varsigma, \Omega) \tag{117}$$

and call it the *local equivariant topological bifurcation invariant* (or simply *local bifurcation invariant*) of the (bifurcation) point (α_o, β_o).

5.2. *Computation of local bifurcation invariant: Reduction to product formula*

Although the computations of the local bifurcation invariant (117) are, in general, parallel to the non-equivariant case (see Subsections 4.4 and 4.5), several (specifically equivariant) steps have substantial differences, which should be carefully explained.

Linearization and finite-dimensional reduction As in the non-equivariant case, we start with the linearization of the map $\mathfrak{F}_\varsigma$. Namely, using (95)–(97) and the linear map $a(\alpha, \beta)$ defined by (88), construct the map $\tilde{\mathfrak{F}}: \overline{\Omega_1} \to \mathbb{R} \oplus W$ by formula (98). Then,

$$G\text{-Deg}^t(\mathfrak{F}_\varsigma, \Omega) = G\text{-Deg}^t(\tilde{\mathfrak{F}}, \Omega_1).$$

Take Σ defined by (101) and construct W^m and $\mathfrak{F}_2: \overline{\Omega_1} \to \mathbb{R} \oplus W$ as in Proposition 4.6. Obviously, W^m is G-invariant, so that $\mathfrak{F}_2$ is G-equivariant. Put $\Omega_o := \Omega_1 \cap (\mathbb{C} \oplus W^m)$ and $\mathfrak{F}_o := \mathfrak{F}_2|_{\overline{\Omega_o}}$. Then,

$$G\text{-Deg}^t(\mathfrak{F}_\varsigma, \Omega) = G\text{-Deg}^t(\mathfrak{F}_2, \Omega_1) = G\text{-Deg}^t(\mathfrak{F}_o, \Omega_o).$$

Product formula Put $W_{o*} := \mathbb{C} \oplus \bigoplus_{l=1}^m W_l$ and $\Omega_{o*} := \Omega_o \cap W_{o*}$. Then, without loss of generality, one can assume that $\Omega_o = B \times \Omega_{o*}$, where B is the unit ball in V, and, in addition, $\mathfrak{F}_o = a_0(\lambda_o) \times \mathfrak{F}_{o*}$, where $a_0(\lambda_o): \overline{B} \to V$ is given by (90), $\lambda_o = \alpha_o + \mathrm{i}\beta_o$, and $\mathfrak{F}_{o*}: \overline{\Omega_{o*}} \to \mathbb{R} \oplus \bigoplus_{l=1}^m W_l$, is given by

$$\mathfrak{F}_{o*}(\lambda, u) = \left(\tilde{\varsigma}(\lambda, u), \bigoplus_{l=1}^m a_l\big(\eta(\lambda)\big)u\right), \tag{118}$$

where a_l is given by (91) (see also (88) and (89)).

Then, apply the multiplicativity property of the twisted G-equivariant degree (see Theorem 3.48) to obtain

$$G\text{-Deg}^t(\mathfrak{F}_\varsigma, \Omega) = \Gamma\text{-Deg}\big(a_0(\lambda_o), B\big) \cdot G\text{-Deg}^t(\mathfrak{F}_{o*}, \Omega_{o*}). \tag{119}$$

Computation of $\Gamma\text{-Deg}(a_0(\lambda_o), B)$ Consider the set $\sigma_- := \{\mu \in \sigma(a_0(\lambda_o)) : \mu < 0\}$, where $\sigma(a_0(\lambda_o))$ stands for the (real) spectrum of $a_0(\lambda_o)$ (observe, by the way, that there is a direct connection between σ_- and the set of positive eigenvalues of $D_x f(\alpha_o, 0)$). For every $\mu \in \sigma_-$, denote by $E(\mu) \subset V$ the eigenspace corresponding to μ. Since $E(\mu)$ is Γ-invariant, one can consider its Γ-isotypical decomposition

$$E(\mu) = E_0(\mu) \oplus E_1(\mu) \oplus \cdots \oplus E_r(\mu),$$

where the component $E_i(\mu)$ is modeled on the irreducible Γ-representation $\mathcal{V}_i$, $i = 0, 1, 2, \dots, r$. Put

$$m_i(\mu) := \dim E_i(\mu)/\dim \mathcal{V}_i, \quad i = 0, 1, 2, \dots, r. \tag{120}$$

The number $m_i(\mu)$ is called $\mathcal{V}_i$*-multiplicity* of the eigenvalue μ.

By applying the homotopy and multiplicativity properties (cf. Theorem 3.46),

$$\Gamma\text{-Deg}\big(a_0(\lambda_o), B\big) = \prod_{\mu\in\sigma_-}\prod_{i=0}^{r}(\deg_{\mathcal{V}_i})^{m_i(\mu)}. \tag{121}$$

5.3. *Computation of local bifurcation invariant: Reduction to crossing numbers and basic degrees*

In view of formulae (119) and (121), it remains to compute $G\text{-Deg}^t(\mathfrak{F}_{o*}, \Omega_{o*})$.

Isotypical representation of a_l Take the isotypical decomposition (114) and put $a_{j,l}(\lambda) := a_l(\eta(\lambda))|_{V_{j,l}} : V_{j,l} \to V_{j,l}$ $(j = 0, 1, \dots, s)$. Observe that for a fixed λ, the operator $a_{j,l}(\lambda)$ is a Γ-equivariant complex isomorphism, which can be represented (cf. Proposition 2.9) by a complex $n_j \times n_j$-matrix, where $n_j := \dim V_{j,l}/\dim \mathcal{V}_{j,l}$ (this number is the same for all $l = 1, 2, \dots$). Put (cf. (113))

$$\Delta_{\alpha,j}(\lambda) := \Delta_\alpha(\lambda)|_{U_j} : U_j \to U_j. \tag{122}$$

Keeping in mind (91), denote by

$$\Delta^{\mathbb{C}}_{\alpha,j}(\lambda) : \mathbb{C}^{n_j} \to \mathbb{C}^{n_j}, \tag{123}$$

the $n_j \times n_j$-matrix representation of the operator $\Delta_{\alpha,j}(\lambda) : U_j \to U_j$ (cf. Proposition 2.9). Then,

$$a_{j,l}(\lambda) = \frac{1}{\mathrm{i}l\beta}\Delta^{\mathbb{C}}_{\alpha,j}(\mathrm{i}l\beta) \quad \text{with } \lambda = \alpha + \mathrm{i}\beta. \tag{124}$$

Isotypical deformations The obtained identification (124), (123) of $a_{j,l}(\cdot)$ and the fact that the linear group $GL^G(V_{j,l})$ is isomorphic to the complex linear group $GL(n_j, \mathbb{C})$, can be used to deform $a_{j,l}(\cdot)$ in a way similar to the non-equivariant case. Namely, put $\tilde{a}_{j,l}(\cdot) := a_{j,l}|_\Sigma$ (cf. (101)). Then, by Proposition 3.35, the homotopy class of the map $\tilde{a}_{j,l} : \Sigma \to GL(n_j, \mathbb{C})$ contains a representative $b_{j,l} : \Sigma \to GL(n_j, \mathbb{C})$ of the following form:

$$b_{j,l}(\lambda) = \begin{bmatrix} (\lambda-\lambda_o)^{\mathfrak{k}_{j,l}} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} : \mathbb{C}^{n_j} \to \mathbb{C}^{n_j}, \tag{125}$$

where $\mathfrak{k}_{j,l}$ is an integer defined by

$$\mathfrak{k}_{j,l} = \deg(\det_{\mathbb{C}} \Delta^{\mathbb{C}}_{\alpha,j}, \widetilde{D}) \tag{126}$$

(cf. (103), (107) and (108)).

Using the identification (124), (123) and formula (125), define the map $\tilde{\mathfrak{F}}_{o*} : \overline{\Omega_{o*}} \to \mathbb{R} \oplus \bigoplus_{l=1}^{m} W_l$ by

$$\tilde{\mathfrak{F}}_{o*}(\lambda, v) := \left(\tilde{\varsigma}(\lambda, v), \bigoplus_{l=1}^{m} b_{j,l}\big(\eta(\lambda)\big) v_l \right) \tag{127}$$

(here v_l is an orthogonal projection of $v \in \bigoplus_{l=1}^{m} W_l$ to W_l). Combining formula (127) with Proposition 3.45 yields:

$$G\text{-Deg}^t(\mathfrak{F}_{o*}, \Omega_{o*}) = G\text{-Deg}^t(\tilde{\mathfrak{F}}_{o*}, \Omega_{o*}) = \sum_{j,l} \mathfrak{k}_{j,l} \deg_{\mathcal{V}_{j,l}} . \tag{128}$$

In this way (cf. (119), (121) and (128)), we have proved the following

PROPOSITION 5.7. *Let* $\mathfrak{F}_{\varsigma}$ *and* Ω *be as in Theorem* 5.4. *Take the isotypical decompositions* (113) *and* (114). *Then,*

$$G\text{-Deg}^t(\mathfrak{F}_{\varsigma}, \Omega) = \prod_{\mu \in \sigma_-} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)} \cdot \sum_{j,l} \mathfrak{k}_{j,l} \deg_{\mathcal{V}_{j,l}}, \tag{129}$$

where σ_- *denotes the* (*real*) *negative spectrum of the operator* $a_0(\lambda_o)$ (*see* (90)), $\deg_{\mathcal{V}_i}$ *is defined by* (63), $m_i(\mu)$ *stands for the* $\mathcal{V}_i$*-multiplicity of* μ (*cf.* (120)), $\mathfrak{k}_{j,l}$ *is given by* (126), $\deg_{\mathcal{V}_{j,l}}$ *is defined in* (60) ($j = 0, \ldots, s$ *and* $l = 1, \ldots, m$)*, and the multiplication "$\cdot$" is taken in* $A_1^t(\Gamma \times S^1)$.

Crossing numbers To complete our computations of the local bifurcation invariant, we need to express the integers $\mathfrak{k}_{j,l}$ from (129) via the symmetric spectral properties of the operator $D_x f(\alpha, 0)$. To this end, we adopt the concept of crossing numbers to the equivariant setting.

Following the scheme used in the non-equivariant case, assume that $\mathrm{i}l\beta_o$ is a root of the characteristic equation and choose a small neighborhood S of $\mathrm{i}l\beta_o$ in the right half-plane of $\mathbb{C}$ (see (110)) such that for $|\alpha - \alpha_o| \leqslant \varepsilon$, the characteristic roots of Eq. (75) can only leave S through the 'exit' at the point $\mathrm{i}l\beta_o$ for $\alpha = \alpha_o$.

DEFINITION 5.8. Using (113) and the identification (124), (123), put

$$\mathfrak{t}^{\pm}_{j,l}(\alpha_o, \beta_o) := \deg(\det_{\mathbb{C}} \Delta^{\mathbb{C}}_{\alpha_\pm, j}, S)$$

and

$$\mathfrak{t}_{j,l}(\alpha_o,\beta_o) = \mathfrak{t}^-_{j,l}(\alpha_o,\beta_o) - \mathfrak{t}^+_{j,l}(\alpha_o,\beta_o). \tag{130}$$

The integer (130) is called the $\mathcal{V}_{j,l}$*-isotypical crossing number* at (α_o,β_o) (cf. (112).

REMARK 5.9. The crossing number (130), if it is positive (resp. negative), indicates the algebraic count (with $\mathcal{U}_j$-multiplicities) of roots of the (isotypical) characteristic equation

$$\det\nolimits_{\mathbb{C}} \Delta_{\alpha,j}(\lambda) = 0 \tag{131}$$

(i.e. the characteristic equation restricted to the isotypical component U_j of the space V^c (cf. (113)), which leave (resp. enter) the neighborhood S through the point $\mathrm{i}l\beta_o$ as α crosses α_0). For the system (71), the characteristic roots are the eigenvalues of the (complex) matrix $D_x f(\alpha,0)$. Thus, in order to determine the crossing number $\mathfrak{t}_{j,l}(\alpha_o,\beta_o)$, one needs to verify what happens to the eigenvalues of $D_x f(\alpha,0)$ when α crosses α_o. For example, if there is one eigenvalue, which exits S through $\mathrm{i}l\beta_o$, and its $\mathcal{U}_j$-multiplicity is k, then $\mathfrak{t}_{j,l}(\alpha_o,\beta_o) = k$.

Clearly,

$$\mathfrak{t}_{j,l}(\alpha_o,\beta_o) = \mathfrak{t}_{j,1}(\alpha_o,l\beta_o). \tag{132}$$

As in the non-equivariant case, a simple Brouwer degree argument leads to the relation

$$\mathfrak{k}_{j,l} = \mathfrak{t}_{j,l}(\alpha_o,\beta_o). \tag{133}$$

In this way, by Proposition 5.7, we obtain the following computational formula for the local bifurcation invariant $\omega(\alpha_o,\beta_o)$.

PROPOSITION 5.10. *Under the notations of Proposition* 5.7, *we have*

$$\omega(\alpha_o,\beta_o) = \prod_{\mu\in\sigma_-}\prod_{i=0}^{r}(\deg_{\mathcal{V}_i})^{m_i(\mu)} \cdot \sum_{j,l} \mathfrak{t}_{j,l}(\alpha_o,\beta_o)\deg_{\mathcal{V}_{j,l}}, \tag{134}$$

where $\mathfrak{t}_{j,l}$ *is given by* (130).

5.4. *Summary of the equivariant degree method*

Theorem 5.4 together with Proposition 5.10 give an effective algorithm to classify symmetric branches of non-trivial periodic solutions to the Γ-symmetric system (77) (and, correspondingly, (71)) bifurcating from an isolated center $(\alpha_o,0)$. For the reader convenience, we sketch below the main steps one should follow.

(a) Take an isolated center $(\alpha_o, 0)$ for the system (71) with $\mathrm{i}\beta_o$ ($\beta_o > 0$) being the corresponding characteristic root (cf. Definition 4.2 and formulae (74) and (75)).

(b) Consider the operator $A := -\frac{1}{\beta_o} D_x f(\alpha_o, 0) : V \to V$, which is clearly Γ-equivariant, and put $\sigma_- := \{\mu \in \sigma(A) : \mu < 0\}$, where $\sigma(A)$ stands for the real spectrum of A. Notice that σ_- is composed of $-\frac{1}{\beta_o}\nu$, where ν is a positive eigenvalue of $D_x f(\alpha_o, 0)$.

(c) For every $\mu \in \sigma_-$ (or, equivalently, one can take the corresponding positive eigenvalue of $D_x f(\alpha_o, 0)$), consider the Γ-isotypical decomposition of the eigenspace $E(\mu)$:

$$E(\mu) = E_0(\mu) \oplus E_1(\mu) \oplus \cdots \oplus E_r(\mu),$$

where the component $E_i(\mu)$ is modeled on the irreducible Γ-representation $\mathcal{V}_i$, and evaluate the $\mathcal{V}_i$-multiplicity of μ, i.e. $m_i(\mu) = \dim E_i(\mu) / \dim \mathcal{V}_i$.

(d) Compute the element $\deg_0(\alpha_o, \beta_o) \in A(\Gamma)$ according to the formula:

$$\deg_0(\alpha_o, \beta_o) := \prod_{\mu \in \sigma_-} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)}. \tag{135}$$

(e) Find the (complex) Γ-isotypical decomposition of the complexification V^c, i.e.

$$V^c = U_0 \oplus U_1 \oplus \cdots \oplus U_s,$$

where the Γ-isotypical component U_j is modeled on the irreducible Γ-representation $\mathcal{U}_j$, and put $A_j(\alpha) := D_x f(\alpha, 0)|_{U_j}$ (here $D_x f(\alpha, 0)$ is considered as the corresponding complexification). Consider the set of all the purely imaginary eigenvalues of $A_j(\alpha_o)$ which are integer multiples of $\mathrm{i}\beta_o$:

$$\{\mathrm{i}\beta_o, \mathrm{i}k_1\beta_o, \ldots, \mathrm{i}k_q\beta_o\},$$

where $k_p > 1$ are integers ($p = 1, \ldots, q$).

(f) Compute the element $\deg_*(\alpha_o, \beta_o) \in A_1^t(G)$ defined as the following sum:

$$\deg_*(\alpha_o, \beta_o) = \sum_{j=0}^{s} \sum_{p=1}^{q} \mathfrak{t}_{j,k_p}(\alpha_o, \beta_o) \deg_{\mathcal{V}_{j,k_p}}, \tag{136}$$

where the numbers $\mathfrak{t}_{j,k_p}(\alpha_o, \beta_o) = \mathfrak{t}_{j,1}(\alpha_o, k_p\beta_o)$ are the $\mathcal{V}_{j,k_p}$-isotypical crossing numbers at (α_o, β_o) (cf. (132)). Obviously, all the numbers $\mathfrak{t}_{j,l}(\alpha_o, \beta_o)$ appearing in formula (134) and satisfying the condition $l \neq k_p$ ($p = 1, \ldots, q$), are equal to zero.

(g) Compute the local bifurcation invariant $\omega(\alpha_o, \beta_o)$ for the system (71) according to the formula:

$$\omega(\alpha_o, \beta_o) = \deg_0(\alpha_o, \beta_o) \cdot \deg_*(\alpha_o, \beta_o) = \sum_{(H)} n_H(H), \tag{137}$$

where '$\cdot$' denotes the product in the $A(\Gamma)$-module $A_1^t(G)$.

(h) Let $\{(H_1), \ldots, (H_n)\}$ be the set of all dominating orbit types in W such that the numbers n_{H_k}, $k = 1, \ldots, n$, appearing in (137) are all non-zero. Then, for each $k = 1, \ldots, n$, there exist at least $|\frac{G}{H_k}|_{S^1}$ different branches of non-trivial periodic solutions to (77) (with the limit frequency $l\beta_o$ for some $l \in \mathbb{N}$) bifurcating from $(\alpha_o, 0)$. Moreover, for each (α, β, u), belonging to these $|\frac{G}{H_k}|_{S^1}$ branches, one has $(G_u) = (H_k)$ (here, as usual, the symbol $|X|_{S^1}$ stands for the number of S^1-orbits in the set X). In particular, the number

$$\left|\frac{G}{H_1}\right|_{S^1} + \left|\frac{G}{H_2}\right|_{S^1} + \cdots + \left|\frac{G}{H_n}\right|_{S^1}$$

gives a lower estimate for the number of branches of non-trivial solutions resulting in the (local) Hopf bifurcation in the system (77). In addition, if in formula (137) one has $n_{H_o} \neq 0$ for some *non-dominating* orbit type (H_o) in W, then Remark 5.5(ii) is applied.

5.5. *Usage of Maple© routines*

The last step of the equivariant degree method is the computation of the exact value of the local bifurcation invariant $\omega(\alpha_o, \beta_o)$ given by (134). Depending on the size of the group Γ, the number of related algebraic computations can be quite substantial and the usage of computer programming seems to be indispensable. We have created several Maple© routines, for selected groups Γ, that are able to complete these tasks, based on the properly prepared spectral data extracted from the characteristic equation (75).

Arranging equivariant spectral data We begin with the information related to the negative spectrum of the operator A (described in (b) and (c)). In order to simplify the computations of the factor (135) in formula (134), one can take advantage of the fact that $(\deg_{\mathcal{V}_i})^2 = (\Gamma)$, which is the unit in the Burnside ring $A(\Gamma)$. Therefore, it is sufficient to consider the numbers

$$\mathfrak{m}_i := \sum_{\mu \in \sigma_-} m_i(\mu), \quad \text{for } i = 0, 1, \ldots, r,$$

and introduce the sequence $\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r\}$, where

$$\varepsilon_i := \mathfrak{m}_i \bmod 2, \quad i = 0, 1, \ldots, r.$$

Folding homomorphism Θ_l Define on generators $(K^{\varphi,k})$ an $A(\Gamma)$-module homomorphism $\Theta_l : A_1^t(\Gamma \times S^1) \to A_1^t(\Gamma \times S^1)$ by

$$\Theta_l\big[(K^{\varphi,k})\big] := (K^{\varphi,lk})$$

Table 2
Isotypical crossing numbers

Representation	Folding 1	Folding 2	$\dots$	Folding l
$\mathcal{U}_0$	$\mathfrak{t}_{0,1}(\alpha_o,\beta_o)$	$\mathfrak{t}_{0,2}(\alpha_o,\beta_o)$	$\dots$	$\mathfrak{t}_{0,l}(\alpha_o,\beta_o)$
$\mathcal{U}_1$	$\mathfrak{t}_{1,1}(\alpha_o,\beta_o)$	$\mathfrak{t}_{1,2}(\alpha_o,\beta_o)$	$\dots$	$\mathfrak{t}_{1,l}(\alpha_o,\beta_o)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{U}_s$	$\mathfrak{t}_{s,1}(\alpha_o,\beta_o)$	$\mathfrak{t}_{s,2}(\alpha_o,\beta_o)$	$\dots$	$\mathfrak{t}_{s,l}(\alpha_o,\beta_o)$

(see Subsection 3.6). Then,

$$\deg_{\mathcal{V}_{j,l}} = \Theta_l(\deg_{\mathcal{V}_{j,1}}). \tag{138}$$

The full exact value of the local bifurcation invariant $\omega(\alpha_o,\beta_o)$ (cf. (134)) can be obtained with the usage of the Maple© routines, by applying the following formula

$$\begin{aligned}&\omega(\alpha_o,\beta_o)\\ &\quad=\sum_l \Theta_l\big[\texttt{showdegree}\,[\Gamma](\varepsilon_0,\varepsilon_1,\dots,\varepsilon_r,\mathfrak{t}_{0,l}(\alpha_o,\beta_o),\mathfrak{t}_{1,l}(\alpha_o,\beta_o),\dots,\\ &\qquad\mathfrak{t}_{s,l}(\alpha_o,\beta_o))\big]\end{aligned} \tag{139}$$

(see Table 2).

5.6. *Bibliographical remarks*

Symmetric Hopf bifurcation was intensively studied during the last decades. The commonly used methods were based on generic approximation, transversality theory (cf. [51–53]), reduction to normal forms and special singularity theory (see the fundamental monographs [64,67], for the recent developments we refer to [31], see also [30,32,63,66]). The case of the symmetric Hopf bifurcation from the so-called "relative equilibria" was considered in [28,100–102,116,147]). One of the advantages of the singularity theory methods is that they allow to predict stability of the bifurcating branches, while the twisted degree method is not sensitive to stability. However, it is our belief that the latter method is easier to apply (in a standard way), allows to treat "large" groups of symmetries and can be completely computerized. The $\Gamma\times S^1$-equivariant settings for an ODEs system, relevant to our discussion, was studied in [18,19,81,91,93]. The notion of a dominating orbit type (which was introduced in [10]) is commonly known in the literature as "maximal". The Maple© library for the computations of the equivariant degree, was developed by A. Biglands.

6. Symmetric Hopf bifurcation for FDEs

In this section, we apply the equivariant degree method to study the Hopf bifurcation problem in systems of symmetric functional differential equations. Since in this case, the

equivariant degree setting is completely parallel to the one that was developed for ordinary differential equations, we only point out the differences that are specific for FDEs and NFDEs. In order to simplify the exposition, we restrict ourselves to Γ being a finite group. In particular, in this case the set $\Phi_1^t(G)$ coincides with $\{(H) \in \mathcal{O}(G)\colon\ \dim W(H) = 1\}$. We also study a continuation of a symmetric branch of non-constant periodic solutions for FDEs.

Throughout this section V stands for an orthogonal Γ-representation (Γ is finite).

6.1. *Symmetric Hopf bifurcation for FDEs with delay: General framework*

Statement of the problem Given a constant $\tau \geqslant 0$, denote by $C_{V,\tau}$ the Banach space of continuous functions from $[-\tau, 0]$ to V equipped with the usual supremum norm

$$\|\varphi\| := \sup_{-\tau \leqslant \theta \leqslant 0} \bigl|\varphi(\theta)\bigr|, \quad \varphi \in C_{V,\tau}.$$

Given a continuous function $x : \mathbb{R} \to V$ and $t \in \mathbb{R}$, define $x_t \in C_{V,\tau}$ by

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-\tau, 0].$$

Clearly, the Γ-action on V induces a natural isometric Banach representation of Γ on the space $C_{V,\tau}$ with the Γ-action given by

$$(\gamma\varphi)(\theta) := \gamma\bigl(\varphi(\theta)\bigr), \quad \gamma \in \Gamma,\ \theta \in [-\tau, 0].$$

Consider the following one-parameter family of differential delayed equations

$$\dot{x} = f(\alpha, x_t), \tag{140}$$

where $x : \mathbb{R} \to V$ is a differentiable function and $f : \mathbb{R} \oplus C_{V,\tau} \to V$ satisfies the following assumptions:

(A1) f is continuously differentiable;

(A2) f is Γ-equivariant, i.e.

$$f(\alpha, \gamma\varphi) = \gamma f(\alpha, \varphi), \quad \varphi \in C_{V,\tau},\ \alpha \in \mathbb{R},\ \gamma \in \Gamma;$$

(A3) $f(\alpha, 0) = 0$ for all $\alpha \in \mathbb{R}$.

For any $x_o \in V$, we use the same symbol to denote the constant function $x_o(t) \equiv x_o$. Clearly, $(x_o)_t = x_o$ for all $t \in \mathbb{R}$. A point $(\alpha, x_o) \in \mathbb{R} \oplus V$ is said to be a *stationary point* of (140) if $f(\alpha, x_o) = 0$. In particular, by condition (A3), $(\alpha, 0)$ is a stationary point of (140) for all $\alpha \in \mathbb{R}$. Moreover, we say that a stationary point (α, x_o) is *nonsingular* if the restriction of f to the space $\mathbb{R} \oplus V \subset \mathbb{R} \oplus C_{V,\tau}$, still denoted by f, has the derivative $D_x f(\alpha, x_o) : V \to V$ (with respect to $x \in V$), which is an isomorphism. The definition of the Hopf bifurcation for the system (140) occurring at $(\alpha_o, 0)$ is exactly the same as the one given in Subsection 4.1.

In what follows, we consider a fixed stationary point $(\alpha_o, 0)$ of (140). In order to prevent the occurrence of a steady-state bifurcation at $(\alpha_o, 0)$, we assume

(A4) The stationary point $(\alpha_o, 0)$ of (140) is non-singular, i.e.

$$\det\bigl(D_x f(\alpha_o, 0)|_V\bigr) \neq 0.$$

Characteristic equation Let V^c be the complexification of the vector space V and let the Γ-isotypical decompositions of V and V^c be given by (113), where $V_0 = V^\Gamma$, V_j is modeled on the real irreducible Γ-representation $\mathcal{V}_j$, $U_0 = (V^c)^\Gamma$ and U_j is modeled on the complex irreducible Γ-representation $\mathcal{U}_j$.

Let (α, x_o) be a stationary point of (140). The linearization of (140) at (α, x_o) leads to the following *characteristic equation* for the stationary point (α, x_o)

$$\det\nolimits_{\mathbb{C}} \Delta_{(\alpha,x_o)}(\lambda) = 0, \tag{141}$$

where

$$\Delta_{(\alpha,x_o)}(\lambda) := \lambda\,\mathrm{Id} - D_x f(\alpha, x_o)(\mathrm{e}^{\lambda\cdot}\cdot)$$

is a complex linear operator from V^c to V^c, with $(\mathrm{e}^{\lambda\cdot}\cdot)(\theta, x) = \mathrm{e}^{\lambda\theta}x$ and $D_x f(\alpha, x_o)(z \otimes x) = z \otimes D_x f(\alpha, x_o)x$ for $z \otimes x \in V^c$ (cf. [145]). Put $\Delta_\alpha(\lambda) := \Delta_{(\alpha,0)}(\lambda)$.

A solution λ_o to (141) is called a *characteristic root* of (141) at the stationary point (α, x_o). It is clear that (α, x_o) is a non-singular stationary point if and only if 0 is not a characteristic root of (141) at the stationary point (α, x_o).

Recall (see Definition 4.2) that a non-singular stationary point (α, x_o) is called a *center* if (141) permits a purely imaginary root. We call (α, x_o) an *isolated center* if it is the only center in some neighborhood of (α, x_o) in $\mathbb{R} \oplus V$.

By (A2) and (A3), the operator $\Delta_\alpha(\lambda) : V^c \to V^c$, $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{C}$, is Γ-equivariant. Consequently, for every isotypical component U_j of V^c, $j = 0, 1, \ldots, s$ (cf. (113)), $\Delta_\alpha(\lambda)(U_j) \subseteq U_j$. Put

$$\Delta_{\alpha,j}(\lambda) := \Delta_\alpha(\lambda)|_{U_j}.$$

Crossing numbers Assume the following condition to be satisfied:

(A5) The stationary point $(\alpha_o, 0)$ is an isolated center for the system (140) such that (141) permits a purely imaginary root $\lambda = \mathrm{i}\beta_o$ with $\beta_o > 0$.

Let λ be a complex root of the characteristic equation $\det_{\mathbb{C}} \Delta_{\alpha_o}(\lambda) = 0$. We use the following notations:

$$E(\lambda) := \ker \Delta_{\alpha_o}(\lambda) \subset V^c,$$

$$E_j(\lambda) := E(\lambda) \cap U_j,$$

$$m_j(\lambda) := \dim_{\mathbb{C}} E_j(\lambda) / \dim_{\mathbb{C}} \mathcal{U}_j.$$

The integer $m_j(\lambda)$ is called the $\mathcal{U}_j$*-multiplicity* of the characteristic root λ.

Define S by formula (110). Under the assumption (A5), the constants $\delta > 0$ and $\varepsilon > 0$ can be chosen so small that the following condition is satisfied:

(*) For every $\alpha \in [\alpha_o - \varepsilon, \alpha_o + \varepsilon]$, if there is a characteristic root $u + \mathrm{i}v \in \partial S$ at $(\alpha, 0)$, then $u + \mathrm{i}v = \mathrm{i}\beta_o$ and $\alpha = \alpha_o$.

Put $\alpha_\pm := \alpha_o \pm \varepsilon$. Note that $\Delta_\alpha(\lambda)$ is analytic in $\lambda \in \mathbb{C}$ and continuous in $\alpha \in [\alpha_-, \alpha_+]$ (see [71]). It follows that $\det_{\mathbb{C}} \Delta_{\alpha_o \pm \varepsilon}(\lambda) \neq 0$ for all $\lambda \in \partial S$. So, the following notation (cf. (123))

$$\mathfrak{t}^{\pm}_{j,1}(\alpha_o, \beta_o) := \deg\bigl(\det_{\mathbb{C}} \Delta^{\mathbb{C}}_{\alpha_\pm, j}(\cdot), S\bigr), \tag{142}$$

is well-defined for $0 \leqslant j \leqslant s$ (see (113)).

Following the same lines as in Subsection 5.3 (see Definition 5.8), introduce

DEFINITION 6.1. Let $\mathcal{U}_j$ be a complex Γ-irreducible representation and $\mathcal{V}_{j,l} := {}^l\mathcal{U}_j$ for $l \in \mathbb{N}$. Put

$$\mathfrak{t}_{j,1}(\alpha_o, \beta_o) := \mathfrak{t}^{-}_{j,1}(\alpha_o, \beta_o) - \mathfrak{t}^{+}_{j,1}(\alpha_o, \beta_o), \tag{143}$$

and for any integer $l > 1$,

$$\mathfrak{t}_{j,l}(\alpha_o, \beta_o) := \mathfrak{t}_{j,1}(\alpha_o, l\beta_o). \tag{144}$$

Then, $\mathfrak{t}_{j,l}(\alpha_o, \beta_o)$ is called the $\mathcal{V}_{j,l}$*-isotypical crossing number* of $(\alpha_o, 0)$ corresponding to the characteristic root $\mathrm{i}\beta_o$.

REMARK 6.2. Observe that $\mathfrak{t}_{j,l}(\alpha_o, \beta_o) = 0$ if $\mathrm{i}l\beta_o$ is not a root of (141) (cf. (142) and (143)).

In order to establish the existence of small amplitude periodic solutions bifurcating from the stationary point $(\alpha_o, 0)$, i.e. the occurrence of Hopf bifurcation at the stationary point $(\alpha_o, 0)$, and to associate with $(\alpha_o, 0)$ a *local bifurcation invariant*, we apply the standard steps for the degree-theoretical approach.

Normalization of the period Normalization of the period is obtained by making the change of variable $u(t) = x(\frac{p}{2\pi} t)$ for $t \in \mathbb{R}$. We obtain the following equation, which is equivalent to (140):

$$\dot{u}(t) = \frac{p}{2\pi} f(\alpha, u_{t,2\pi/p}), \tag{145}$$

where $u_{t,2\pi/p} \in C_{V,\tau}$ is defined by

$$u_{t,2\pi/p}(\theta) = u\left(t + \frac{2\pi}{p}\theta\right), \quad \theta \in [-\tau, 0].$$

Evidently, $u(t)$ is a 2π-periodic solution of (145) if and only if $x(t)$ is a p-periodic solution of (140). We can also introduce $\beta := \frac{2\pi}{p}$ into Eq. (145) to obtain

$$\dot{u}(t) = \frac{1}{\beta} f(\alpha, u_{t,\beta}). \tag{146}$$

$\Gamma \times S^1$-equivariant setting in functional spaces We use the standard identification $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ (with $t \leftrightarrow e^{it}$) and the operators L, j, K defined by (81), (82) and (85). Put $\mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}_+$. Take $(L+K)^{-1} : L^2(S^1; V) \to H^1(S^1; V)$ (cf. Subsection 4.2). It can be easily shown that the map $\mathcal{F} : \mathbb{R}^2_+ \times H^1(S^1; V) \to H^1(S^1; V)$ defined by

$$\mathcal{F}(\alpha, \beta, u) = (L+K)^{-1}\left[Ku + \frac{1}{\beta} N_f\big(\alpha, \beta, j(u)\big)\right], \tag{147}$$

is completely continuous, where $N_f : \mathbb{R}^2_+ \times C(S^1; V) \to L^2(S^1; V)$ is defined by

$$N_f(\alpha, \beta, v)(t) = f(\alpha, v_{t,\beta}), \tag{148}$$

and $e^{it} \in S^1$, $(\alpha, \beta, v) \in \mathbb{R}^2_+ \times C(S^1; V)$.

REMARK 6.3. Notice that, because of the presence of the delay in (140), the map N_f depends on the parameter β, which is the only difference from the setting that was discussed in Subsection 4.2.

Put $W := H^1(S^1; V)$. The space W is an isometric Hilbert $\Gamma \times S^1$-representation of the group $G = \Gamma \times S^1$ (see Subsection 5.1) and the non-linear operator $\mathcal{F}$, defined by (147), is clearly G-equivariant.

REMARK 6.4. Notice that $(\alpha, \beta, u) \in \mathbb{R}^2_+ \times W$ is a 2π-periodic solution of (146) if and only if $u = \mathcal{F}(\alpha, \beta, u)$. Consequently, the occurrence of a Hopf bifurcation at $(\alpha_o, 0)$ for Eq. (140) is equivalent to a bifurcation of 2π-periodic solutions of (146) from $(\alpha_o, \beta_o, 0)$ for some $\beta_o > 0$. On the other hand, if a bifurcation at $(\alpha_o, \beta_o, 0) \in \mathbb{R}^2_+ \times W$ takes place in (146), then we *necessarily* have that the operator $\mathrm{Id} - D_u\mathcal{F}(\alpha_o, \beta_o, 0) : W \to W$ is not an isomorphism, or equivalently, $il\beta_o$ is a purely imaginary characteristic root at $(\alpha_o, 0)$, for some $l \in \mathbb{N}$, i.e. $\det_{\mathbb{C}} \Delta_{\alpha_o}(il\beta_o) = 0$.

Sufficient condition for Hopf bifurcation Bearing in mind Remarks 6.3 and 6.4 and literally follow Subsection 5.1 (see also Subsection 4.3), one can establish

THEOREM 6.5. *Given the system* (140), *assume conditions* (A1)–(A5) *to be satisfied. Take* $\mathcal{F}$ *defined by* (147) *and construct* Ω *according to* (93). *Let* $\varsigma : \overline{\Omega} \to \mathbb{R}$ *be a* G*-invariant auxiliary function* (*see Definition* 4.3) *and let* $\mathfrak{F}_\varsigma$ *be defined by* (94). *Then, a complete analogue of the conclusion of Theorem* 5.4 *is true for the systems* (140) *and* (146).

REMARK 6.6. Following the scheme described in Subsections 5.2 and 5.3, one can establish a complete analogue of formula (134) in the setting relevant to Theorem 6.5.

6.2. *Symmetric Hopf bifurcation for neutral FDEs*

Statement of the problem We use the same notations as in Subsection 6.1.

Consider the following one-parameter family of *neutral equations*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t) - b(\alpha, x_t)\big] = f(\alpha, x_t), \tag{149}$$

where $x : \mathbb{R} \to V$ is a continuous function,[2] and $b,\ f : \mathbb{R} \oplus C_{V,\tau} \to V$ satisfy the following assumptions:

(A1) $b,\ f$ are continuously differentiable;

(A2) $b,\ f$ are Γ-equivariant;

(A3) $b(\alpha, 0) = 0,\ f(\alpha, 0) = 0$ for all $\alpha \in \mathbb{R}$.

Also, to prevent the occurrence of the steady-state bifurcation, assume

(A4) $\det(D_x f(\alpha, 0)|_V) \neq 0$ for all $\alpha \in \mathbb{R}$.

In addition, assume that

(L) b satisfies the Lipschitz condition with respect to the second variable, i.e.

$$\exists_\kappa\ 0 \leqslant \kappa < 1,\ \text{s.t.}\ \big\|b(\alpha, \varphi) - b(\alpha, \psi)\big\| \leqslant \kappa \|\varphi - \psi\|_\infty \tag{150}$$

for all $\varphi,\ \psi \in C_{V,\tau},\ \alpha \in \mathbb{R}$.

Literally following Section 6.1, one can define the notions of stationary/singular points to (149).

Characteristic equation Let (α, x_o) be a stationary point of (149). The linearization of (149) at (α, x_o) leads to the following *characteristic equation*,

$$\det_{\mathbb{C}} \Delta_{(\alpha, x_o)}(\lambda) = 0, \tag{151}$$

where

$$\Delta_{(\alpha, x_o)}(\lambda) := \lambda\big[\mathrm{Id} - D_x b(\alpha, x_o)(\mathrm{e}^{\lambda \cdot} \cdot)\big] - D_x f(\alpha, x_o)(\mathrm{e}^{\lambda \cdot} \cdot). \tag{152}$$

In what follows, assume that

(A5) The system (149) has an isolated center $(\alpha_o, 0)$ for some $\alpha_o \in \mathbb{R}$, with the corresponding purely imaginary characteristic root $\mathrm{i}\beta_o$, for $\beta_o > 0$.

We associate to the problem (149) the *local bifurcation invariant* and follow the standard steps (similar to those in Section 6.1) of the degree-theoretical treatment of the Hopf bifurcation phenomenon.

Normalization of period By making a change of variable $x(t) = u(\beta t)$ with $\beta := \frac{2\pi}{p}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[u(t) - b(\alpha, u_{t,\beta})\big] = \frac{1}{\beta} f(\alpha, u_{t,\beta}), \tag{153}$$

[2] Formally speaking, we do not require in (149) that $x(t)$ is differentiable, but only $x(t) - b(\alpha, x_t)$ to be continuously differentiable.

where $u_{t,\beta} \in C_{V,\tau}$ is defined by

$$u_{t,\beta}(\theta) = u(t + \beta\theta), \quad \theta \in [-\tau, 0].$$

Evidently, $u(t)$ is a 2π-periodic solution of (153) if and only if $x(t)$ is p-periodic solution of (149).

$\Gamma \times S^1$-equivariant setting in functional spaces Put $W := H^1(S^1; V)$ and let L, j, K and N_f be defined by (81), (82), (85) and (148) respectively. Define $N_b : \mathbb{R}^2_+ \times C(S^1; V) \to L^2(S^1; V)$ by

$$N_b(\alpha, \beta, v)(t) = b(\alpha, v_{t,\beta}).$$

Take $(L + K)^{-1} : L^2(S^1; V) \to W$ and the map $\mathcal{F} : \mathbb{R}^2_+ \times W \to W$ defined by

$$\mathcal{F}(\alpha, \beta, u) = (L + K)^{-1}\left[\frac{1}{\beta} N_f(\alpha, \beta, u) + K\big(u - N_b(\alpha, \beta, u)\big)\right] + N_b(\alpha, \beta, u). \tag{154}$$

Observe that the map $\mathcal{F}$ is a sum of two maps, where the first map

$$(\alpha, \beta, u) \mapsto (L + K)^{-1}\left[\frac{1}{\beta} N_f(\alpha, \beta, u) + K\big(u - N_b(\alpha, \beta, u)\big)\right],$$

is completely continuous, and the second map $(\alpha, \beta, u) \mapsto N_b(\alpha, \beta, u)$ is a Banach contraction with constant κ $(0 \leqslant \kappa < 1)$ (see assumptions (A1) and (L)). Therefore, $\mathcal{F}$ is a condensing map (see Definition 3.54 and Remark 3.55).

Finally, an analogue of Remark 6.4 is applied to Eq. (149).

Sufficient condition for Hopf bifurcation Following the same lines as in Subsections 4.3 and 6.1, define Ω and $\mathfrak{F}_\varsigma$ by (93) and (94) respectively. Obviously, $\mathfrak{F}_\varsigma$ is an Ω-admissible G-equivariant condensing field. Therefore, the standard Nussbaum–Sadovskii extension of the twisted degree (see Definition 3.57, Lemma 3.59 and formula (70)) can be applied to the admissible pair $(\mathfrak{F}_\varsigma, \Omega)$. In a standard way, one can easily establish

THEOREM 6.7. *Given the system* (149), *assume conditions* (A1)–(A5) *and* (L) *to be satisfied. Take $\mathcal{F}$ defined by* (154) *and construct Ω according to* (93). *Let $\varsigma : \overline{\Omega} \to \mathbb{R}$ be a G-invariant auxiliary function* (*see Definition* 4.3) *and let $\mathfrak{F}_\varsigma$ be defined by* (94). *Then, a complete analogue of the conclusion of Theorem* 5.4 *is true for the systems* (149) *and* (153). *In addition, an analogue of Remark* 6.6 *is valid.*

6.3. *Global bifurcation problems*

In this subsection, we apply the twisted degree method to a global Hopf bifurcation problem for a system of symmetric functional differential equations to analyze a continuation of

symmetric branches of non-constant periodic solutions. To this end, we translate/adapt conditions (A1)–(A5) from Subsection 6.1 to the abstract setting related to symmetric global Hopf bifurcation occurring in the system (140).

To begin with, assume that the system (140) satisfies conditions (A1)–(A3). Take the Hilbert G-representation $W := H^1(S^1; V)$ and define $\mathcal{F}: \mathbb{R}^2 \oplus W \to W$ by formula (147). Put $\mathfrak{F}(\alpha, \beta, u) := u - \mathcal{F}(\alpha, \beta, u)$. Then, the conditions (A1)–(A3) translate as:

(F1) $\mathfrak{F}$ is a completely continuous G-equivariant field of class C^1 and $\mathfrak{F}(\lambda, 0) = 0$ for all $(\lambda, 0) \in \mathbb{R}^2 \oplus W$.

Put $\bar{\mathfrak{F}} := \mathfrak{F}|_{\mathbb{R}^2 \oplus W^{S^1}} : \mathbb{R}^2 \oplus W^{S^1} \to W^{S^1}$. Then, the condition (A4) translates as:

(F2) $D_w \bar{\mathfrak{F}}(\lambda, 0)$ is an isomorphism from W^{S^1} to W^{S^1} for all $\lambda \in \mathbb{R}^2$.

Condition (A5), being necessary for the occurrence of the (local) Hopf bifurcation (cf. Proposition 4.1), translates as: $D_w \mathfrak{F}(\lambda_o, 0): W \to W$ is not an isomorphism. To study the *global* behavior of branches bifurcating from $(\lambda_o, 0)$, we need the following condition to be satisfied:

(F3) The set $\Lambda := \{\lambda \in \mathbb{R}^2\colon\ D_w \mathfrak{F}(\lambda, 0): W \to W$ is not an isomorphism$\}$ is discrete in $\mathbb{R}^2$.

We are interested in a solution set to the equation

$$\mathfrak{F}(\lambda, w) = 0, \quad (\lambda, w) \in \mathbb{R}^2 \oplus W. \tag{155}$$

In the same way, as it was done in Subsection 4.3, one can define trivial/nontrivial solutions to (155) as well as its bifurcation points. Bearing in mind condition (F3), for each $\lambda_o \in \Lambda$, one can associate the local bifurcation invariant $\omega(\lambda_o)$ (cf. Section 6.1) and the *local bifurcation Theorem 6.5* can be applied.

Below we discuss the global bifurcation phenomenon for Eq. (155). Let $\mathcal{S}$ be the closure of the set of all non-trivial solutions to (155), i.e.

$$\mathcal{S} := \overline{\left\{(\lambda, w) \in \mathbb{R}^2 \oplus W\colon\ \mathfrak{F}(\lambda, w) = 0, w \neq 0\right\}}. \tag{156}$$

Notice that $(\lambda_o, 0)$ is a bifurcation point of (155) if and only if $(\lambda_o, 0) \in \mathcal{S}$. Take a connected component $\mathcal{C} \subset \mathcal{S}$. If $\mathcal{C}$ contains a bifurcation point $(\lambda_o, 0)$, then it is clearly G-invariant. Notice that, in general, $\mathcal{C}$ may be composed of several orbit types, i.e.

$$\mathcal{C} = \bigcup_{(H)} \mathcal{C}_{(H)},$$

and the global behavior of $\mathcal{C}_{(H)}$ can be different for different orbit types (H), for example some of the branches $\mathcal{C}_{(H)}$ may be bounded, while the other are unbounded.

The result following below can be considered as a *global Hopf bifurcation theorem* for the system (140).

THEOREM 6.8. *Suppose that* $\mathfrak{F}: \mathbb{R}^2 \oplus W \to W$ *satisfies the assumptions* (F1)–(F3) *and let* $\mathcal{C}$ *be a bounded connected component of* $\mathcal{S}$ (*cf.* (156)) *such that* $\mathcal{C} \cap \mathbb{R}^2 \times \{0\} \neq \emptyset$. *Then,*

$$\mathcal{C} \cap \mathbb{R}^2 \times \{0\} = \left\{(\lambda_1, 0), (\lambda_2, 0), \ldots, (\lambda_N, 0)\right\},$$

and

$$\sum_{k=1}^{N} \omega(\lambda_k) = 0.$$

The proof of Theorem 6.8 can be easily obtained by combining the methods developed to prove Theorems 4.4, 5.4 and 6.5 with the standard global bifurcation techniques (see, for instance, [89,93,95,128]).

In order to present global bifurcation results for symmetric systems we will be dealing with in what follows, we need a slightly more general

THEOREM 6.9. *Suppose that* $\mathfrak{F}: \mathbb{R}^2 \oplus W \to W$ *satisfies the assumptions* (F1)–(F3) *and let* $\mathcal{C}^{(H_o)}$ *be a bounded connected component of* $\mathcal{S}^{(H_o)}$ (*cf.* (156)) *such that* $\mathcal{C}^{(H_o)} \cap \mathbb{R}^2 \times \{0\} = \{(\lambda_1, 0), (\lambda_2, 0), \dots, (\lambda_N, 0)\} \neq \emptyset$. *Suppose that*

$$\omega(\lambda_k) = \sum_{(H)} n_H^k (H).$$

Then,

$$\sum_{k=1}^{N} n_{H_o}^k = 0.$$

PROOF. Consider the map

$$\mathfrak{F}^{H_o}: \mathbb{R}^2 \oplus W^{H_o} \to W^{H_o}.$$

Put $G_o = N(H_o)$. Since the map $\mathfrak{F}^{H_o}$ is G_o-equivariant and satisfies similar to (F1)–(F3) conditions, we can define for every $\lambda_o \in \Lambda$ the local bifurcation invariant

$$\omega_o(\lambda_o) = \sum_{(K)} m_K (K) \in A_1^t(G_o)$$

associated with $\mathfrak{F}^{H_o}$. Since $\mathcal{C}^{H_o}$ is a bounded connected component of $\mathcal{S}^{H_o}$, we can apply Theorem 6.8 to obtain

$$\sum_{k=1}^{N} \omega_o(\lambda_k) = 0, \quad \omega_o(\lambda_k) = \sum_{K} m_K^k (K).$$

On the other hand, (H_o) is an orbit type in W^{H_o} and, it is easy to see that $m_{H_o}^k = n_{H_o}^k$ for all $k = 1, 2, \dots, N$. Consequently,

$$\sum_{k=1}^{N} n_{H_o}^k = 0,$$

and the statement follows. □

COROLLARY 6.10. *Suppose as in Theorem* 6.9 *that* $\mathfrak{F}:\mathbb{R}^2 \oplus W \to W$ *satisfies the assumptions* (F1)–(F3) *and let* $\mathcal{C}^{(H_o)}$ *be a connected component of* $\mathcal{S}^{(H_o)}$ (*cf.* (156)) *such that* $\mathcal{C}^{(H_o)} \cap \mathbb{R}^2 \times \{0\} = \{(\lambda_1, 0)\}$, *where* (H_o) *is an orbit type in* W. *If* $n^1_{H_o} \neq 0$, *then* $\mathcal{C}^{(H_o)}$ *is unbounded.*

REMARK 6.11.

(i) Observe that the conclusions of Theorems 6.8 and 6.9 and Corollary 6.10 are still true if we replace condition (F2) with the following one:
  (F2′) (a) $D_w\bar{\mathfrak{F}}(\lambda, 0)$ is an isomorphism from W^{S^1} to W^{S^1} for all $\lambda = (\alpha, \beta) \neq (\alpha_o, \beta)$, where α_o is a certain fixed number;
  (b) there is no Hopf bifurcation in (140) from the point $(\alpha_o, 0)$.

(ii) It should be pointed out that the global bifurcation results described in this subsection can be easily extended to almost all the classes of parametrized symmetric dynamical systems considered in this paper (as long as the corresponding local result is established).

6.4. *Bibliographical remarks*

The approach outlined in Section 6.1 (resp. Section 6.2) was discussed in [11] (resp. [15]) (see also [19]). The global bifurcation results given in Section 6.3 were published in [13]. Similar systems of symmetric DEs and FDEs were considered in [67,65,66,68,92,94,95, 129,140,145,146,148]. For the singularity theory treatment of the symmetric Hopf bifurcation in NFDEs, we refer to [69]. In the non-symmetric case, local and global Hopf bifurcation problems in FDEs were studied in [36,110,111,124,125,122,123].

7. Symmetric Hopf bifurcation problems for functional parabolic systems of equations

As we have seen in Sections 4–6, the twisted degree provides an effective and direct method for the classification of symmetric Hopf bifurcations in ODEs and FDEs, based on the equivariant spectral properties of the linearized equation at a stationary point. In this section, we present a setting for applications of the twisted degree to study the Hopf bifurcation problem for a continuously parametrized Γ-symmetric Fredholm coincidence equation, which is next associated to a system of non-linear functional parabolic equations with Γ-symmetries.

7.1. *Symmetric bifurcation in parameterized equivariant coincidence problems*

Delayed functional parabolic systems of our interest fall into a category of the so-called equivariant *coincidence* problems. Therefore, we start with a brief discussion of an abstract functional setting for such equivariant coincidence problems. We use the notations introduced in Subsections 2.3 and 2.4 ($G = \Gamma \times S^1$).

Functional setting for equivariant coincidence problems Let $\mathbb{E}$ and $\mathbb{F}$ be real isometric Banach G-representations, $\mathcal{P}$ a topological space equipped with the trivial G-action, and $\{L_\lambda\}_{\lambda\in\mathcal{P}} \subset \mathrm{Op}^G$ a *continuous family of equivariant Fredholm operators of index zero, parametrized by* $\mathcal{P}$, i.e. $L_\lambda \in \mathfrak{F}_0^G$ for each $\lambda \in \mathcal{P}$, and the mapping $\eta : \mathcal{P} \to \mathrm{Op}^G$ defined by $\eta(\lambda) = L_\lambda$ for $\lambda \in \mathcal{P}$, is continuous.

Define ξ by

$$\xi := \big\{(\lambda, u, y) \in \mathcal{P} \times (\mathbb{E} \oplus \mathbb{F})\colon\ u \in \mathrm{Dom}(L_\lambda),\ \ y = L_\lambda u\big\},$$

and consider $\pi : \xi \to \mathcal{P}$, $\pi(\lambda, u, y) = \lambda$ for $(\lambda, u, y) \in \xi$. It has been shown in [50] that $\pi : \xi \to \mathcal{P}$ determines a locally trivial G-vector bundle.

Let

$$\mathcal{E} := \big\{(\lambda, u) \in \mathcal{P} \times \mathbb{E}\colon\ u \in \mathbb{E}_{L_\lambda}\big\} \tag{157}$$

and let $p_1 : \xi \to \mathcal{E}$ be given by $p_1(\lambda, u, y) = (\lambda, u)$, for $(\lambda, u, y) \in \xi$. Since, for every $\lambda \in \mathcal{P}$, the projection $pr_1 : \mathrm{Gr}(L_\lambda) \to \mathbb{E}_\lambda := \mathbb{E}_{L_\lambda}$ is an equivariant isometry, the mapping $p_1 : \xi \to \mathcal{E}$ gives us the natural identification of the G-bundles ξ and $\mathcal{E}$ (we use the same symbol for a bundle and its total space).

Define the vector bundle morphism $L : \mathcal{E} \to \mathbb{F}$, where $\mathbb{F}$ is viewed as a bundle over a one-point space, by

$$L(\lambda, u) = L_\lambda u, \quad (\lambda, u) \in \mathcal{E}. \tag{158}$$

DEFINITION 7.1. Let X be a subset of $\mathcal{P}$ and let L be given by (157) and (158). An *equivariant resolvent* of L over X is a G-vector bundle morphism $K : X \times \mathbb{E} \to \mathbb{F}$ such that

(i) for every $\lambda \in X$, $K_\lambda : \mathbb{E} \to \mathbb{F}$ is a finite-dimensional linear operator;
(ii) for every $\lambda \in X$, $L_\lambda + K_\lambda : \mathbb{E}_\lambda \to \mathbb{F}$ is an isomorphism.

Denote by $\mathcal{R}^G(L, X)$ the set of all equivariant resolvents of L over X.

REMARK 7.2.

(i) In contrast to the non-equivariant case, given $\lambda_o \in \mathcal{P}$, one may have $\mathcal{R}^G(L, \{\lambda_o\}) = \emptyset$.
(ii) Also, in general it might happen that $\mathcal{R}^G(L, X) = \emptyset$, while $\mathcal{R}^G(L, \{\lambda_o\}) \neq \emptyset$ for each $\lambda_o \in X$.

In the light of Remark 7.2, the following result turns out to be useful.

LEMMA 7.3. (Cf. [90].) *Let $X \subset \mathcal{P}$ be a compact contractible set containing a point λ^* such that $\mathcal{R}^G(L, \{\lambda^*\}) \neq \emptyset$. Then, $\mathcal{R}^G(L, X) \neq \emptyset$.*

Assume the following condition to be satisfied:

(H1) There exists a compact subset $X \subset \mathcal{P}$ such that $\mathcal{R}^G(L, X) \neq \emptyset$.

Fix $K \in \mathcal{R}^G(L, X)$ and put

$$R_\lambda := (L_\lambda + K_\lambda)^{-1} \tag{159}$$

(cf. Definition 7.1(ii)).

Given a completely continuous G-equivariant map $F : \mathcal{E} \to \mathbb{F}$, consider the associated *parametrized equivariant coincidence problem* (cf. [90])

$$L_\lambda u = F(\lambda, u), \quad (\lambda, u) \in \mathcal{E}|_{X \times \mathrm{Dom}(L_\lambda)}. \tag{160}$$

Using the resolvent K, one can transform (160) into the following fixed-point problem:

$$y = \mathcal{F}(\lambda, y), \quad (\lambda, y) \in X \times \mathbb{F}, \tag{161}$$

where

$$\mathcal{F}(\lambda, y) = F(\lambda, R_\lambda y) + K_\lambda(R_\lambda y), \quad (\lambda, y) \in X \times \mathbb{F}.$$

By assumption (H1), X is compact, therefore, $\mathcal{F}$ is completely continuous.

Bifurcation invariant for the equivariant coincidence problem In what follows, $\mathbb{E}$, $\mathbb{F}$ stand for isometric Banach G-representations, $\mathcal{P} = \mathbb{R} \times \mathbb{R}_+$ and $\{L_\lambda\}_{\lambda \in \mathcal{P}}$ is a continuous family of G-equivariant Fredholm G-equivariant operators of index zero satisfying condition (H1). Fix $K \in \mathcal{R}^G(L, X)$ with R_λ, $\lambda \in \mathcal{P}$, defined by (159).

Keeping in mind the setting relevant to parametrized parabolic systems discussed in the next subsections, we will specify F from (160), assuming

(H2) $\begin{cases} \text{(i) there exists a real isometric Banach } G\text{-representation } \widehat{\mathbb{E}} \text{ and an} \\ \quad \text{injective } G\text{-vector bundle morphism } J : \mathcal{E} \to \mathcal{P} \times \widehat{\mathbb{E}} \text{ such that} \\ \quad J_\lambda := J(\lambda, \cdot) \text{ is a compact linear operator for every } \lambda \in \mathcal{P}; \\ \text{(ii) there exists an equivariant } C^1\text{-map } \widehat{F} : \mathcal{P} \times \widehat{E} \to \mathbb{F}. \end{cases}$

Put

$$F := \widehat{F} \circ J. \tag{162}$$

Obviously, F is G-equivariant and completely continuous.

Consider now the coincidence problem (160) with F defined by (162) (see also condition (H2)). In addition, assume that there exists a two-dimensional submanifold $M \subset \mathcal{P} \times \mathbb{E}^G$ (thought of as a “bifurcation surface”) satisfying the following two conditions:

(H3) M is a subset of the solution set to (160);

(H4) if $(\lambda_o, u_o) \in M$, then there exist open neighborhoods U_{λ_o} of λ_o in $\mathcal{P}$ and U_{u_o} of u_o in $\mathbb{E}^G$, and a C^1-map $\chi : U_{\lambda_o} \to \mathbb{E}^G$ such that

$$M \cap (U_{\lambda_o} \times U_{u_o}) = \mathrm{Gr}(\chi).$$

Bearing in mind condition (H3), $(\lambda, u) \in M$ is called a *trivial solution* to (160) while the other solutions to (160) are said to be *non-trivial solutions*. A point $(\lambda_o, u_o) \in M$ is called a *bifurcation point* if in each neighborhood of (λ_o, u_o), there exists a nontrivial solution to (160). In what follows, we study the existence and multiplicity of branches of non-trivial solutions bifurcating from M, and classify their symmetries.

REMARK 7.4. (i) It is clear that (λ, u) is a solution to (160) if and only if (λ, y) is a solution to the system (161), where $y = (L_\lambda + K_\lambda)u$. Moreover, the set of all trivial solutions to (161) can be represented as

$$\widetilde{M} := \big\{(\lambda, y) \in X \times \mathbb{F}\colon \big(\lambda, R_\lambda(y)\big) \in M\big\}.$$

Then, condition (H4) translates as:

(H4′) if $(\lambda_o, y_o) \in \widetilde{M}$, then there exist open neighborhoods U_{λ_o} of λ_o in $\mathcal{P}$ and U_{y_o} of y_o in $\mathbb{F}^G$ and a C^1-map $\tilde{\chi} : U_{\lambda_o} \to \mathbb{F}^G$ such that

$$\widetilde{M} \cap (U_{\lambda_o} \times U_{y_o}) = \mathrm{Gr}(\tilde{\chi}).$$

(ii) Also, $(\lambda_o, u_o) \in M$ is a bifurcation point of (160) if and only if $(\lambda_o, y_o) \in \widetilde{M}$ is a bifurcation point of (161), where $y_o = (L_{\lambda_o} + K_{\lambda_o})u_o$.

Rewrite (161) as

$$(\pi - \mathcal{F})(\lambda, y) = 0, \quad (\lambda, y) \in X \times \mathbb{F}, \tag{163}$$

where π is the projection map on $\mathbb{F}$, $\pi(\lambda, y) = y$. Clearly, $\pi - \mathcal{F}$ is a G-equivariant completely continuous field of class C^1, and

$$D_y(\pi - \mathcal{F})(\lambda, y) = \mathrm{Id} - \big(D_u F\big(\lambda, R_\lambda(y)\big)R_\lambda + K_\lambda R_\lambda\big).$$

Thus, by assumption (H2) (see also (162)), $D_y(\pi - \mathcal{F})$ is a bounded Fredholm operator of index zero. Since G acts trivially on $X \times \mathbb{F}^G$, for $(\lambda, y) \in X \times \mathbb{F}^G$, $D_y(\pi - \mathcal{F})(\lambda, y)$ is G-equivariant (in particular, it is also G-equivariant for $(\lambda, y) \in \widetilde{M}$). By the implicit function theorem, a *necessary* condition for $(\lambda_o, y_o) \in \widetilde{M}$ to be a bifurcation point, is that the derivative $D_y(\pi - \mathcal{F})(\lambda_o, y_o)$ is not an isomorphism of $\mathbb{F}$. Such a point $(\lambda_o, y_o) \in \widetilde{M}$ is called *L-singular*. An L-singular point (λ_o, y_o) is said to be *isolated*, if it is the only L-singular point in some neighborhood of (λ_o, y_o) in $\widetilde{M}$.

Finally, assume that

(H5) there exists an isolated L-singular point $(\lambda_o, y_o) \in \widetilde{M}$.

In order to associate a local bifurcation invariant $\omega(\lambda_o, u_o) \in A_1^t(G)$ to (160), take a neighborhood $\mathcal{D}_{\lambda_o}$ of (λ_o, y_o) in $\widetilde{M}$ such that (i) (λ_o, y_o) is the only L-singular point in $\mathcal{D}_{\lambda_o}$ and (ii) $\overline{\mathcal{D}}_{\lambda_o} \subset \widetilde{M} \cap (U_{\lambda_o} \times U_{y_o})$ (see (H4′)).

Choose a small number $r > 0$, define

$$U(r) := \big\{(\lambda, y) \in \mathcal{P} \times \mathbb{F}\colon \big(\lambda, \tilde{\chi}(\lambda)\big) \in \mathcal{D}_{\lambda_o},\ \big\|y - \tilde{\chi}(\lambda)\big\| < r\big\}, \tag{164}$$

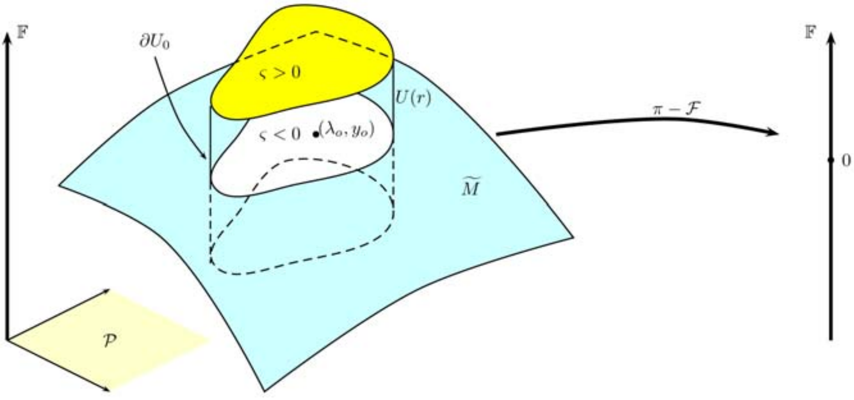

Fig. 5. Functional setting and auxiliary function for the bifurcation problem related to Eq. (163).

and introduce a G-invariant *auxiliary* function $\varsigma : \overline{U(r)} \to \mathbb{R}$ satisfying the properties

$$\begin{cases} \varsigma(\lambda, y) > 0 & \text{if } \|y - \tilde{\chi}(\lambda)\| = r, \\ \varsigma(\lambda, y) < 0 & \text{if } (\lambda, y) \in \mathcal{D}_{\lambda_o}. \end{cases} \tag{165}$$

Put

$$\partial U_0 := \left\{(\lambda, y) \in \overline{U(r)} \colon \left(\lambda, \tilde{\chi}(\lambda)\right) \in \partial \mathcal{D}_{\lambda_o}\right\} \subset \partial U(r).$$

Then, by the implicit function theorem, the above $r > 0$ can be chosen to be so small that

$$y - \mathcal{F}(\lambda, y) \neq 0, \quad \text{for } (\lambda, y) \in \partial U_0 \setminus \widetilde{M}$$

(see Fig. 5).

Define the map $\mathfrak{F}_\varsigma : \overline{U(r)} \to \mathbb{R} \oplus \mathbb{F}$ by

$$\mathfrak{F}_\varsigma(\lambda, y) := \left(\varsigma(\lambda, y), (\pi - \mathcal{F})(\lambda, y)\right), \tag{166}$$

which is clearly a $U(r)$-admissible G-equivariant completely continuous vector field. Therefore, the following *local bifurcation invariant*

$$\omega(\lambda_o, u_o) := G\text{-Deg}^t\left(\mathfrak{F}_\varsigma, U(r)\right) \in A_1^t(G) \tag{167}$$

is well-defined.

Using the standard scheme discussed in Sections 4–6, one can prove the following G-symmetric (local) bifurcation result for the equivariant coincidence problem (160).

THEOREM 7.5 (Local bifurcation theorem). *Suppose that the assumptions* (H1)–(H5) *are satisfied,* $\omega(\lambda_o, u_o)$ *is given by* (167) (*with* $\mathfrak{F}_\varsigma$ *defined by* (166), $U(r)$ *by* (164) *and* ς *satisfying* (165)). *If*

$$\omega(\lambda_o, u_o) = \sum_{(H)} n_H\,(H) \neq 0,$$

i.e., there is $n_{H_o} \neq 0$ *for some orbit type* (H_o), *then there exists a branch of non-trivial solutions* (λ, u) *to Eq.* (160) *bifurcating from* (λ_o, u_o) *such that* $G_u \supset H_o$.

7.2. *Hopf bifurcation for FPDEs with symmetries: Reduction to local bifurcation invariant*

Throughout this section, assume $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_1 and Γ_2 are compact Lie groups. Let $V' := \mathbb{R}^m$ (resp. $V := \mathbb{R}^n$) be an orthogonal Γ_1-representation (resp. Γ_2-representation). Assume that $\Omega \subset V'$ is an open Γ_1-invariant bounded set such that $\partial\Omega$ is C^2-smooth. Clearly, the space $L^2(\mathbb{R} \times \overline{\Omega}; V)$ is an isometric Banach Γ-representation with the Γ-action given by

$$(\gamma u)(t, x) = \gamma_2\big(u(t, \gamma_1 x)\big), \quad \gamma = (\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2.$$

Statement of the problem Consider a system of functional parabolic differential equations on $\mathbb{R} \times \overline{\Omega}$

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) + P(\alpha, x)u = f(\alpha, u_t)(x), & (t, x) \in \mathbb{R} \times \Omega, \\ B(\alpha, x)u(t, x) = 0, & (t, x) \in \mathbb{R} \times \partial\Omega, \end{cases} \tag{168}$$

where $u \in L^2(\mathbb{R} \times \overline{\Omega}; V)$ satisfies appropriate differentiability requirements,[3] $u_t(\theta, x) := u(t + \theta, x)$ for $\theta \in [-\tau, 0]$ ($\tau > 0$ is a fixed constant), $\alpha \in \mathbb{R}$ is a (bifurcation) parameter, $f : \mathbb{R} \times C([-\tau, 0]; L^2(\Omega; V)) \to L^2(\Omega; V)$ is a map of class C^1, which is bounded on bounded sets, $P(\alpha, x) = [P_i(\alpha, x)]_{i=1}^n$ is a vector with components being second-order uniformly elliptic operators, i.e.

$$P_i(\alpha, x) = \nabla^T A_i(\alpha, x)\nabla + a_i(\alpha, x),$$

with $A_i(\alpha, x)$ being a continuously differentiable (with respect to α and x) $n \times n$ symmetric positive definite matrix satisfying the condition

$$\exists c_1, c_2 > 0\ \forall (\alpha, x) \in \mathbb{R} \times \overline{\Omega}\ \forall y \in V' \quad c_1\|y\| \leqslant y^T A_i(\alpha, x) y \leqslant c_2\|y\|,$$

[3] u is weakly differentiable with respect to t and has weak derivatives of order 2 with respect to $x \in \Omega$. More precisely, assume here that u is an element of the Sobolev space $H^{1,2}(\mathbb{R} \times \Omega; V)$ of L^2-integrable V-valued functions from $\mathbb{R} \times \Omega$ with weak L^2-integrable derivative in $\mathbb{R}$ and weak L^2-integrable derivatives of order 2 in Ω.

where ∇ stands for the gradient operator, and $a_i(\alpha, x)$ is continuous. The boundary operator $B(\alpha, x)$ is defined by either (Dirichlet conditions)

$$B(\alpha, x)u(t, x) = u(t, x)$$

or (mixed Dirichlet/Neumann conditions)

$$B(\alpha, x)u(t, x) = b(\alpha, x)u(t, x) + \frac{\partial}{\partial n}(\alpha, x)\, u(t, x),$$

where $b \in C^1(\mathbb{R} \times \partial\Omega; \mathbb{R})$, $\frac{\partial}{\partial n}(\alpha, x) = [\nu^T(x)A_i(\alpha, x)\nabla]_{i=1}^n$ ($\nu(x)$ is the outward normal vector to $\partial\Omega$ at x).

Assume that

(C1) the operators P, B and the map f are Γ-equivariant, i.e. for $\gamma \in \Gamma$,

$$\begin{aligned}
&\gamma P(\alpha, x)u = P(\alpha, x)\gamma u, && x \in \Omega,\ \theta \in [-\tau, 0],\\
&\gamma f\big(\alpha, v(\theta, x)\big) = f\big(\alpha, \gamma v(\theta, x)\big), && x \in \Omega,\ \theta \in [-\tau, 0],\\
&\gamma B(\alpha, x)u = B(\alpha, x)\gamma u, && x \in \partial\Omega.
\end{aligned}$$

We use the standard identification $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ and introduce the following notation

$$\mathcal{H}^{1,2}_{B(\alpha)} = \big\{\varphi \in H^{1,2}(S^1 \times \Omega; V)\colon\ B(\alpha, x)\varphi = 0\big\}, \tag{169}$$

where $H^{k,\ell}(S^1 \times \Omega; V)$ stands for the Sobolev space of V-valued functions with weak (L^2-integrable) derivatives of order k in S^1 and of order l in Ω. Put

$$\mathbb{E} = \mathbb{F} = L^2(S^1 \times \Omega; V), \quad \mathcal{P} = \mathbb{R} \times \mathbb{R}_+, \quad \widehat{\mathbb{E}} = C\big(S^1; L^2(\Omega; V)\big), \tag{170}$$

where $\widehat{\mathbb{E}}$ is equipped with the usual supremum norm.

We are looking for p-periodic solutions to the system (168) for an unknown period $p > 0$.

Normalization of the period As usual, we replace the original problem with the equivalent one, for which we are looking for 2π-periodic solutions. Namely, let $\beta := \frac{2\pi}{p}$ and

$$v(t, x) = u\left(\frac{1}{\beta}t, x\right).$$

Then, the original problem is reduced to finding non-trivial solutions (α, β, v) for the system

$$\begin{cases}
\frac{\partial}{\partial t}v(t, x) + \frac{1}{\beta}P(\alpha, x)v = \frac{1}{\beta}f(\alpha, v_{t,\beta})(x), & (t, x) \in \mathbb{R} \times \Omega,\\
B(\alpha, x)v(t, x) = 0, & (t, x) \in \mathbb{R} \times \partial\Omega,\\
v(t, x) = v(t + 2\pi, x), & (t, x) \in \mathbb{R} \times \Omega,
\end{cases} \tag{171}$$

where

$$v_{t,\beta}(\theta, x) := v(t + \beta\theta, x) \quad \text{for } (\theta, x) \in [-\tau, 0] \times \Omega.$$

$\Gamma \times S^1$-setting in functional spaces Below we reformulate the system (171) as an equivariant parametrized coincidence problem based on the general discussion in Subsection 7.1.

For every $\lambda := (\alpha, \beta) \in \mathcal{P}$, define the subspace

$$\mathrm{Dom}(L_\lambda) := \big\{u \in \mathbb{E}\colon\ u \in \mathcal{H}^{1,2}_{B(\alpha)}\big\},$$

and the operator

$$L_\lambda : \mathrm{Dom}(L_\lambda) \subset \mathbb{E} \to \mathbb{E}$$

(cf. (170)) by

$$L_\lambda v(t, x) := \frac{\partial}{\partial t} v(t, x) + \frac{1}{\beta} P(\alpha, x) v$$

(cf. (169)–(171)).

Notice that $\mathbb{E}$, $H^{1,2}(S^1 \times \Omega; V)$ and $\widehat{\mathbb{E}}$ are isometric Banach G-representations, where S^1 acts in a standard way by shifting the time argument t. It is also clear (see [106]) that each (unbounded) linear operator L_λ, for $\lambda \in \mathcal{P}$, is a closed G-equivariant Fredholm operator of index zero, and the orthogonal projection on the (finite-dimensional) kernel of L_λ is a G-equivariant resolvent K of L_λ. Therefore, $R^G(L, \{\lambda\}) \neq \emptyset$ for any $\lambda \in \mathcal{P}$, and, by Lemma 7.3, the condition (H1) is satisfied for every compact subset $X \subset \mathcal{P}$.

On the other hand, since

$$\frac{1}{\beta} f(\alpha, v_{t,\beta}) \in L^2(\Omega; V) \quad \text{for } v_{t,\beta} \in C\big([-\tau, 0]; L^2(\Omega; V)\big),$$

we have the continuous map $N_f : \mathcal{P} \times \widehat{\mathbb{E}} \to L^2(\Omega, V)$ defined by

$$N_f(\alpha, \beta, v)(t) := \frac{1}{\beta} f(\alpha, v_{t,\beta}).$$

Define $\widehat{F} : \mathcal{P} \times \widehat{\mathbb{E}} \to \mathbb{F}$ by

$$\widehat{F}(\lambda, v)(t, x) := i \circ N_f(\alpha, \beta, v)(t)(x) = \frac{1}{\beta} f(\alpha, v_{t,\beta})(x), \quad \lambda = (\alpha, \beta),$$

where i denotes the natural embedding $\widehat{\mathbb{E}} \hookrightarrow \mathbb{F}$. The continuous differentiability of f implies that $\widehat{F}$ is continuously differentiable. Since the following composition of the embeddings

$$H^{1,2}(S^1 \times \Omega; V) \hookrightarrow H^{2/3,0}(S^1 \times \Omega; V) \hookrightarrow C\big(S^1; L^2(\Omega; V)\big) = \widehat{\mathbb{E}}$$

is compact (cf. [106]), we have the following embedding

$$J:\mathcal{E}\longrightarrow\mathcal{P}\times\widehat{\mathbb{E}}$$

where $J_\lambda:\mathbb{E}_\lambda\to\widehat{\mathbb{E}}$ is a compact operator for all $\lambda\in\mathcal{P}$. Therefore, $\widehat{F}$ and J satisfy the condition (H2) from Subsection 7.1. In particular, $F:\mathcal{E}\to\mathbb{F}$ defined by $F=\widehat{F}\circ J$ is a G-equivariant completely continuous map of class C^1.

As a consequence, finding a periodic solution $v\in H^{1,2}(S^1\times\Omega;V)$ for the system (171) is equivalent to solving the following parametrized coincidence problem (cf. (160))

$$L_\lambda v=F(\lambda,v),\quad \lambda\in X, \tag{172}$$

where X is a given compact subset of $\mathcal{P}$.

Γ-symmetric steady-state solutions to (168) In order to describe a manifold of trivial solutions to (171) (cf. conditions (H3) and (H4) from Subsection 7.1), at which we expect the occurrence of a Hopf bifurcation, observe that the functions $u(t,x)$ from $H^{1,2}(S^1\times\Omega;V)$, constant with respect to the first variable, can be identified with functions from $H^2(\Omega;V)$, where $H^2(\Omega;V)$ denotes the Sobolev space of V-valued functions with weak (L^2-integrable) derivatives of order 2 in Ω. Clearly, for a function $u(t,x)$, constant with respect to t, we have $u_t(\theta,x)=u(\theta,x)$ for all $t\in\mathbb{R}$.

With these preliminaries on hands, introduce the following

DEFINITION 7.6. Let (α_o,u_o) be a solution to (168) satisfying the following conditions:

(i) $u_o\in H^2(\Omega,V)$;

(ii) $\gamma u_o=u_0$ for all $\gamma\in\Gamma$;

(iii) $\begin{cases} P(\alpha_o,x)u_o=f(\alpha_o,u_o)(x) & \text{in } \Omega,\\ B(\alpha_o,x)u_o=0 & \text{on } \partial\Omega.\end{cases}$

Then, (α_o,u_o) is called a Γ-*symmetric steady-state* solution to (168).

Introduce the following spaces

$$\mathfrak{X}_{\alpha_o}:=\big\{\omega\in H^2(\Omega;V):\ B(\alpha_o,x)\omega=0\big\},$$

$$\mathfrak{X}^c_{\alpha_o}:=\big\{\omega\in H^2(\Omega;V^c):\ B(\alpha_o,x)\omega=0\big\},$$

$$\mathfrak{C}:=C\big([-\tau,0];L^2(\Omega;V)\big),$$

$$\mathfrak{C}^c:=C\big([-\tau,0];L^2(\Omega;V^c)\big),$$

where V^c stands for the complexification of the Γ-representation V. Identify $L^2(\Omega;V)$ with the subspace of $\mathfrak{C}$ consisting of constant $L^2(\Omega;V)$-valued functions, and $L^2(\Omega;V^c)$ with the subspace of $\mathfrak{C}^c$ consisting of constant $L^2(\Omega;V^c)$-valued functions. Denote by $\bar{f}(\alpha,\cdot)$ the restriction of $f(\alpha,\cdot)$ to $L^2(\Omega;V)$ and define

$$\mathcal{L}_{\alpha_o}:=P(\alpha_o,x)-\mathfrak{d}f_{\alpha_o}:\mathfrak{X}_{\alpha_o}\subset L^2(\Omega;V)\to L^2(\Omega;V), \tag{173}$$

where $\mathfrak{d}f_{\alpha_o}(\varphi):=D_u\bar{f}(\alpha_o,u_o)\varphi$ for $\varphi\in\mathfrak{X}_{\alpha_o}$.

In what follows, we apply the same symbols to denote the complexified operators $P(\alpha, x)$, $\mathfrak{d} f_\alpha$, and $B(\alpha_o, x)$, i.e. the operators:

$$
\begin{aligned}
&P(\alpha, x) : \mathfrak{X}_\alpha^c \subseteq L^2(\Omega; V^c) \to L^2(\Omega; V^c),\\
&\mathfrak{d} f_\alpha : \mathfrak{C}^c \to L^2(\Omega; V^c),\\
&B(\alpha_o, x) : H^2(\Omega; V^c) \to L^2(\partial\Omega, V^c).
\end{aligned}
$$

Suppose that (α_o, u_o) is a Γ-symmetric steady-state solution to (168) with $\alpha = \alpha_o$. We say that (α_o, u_o) is *non-singular* if $0 \notin \sigma(\mathcal{L}_{\alpha_o})$, where $\sigma(\mathcal{L}(\alpha_o))$ denotes the spectrum of $\mathcal{L}_{\alpha_o}$. Assume that

(C2) (α_o, u_o) is a nonsingular Γ-symmetric steady-state solution.

Then, by the implicit function theorem, there exists a continuously differentiable function $u(\alpha)$ for $\alpha \in (\alpha_o - \eta, \alpha_o + \eta)$ (for a sufficiently small $\eta > 0$) such that $(\alpha, u(\alpha))$ is a Γ-symmetric steady-state solution to (168) for each α (the condition $\gamma u(\alpha) = u(\alpha)$ for all $\gamma \in \Gamma$ (cf. Definition 7.6(ii)) is provided by the Γ-equivariance of $\mathcal{L}_\alpha$). In what follows, assume that $\{(\alpha, u(\alpha)) \colon \alpha \in (\alpha_o - \eta, \alpha_o + \eta)\}$ is a fixed family of steady-state Γ-symmetric solutions near (α_o, u_o). Since $(\alpha, \beta, u(\alpha))$, $\alpha \in (\alpha_o - \eta, \alpha_o + \eta)$, $\beta \in \mathbb{R}_+$, is clearly a solution to (171) belonging to $\mathcal{P} \times \mathbb{E}^G$, consider it as a *trivial solution*. Moreover, define the map $\chi \colon (\alpha_o - \eta, \alpha_o + \eta) \times \mathbb{R}_+ \to \mathbb{E}^G$ by $\chi(\alpha, \beta) = (\alpha, \beta, u(\alpha))$. Then, the set of (nonsingular) Γ-symmetric steady-state solutions to (168) (cf. condition (C2)), gives rise to a manifold $M \subset \mathcal{P} \times \mathbb{E}^G$, $M := \{(\alpha, \beta, u(\alpha)) \colon \alpha \in (\alpha_o - \eta, \alpha_o + \eta),\ \beta \in \mathbb{R}_+\}$ (defined locally) satisfying the conditions (H3) and (H4).

Characteristic equation Let $(\alpha, u(\alpha))$ be a non-singular Γ-symmetric steady-state solution to (168). The linearization of (168) at $(\alpha, u(\alpha))$ leads to the following *characteristic equation* (near $(\alpha_o, u(\alpha_o))$)

$$
\Delta_{\alpha; u(\alpha)}(\lambda) w := \lambda w + P(\alpha, x) w - \mathfrak{d} f_\alpha(\mathrm{e}^{\lambda \cdot} w) = 0, \quad \lambda \in \mathbb{C}, \tag{174}
$$

where the *characteristic operator* $\Delta_{\alpha; u(\alpha)} : \mathfrak{X}_\alpha^c \to L^2(\Omega; V^c)$ is defined (by (174)) using the standard complexifications of $P(\alpha, x)$ and $\mathfrak{d} f_\alpha(\varphi)$.

Observe that $\mathfrak{X}_\alpha^c$ equipped with the H^2-norm is a complex Hilbert space such that the embedding $\mathfrak{X}_\alpha^c \hookrightarrow L^2(\Omega; V^c)$ is compact, $P(\alpha, x)$ is an elliptic self-adjoint operator (thus, it is a bounded Fredholm operator of index zero from $\mathfrak{X}_\alpha^c$ to $L^2(\Omega; V^c)$), and $\mathfrak{d} f_\alpha(\mathrm{e}^{\lambda \cdot} \cdot)$ is a bounded linear operator for all $\lambda \in \mathbb{C}$, therefore $\Delta_{\alpha; u(\alpha)}(\lambda) : \mathfrak{X}_\alpha^c \to L^2(\Omega; V^c)$, where $\mathfrak{X}_\alpha^c$ is equipped with the H^2-norm, is a bounded Fredholm operator of index zero.[4] Consequently, $\Delta_{\alpha; u(\alpha)}(\lambda)$ is a closed (unbounded) Fredholm operator of index zero from $L^2(\Omega; V^c)$ to itself.

A number $\lambda \in \mathbb{C}$ is called a *characteristic root* of the system (168) at a Γ-symmetric steady-state solution $(\alpha, u(\alpha))$ if $\ker \Delta_{\alpha; u(\alpha)}(\lambda) \neq \{0\}$. It is clear that a Γ-symmetric steady-state solution $(\alpha, u(\alpha))$ is non-singular if and only if 0 is not a characteristic root of

[4] $\Delta_{\alpha; u(\alpha)}(\lambda)$ is a sum of the compact operator $\lambda\,\mathrm{Id} - \mathfrak{d} f_\alpha(\mathrm{e}^{\lambda \cdot} \cdot) : \mathfrak{X}_\alpha^c \to L^2(\Omega; V^c)$ and the Fredholm operator $P(\alpha, x)$ of index zero.

(168) at $(\alpha, u(\alpha))$. We say that a non-singular Γ-symmetric steady-state solution (α_o, u_o), $u_o = u(\alpha_o)$, is a *center* if it has a purely imaginary characteristic root $\mathrm{i}\beta_o$, $\beta_o > 0$, i.e. $\ker \Delta_{\alpha_o;u_o}(\mathrm{i}\beta_o) \neq \{0\}$. A center (α_o, u_o) is called *isolated* if it is the only center in some neighborhood of (α_o, u_o) in $\mathbb{R} \oplus L^2(\Omega; V)$.

For the purpose of studying the local Hopf bifurcation problem for (168), assume:

(C3) There exists a Γ-symmetric steady-state solution $(\alpha_o, u_o) \in \mathbb{R} \oplus L^2(\Omega; V)$ to (168), which is an isolated center such that $\mathrm{i}\beta_o$ (for $\beta_o > 0$) is a characteristic root of (168) for $\alpha = \alpha_o$.

Thus, condition (H5) from Subsection 7.1 is satisfied. Also, notice that (C3) is the necessary condition for the occurrence of the Hopf bifurcation at (α_o, u_o), while (C2) excludes the appearance of the "steady-state" bifurcation.

Apply the "eigenspace" reduction to describe the characteristic roots of the system (168) at the Γ-symmetric steady-state solution $(\alpha, u(\alpha))$. Denote by $\sigma_\alpha \subset \mathbb{R}$ the spectrum of the self-adjoint operator $P(\alpha, x) : \mathfrak{X}^c_\alpha \subseteq L^2(\Omega; V^c) \to L^2(\Omega; V^c)$. Since $P(\alpha, x)$ is a uniformly elliptic differential operator, the spectrum σ_α is discrete and all the eigenvalues $\mu^\alpha_\mathfrak{k}$ are real of finite multiplicity and such that

$$\mu^\alpha_o < \mu^\alpha_1 < \ldots < \mu^\alpha_\mathfrak{k} < \cdots .$$

Using the fact that for every $r > 0$, the number ir is not in the spectrum σ_α of $P(\alpha, x)$, the (auxiliary) operator $S : L^2(\Omega; V^c) \to L^2(\Omega; V^c)$ defined by

$$Sw = \mathrm{i}rw, \quad w \in L^2(\Omega; V^c),$$

is a Γ-equivariant resolvent (cf. Remark and Notation 2.17 and Definition 7.1) of $P(\alpha, x)$, i.e. the (bounded) inverse $\widetilde{R}_{\alpha,r} := [P(\alpha, x) + S]^{-1} : L^2(\Omega; V^c) \to L^2(\Omega; V^c)$ exists for all $\alpha \in \mathbb{R}$ and is Γ-equivariant. On the other hand, since $P(\alpha, x) + S$, as a (bounded) operator from the space $\mathfrak{X}^c_\alpha$ (equipped with the Sobolev H^2-norm) is also invertible and the embedding $\mathfrak{X}^c_\alpha \hookrightarrow L^2(\Omega; V^c)$ is compact, we obtain that the inverse $\widetilde{R}_{\alpha,r} : L^2(\Omega; V^c) \to L^2(\Omega; V^c)$ (i.e. $\widetilde{R}_{\alpha,r}$ is considered here as the inverse of the *unbounded* operator $P(\alpha, x) + S$ from $L^2(\Omega; V^c)$ into itself) is a compact Γ-equivariant operator. Eq. (174) can be rewritten as

$$\tilde{\Delta}^r_{\alpha;u(\alpha)}(\lambda)w := w - \mathfrak{d} f_\alpha\big(\mathrm{e}^{\lambda\cdot} \widetilde{R}_{\alpha,r}(w)\big) + (\lambda - \mathrm{i}r)\widetilde{R}_{\alpha,r}(w) = 0. \tag{175}$$

It is clear that $\lambda \in \mathbb{C}$ is a characteristic root of the system (168) at the Γ-symmetric steady-state solution $(\alpha, u(\alpha))$ if and only if $\ker \tilde{\Delta}^r_{\alpha;u(\alpha)}(\lambda) \neq \{0\}$. Also, $\tilde{\Delta}^r_{\alpha;u(\alpha)}(\lambda)$ is an analytic function in λ (cf. [145]), hence all the characteristic roots λ are isolated. Moreover, $\tilde{\Delta}^r_{\alpha;u(\alpha)}(\lambda)$ is a Γ-equivariant compact field, thus it is a bounded Γ-equivariant Fredholm operator of index zero.

Denote by $E^\alpha_\mathfrak{k} \subset L^2(\Omega; V^c)$ the eigenspace of $P(\alpha, x)$ corresponding to $\mu^\alpha_\mathfrak{k} \in \sigma_\alpha$, and let $\mathfrak{p}^\alpha_\mathfrak{k} : L^2(\Omega; V^c) \to E^\alpha_\mathfrak{k}$ be the orthogonal projection. Consequently, for every $w \in$

$L^2(\Omega; V^c)$, we have $w = \sum_{\mathfrak{k}=0}^{\infty} \mathfrak{p}_{\mathfrak{k}}^{\alpha}(w)$. By substituting $w = \sum_{\mathfrak{k}=0}^{\infty} \mathfrak{p}_{\mathfrak{k}}^{\alpha}(w)$ into (175), we obtain

$$\sum_{\mathfrak{k}=0}^{\infty}\left[\mathfrak{p}_{\mathfrak{k}}^{\alpha}(w) - \frac{1}{\mu_{\mathfrak{k}}^{\alpha} + \mathrm{i}r}\mathfrak{d} f_{\alpha}\left(\mathrm{e}^{\lambda\cdot}\mathfrak{p}_{\mathfrak{k}}^{\alpha}(w)\right) + \frac{\lambda - \mathrm{i}r}{\mu_{\mathfrak{k}}^{\alpha} + \mathrm{i}r}\mathfrak{p}_{\mathfrak{k}}^{\alpha}(w)\right] = 0. \tag{176}$$

Denote by $F_{\mathfrak{k}}^{\alpha}$ the subspace of $\mathfrak{C}$ spanned by functions of the type $t \to \varphi(t)w$, where $\varphi \in C([-\tau, 0]; \mathbb{C})$ and $w \in E_{\mathfrak{k}}^{\alpha}$. We need the following hypothesis (cf. [90,119])

(C4) $\mathfrak{d} f_{\alpha}(F_{\mathfrak{k}}^{\alpha}) \subset E_{\mathfrak{k}}^{\alpha}$ for all Γ-symmetric steady-state solutions $(\alpha, u(\alpha))$ and $\mathfrak{k} = 0, 1, 2, \ldots$.

REMARK 7.7. The condition (C4) is required mainly to simplify the computation of the characteristic roots through a reduction to isotypical components of $L^2(\Omega, V^c)$ (see also [117,118]). One can check that the reaction–diffusion systems with delay of the type considered in [34–36] satisfy (C4). In the case of a parabolic system of Γ-symmetric PDEs (without delay), or the reaction–diffusion logistic equation with delay, (C4) is automatically satisfied.

Under the assumption (C4), Eq. (176) can be reduced to the following sequence of equations

$$\mathfrak{p}_{\mathfrak{k}}^{\alpha}(w) - \frac{1}{\mu_{\mathfrak{k}}^{\alpha} + \mathrm{i}r}\mathfrak{d} f_{\alpha}\left(\mathrm{e}^{\lambda\cdot}\mathfrak{p}_{\mathfrak{k}}^{\alpha}(w)\right) + \frac{\lambda - \mathrm{i}r}{\mu_{\mathfrak{k}}^{\alpha} + \mathrm{i}r}\mathfrak{p}_{\mathfrak{k}}^{\alpha}(w) = 0, \quad \mathfrak{k} = 0, 1, \ldots. \tag{177}$$

Equation (177) can be written in the equivalent form as

$$(\mu_{\mathfrak{k}}^{\alpha} + \lambda)\mathfrak{p}_{\mathfrak{k}}^{\alpha}(w) + \mathfrak{d} f_{\alpha}\left(\mathrm{e}^{\lambda\cdot}\rho_{\mathfrak{k}}^{\alpha}(w)\right) = 0, \quad \mathfrak{k} = 0, 1, \ldots. \tag{178}$$

Under the assumptions (C1)–(C4), for any compact subset $X \subset \mathcal{P}$, the system (168) leads to a parametrized equivariant coincidence problem of the type (160) satisfying the conditions (H1)–(H5) (cf. (172)). Hence, given an isolated center (α_o, u_o) with the corresponding characteristic root $i\beta_o$ (cf. condition (C3)), and following the scheme outlined in Subsection 7.1 (cf. (164)–(167)), one can associate to (α_o, β_o, u_o) the local bifurcation invariant $\omega(\lambda_o, u_o) = \omega(\alpha_o, \beta_o, u_o) \in A_1^t(\Gamma \times S^1)$.

7.3. *Computation of local bifurcation invariant*

In order to obtain an effective computational formula for $\omega(\alpha_o, \beta_o, u_o)$, we need to discuss the so-called *negative spectrum* and *crossing numbers*.

Negative spectrum For the sake of simplicity, assume that Γ is a finite group and, moreover, Γ_1 acts trivially on V'. Assume that V and V^c admit the isotypical decompositions (113) with $V_0 = V^{\Gamma}$ and $U_0 = (V^c)^{\Gamma}$. Moreover, these decompositions are required to correspond to a complete list of the irreducible Γ-representations $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_r$ (resp.

$\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_s$). As in the previous sections, we do not exclude the possibility that some of these Γ-isotypical components are trivial. Clearly, (113) induces the Γ-isotypical decompositions

$$L^2(\Omega; V) = \bigoplus_{i=0}^{r} \mathfrak{V}_i, \qquad L^2(\Omega; V^c) = \bigoplus_{j=0}^{s} \mathfrak{U}_j, \tag{179}$$

where $\mathfrak{V}_i$ (resp. $\mathfrak{U}_j$) denotes the Γ-isotypical component of $L^2(\Omega; V)$ (resp. $L^2(\Omega; V^c)$) modeled on $\mathcal{V}_i$ (resp. $\mathcal{U}_j$).

Consider the operator $P(\alpha_o, x) : \mathfrak{X}_{\alpha_o} \subseteq L^2(\Omega; V) \to L^2(\Omega; V)$ and let K be the orthogonal projection on its kernel. Then, K is a Γ-equivariant resolvent of $P(\alpha_o, x)$. Put $\widetilde{R}_{\alpha_o} := [P(\alpha_o, x) + K]^{-1}$ and define

$$\mathcal{A} := \mathrm{Id} - \frac{1}{\beta_o} \widetilde{R}_{\alpha_o} \circ D_u \bar{f}(\alpha_o, u_o) - \widetilde{R}_{\alpha_o} K : L^2(\Omega; V) \to L^2(\Omega; V), \tag{180}$$

which is clearly a compact field. Put $\mathcal{A}^i := \mathcal{A}_{|\mathfrak{V}_i} : \mathfrak{V}_i \to \mathfrak{V}_i$ and let $\sigma_-(\mathcal{A})$ denote the set of all negative eigenvalues of the operator $\mathcal{A}$. Since $\mathcal{A}$ is a compact field, it is a Fredholm operator of index 0, the set $\sigma_-(\mathcal{A})$ is finite and all the eigenvalues in $\sigma_-(\mathcal{A})$ are of finite multiplicity. Thus, for $\mu \in \sigma_-(\mathcal{A})$, define

$$\begin{aligned} E(\mu) &:= \bigcup_{k=1}^{\infty} \ker[\mathcal{A} - \mu\, \mathrm{Id}]^k, \\ E_i(\mu) &:= \bigcup_{k=1}^{\infty} \ker[\mathcal{A}^i - \mu\, \mathrm{Id}_{|\mathfrak{V}_i}]^k, \\ m_i(\mu) &:= \dim E_i(\mu) / \dim \mathcal{V}_i, \end{aligned} \tag{181}$$

where $E_i(\mu)$ is referred to as a $\mathcal{V}_i$*-isotypical* eigenspace of the operator $\mathcal{A}$ and the integer $m_i(\mu)$ is called the $\mathcal{V}_i$*-multiplicity* of μ.

Crossing numbers Put $\tilde{\Delta}^r_{\alpha;u(\alpha),j}(\lambda) := \tilde{\Delta}^r_{\alpha;u(\alpha)}(\lambda)|_{\mathfrak{U}_j}$ (cf. (175) and (179)). For a characteristic root λ of the system (168) at the Γ-symmetric steady-state solution (α_o, u_o), we use the following notations:

$$\begin{aligned} E_j(\lambda) &:= \bigcup_{k=1}^{\infty} \ker\big[\tilde{\Delta}^r_{\alpha;u(\alpha),j}(\lambda)\big]^k, \\ m_j(\lambda) &:= \dim E_j(\lambda) / \dim \mathcal{U}_j, \end{aligned} \tag{182}$$

where the integer $m_j(\lambda)$ is called the $\mathcal{U}_j$*-multiplicity* of the characteristic root λ. Notice that, since $\tilde{\Delta}^r_{\alpha;u(\alpha),j}(\lambda)$ is a Fredholm operator of index 0, $m_j(\lambda) < \infty$ for all characteristic roots λ.

Let $(\alpha_o, u_o) \in \mathbb{R} \oplus L^2(\Omega; V)$ be an isolated center with $\mathrm{i}\beta_o$ ($\beta_o > 0$) being the corresponding characteristic root as described in condition (C3). Define the set

$$\mathcal{S} = \left\{\tau + \mathrm{i}\beta\colon\ 0 < \tau < \delta, |\beta - \beta_o| < \varepsilon\right\} \subset \mathbb{C},$$

where $\delta > 0$ and $\varepsilon > 0$ are so small numbers that for all $\tau + \mathrm{i}\beta \in \partial\mathcal{S}$ and $\alpha \in [\alpha_o - \varepsilon, \alpha_o + \varepsilon]$, $\ker \Delta_{\alpha;u(\alpha)}(\tau + \mathrm{i}\beta) \neq \{0\}$ implies $\alpha = \alpha_o$ and $\tau + \mathrm{i}\beta = \mathrm{i}\beta_o$. Put $\alpha_\pm := \alpha_o \pm \varepsilon$ and denote by $\mathfrak{s}_\pm$ the set of all characteristic roots $\lambda \in \mathcal{S}$ for $\alpha = \alpha_\pm$, i.e.

$$\mathfrak{s}_\pm := \left\{\lambda \in \mathcal{S}\colon\ \ker \Delta_{\alpha_\pm;u(\alpha_\pm)}(\lambda) \neq \{0\}\right\}.$$

Since $\ker \Delta_{\alpha_\pm;u(\alpha_\pm)}(\lambda) = \ker \tilde{\Delta}^r_{\alpha_\pm;u(\alpha_\pm)}(\lambda)$ and $\tilde{\Delta}^r_{\alpha_\pm;u(\alpha_\pm)}(\lambda)$ is an analytic function in λ, the sets $\mathfrak{s}_\pm$ are finite. Then, for $j = 0, 1, 2, \ldots, s$ (corresponding to the complex Γ-irreducible representations $\mathcal{U}_j$), put

$$\mathfrak{t}^\pm_j(\alpha_o, \beta_o, u_o) := \sum_{\lambda \in \mathfrak{s}_\pm} m_j(\lambda) \tag{183}$$

(cf. (182)).

DEFINITION 7.8. The $\mathcal{V}_{j,l}$*-isotypical crossing numbers* of (α_o, β_o, u_o) are defined as follows:

$$\mathfrak{t}_{j,1}(\alpha_o, \beta_o, u_o) := \mathfrak{t}^-_j(\alpha_o, \beta_o, u_o) - \mathfrak{t}^+_j(\alpha_o, \beta_o, u_o), \tag{184}$$

where $\mathfrak{t}^\pm_j(\alpha_o, \beta_o, u_o)$ are given by (183), and in the case $\mathrm{i}l\beta_o$ is also a characteristic root of (168) at (α_o, u_o) for some integer $l > 1$ (cf. Definition 4.8),

$$\mathfrak{t}_{j,l}(\alpha_o, \beta_o, u_o) := \mathfrak{t}_{j,1}(\alpha_o, l\beta_o, u_o).$$

Local bifurcation result By applying the standard steps, one can establish the following formula for the local bifurcation invariant:

$$\begin{aligned}&\omega(\alpha_o, \beta_o, u_o)\\ &\quad := \left(\prod_{\mu \in \sigma_-(\mathcal{A})} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)}\right) \cdot \sum_{j,l} \mathfrak{t}_{j,l}(\alpha_o, \beta_o, u_o) \deg_{\mathcal{V}_{j,l}}.\end{aligned} \tag{185}$$

Combining the concept of the dominating orbit types with Theorem 7.5 (see also Remark 5.2) and using the same argument as in Sections 4 and 5 (cf. Theorems 4.9 and 5.4), one can prove the following

THEOREM 7.9. *Suppose that the system* (168) *satisfies the assumption* (C1) *and* (C4), *and suppose that* (α_o, u_o) *is a* Γ*-symmetric steady-state solution to* (168) (*cf. Definition* 7.6)

satisfying (C2)–(C3), $\omega(\alpha_o, \beta_o, u_o)$ *is given by* (167) (*with* $\lambda_o = (\alpha_o, \beta_o)$, $\mathfrak{F}_\varsigma$ *defined by* (166), $U(r)$ *by* (164) *and* ς *satisfying* (165)). *Assume* (*cf.* (185)) $\omega(\alpha_o, \beta_o, u_o) \neq 0$, *i.e.*

$$\omega(\alpha_o, \beta_o, u_o) = \sum_{(H)} n_H(H) \quad \text{and} \quad n_{H_o} \neq 0 \tag{186}$$

for some $(H_o) \in \Phi_1^t(G)$.

(i) *Then, there exists a branch of non-trivial solutions to* (168) *with symmetries at least* H_o (*considered in the space* $\mathbb{F}$ (*cf.* (170))) *bifurcating from the point* (α_o, u_o) (*with the limit frequency* $l\beta_o$ *for some* $l \in \mathbb{N}$).

(ii) *If, in addition,* (H_o) *is a dominating orbit type in* $\mathbb{F}$, *then there exist at least* $|G/H_o|_{S^1}$ *different branches of periodic solutions to Eq.* (168) *bifurcating from* (α_o, u_o). *Moreover, for each* (α, β, u) *belonging to these branches of* (*nontrivial*) *solutions, one has* $(G_u) = (H_o)$ (*considered in the space* $\mathbb{F}$).

7.4. *Bibliographical remarks*

The general functional framework for abstract coincidence problem for unbounded parametrized Fredholm operators was studied in [90]. The equivariant situation was considered in [16]. For additional references, see [34–36,117,118].

8. Applications

In this section we consider three concrete models of practical meaning, which can be studied using the methods developed in the previous sections.

Throughout this section, V stands for an orthogonal representation of a group Γ which, for the sake of simplicity, is supposed to be one of the following *finite* groups: the dihedral groups D_8, D_{12} and the icosahedral group (i.e. the alternating group of five elements) A_5. The notations used in this section are explained in Appendix A. For all other details, we refer to [19].

8.1. Γ*-symmetric FDEs describing configurations of identical oscillators*

Model Consider a ring of 8 identical cells (for example, being chemical oscillators) coupled symmetrically by diffusion between the adjacent cells. Denote by $x^j(t)$ the concentration of the chemical substance in the j-th cell. The coupling is taking place between adjacent cells connected by the edges of a regular 8-gon. More precisely, the coupling strength between cells is represented by a function $h : \mathbb{R} \to \mathbb{R}$, in general non-linear. Since the coupling strength between the adjacent j-th and i-th cells may be non-linear as well, and depending on the concentrations x^j and x^i, we assume that it is of the form

$$h\big(x^j(t)\big)\big(g(\alpha, x_t^j) - g(\alpha, x_t^i)\big).$$

This term is supported by the ordinary law of diffusion, which simply means that the chemical substance moves from a region of greater concentration to a region of less concentration, at rate proportional to the gradient of the concentration.

The following system of delayed differential equations represents the simplest mathematical model for the above process:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\alpha x(t) + \alpha H\big(x(t)\big) \cdot C\Big(G\big(x(t-1)\big)\Big), \tag{187}$$

where

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^8 \end{bmatrix}, \quad H(x) = \begin{bmatrix} h(x^1) \\ h(x^2) \\ \vdots \\ h(x^8) \end{bmatrix}, \quad G(x) = \begin{bmatrix} g(x^1) \\ g(x^2) \\ \vdots \\ g(x^8) \end{bmatrix},$$

the product '$\cdot$' is defined on the vectors by component-wise multiplication, and

(G1) The functions $h, g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable, $h(t) \neq 0$ for all $t \in \mathbb{R}$, $g(0) = 0$, $g'(0) > 0$ and C is a symmetric 8×8-matrix of the type

$$C = \begin{bmatrix} c & d & 0 & 0 & 0 & 0 & 0 & d \\ d & c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & d & c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & d & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & d & c & d \\ d & 0 & 0 & 0 & 0 & 0 & d & c \end{bmatrix} \tag{188}$$

which clearly commutes with an orthogonal D_8-representation.

Therefore, conditions (A1)–(A3) from Subsection 6.1 are satisfied for the system (187).

Characteristic values To verify the remaining conditions (A4)–(A5), consider the linearization of the system (187) at $(\alpha, 0)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\alpha x(t) + \alpha h(0)g'(0)C\big(x(t-1)\big),$$

and put

$$\eta := h(0)g'(0). \tag{189}$$

Thus, the condition (A4) for the system (187) amounts to

$$\prod_{i=0}^{4}[-\alpha + \alpha\eta\xi_i] \neq 0, \tag{190}$$

where $\xi_0, \xi_1, \xi_2, \xi_3, \xi_4$ are the eigenvalues of the matrix C given by

$$\xi_i := c + 2d \cos \frac{i2\pi}{8}, \quad i = 0, 1, 2, 3, 4, \tag{191}$$

i.e.

$$\begin{aligned} &\xi_0 := c + 2d, && \xi_1 := c + d\sqrt{2}, \\ &\xi_2 := c, && \xi_3 := c - d\sqrt{2}, \\ &\xi_4 := c - 2d, \end{aligned}$$

and the eigenspace $E(\xi_i)$, for $i = 0, 1, 2, 3$ corresponding to the eigenvalue ξ_i is equivalent to the irreducible D_8-representation $\mathcal{V}_i$, while $E(\xi_4)$ is equivalent to $\mathcal{V}_5$ (cf. Appendix A and [19]). Moreover,

$$\Delta_\alpha(\lambda) = (\lambda + \alpha)\,\mathrm{Id} - \alpha\eta \mathrm{e}^{-\lambda} C.$$

Therefore, a number $\lambda \in \mathbb{C}$ is a characteristic root of (141) at the stationary point $(\alpha_o, 0)$ if and only if

$$\det\nolimits_{\mathbb{C}} \Delta_\alpha(\lambda) = \prod_{i=1}^{k} [\lambda + \alpha - \alpha\eta\xi_i \mathrm{e}^{-\lambda}] = 0. \tag{192}$$

To find a root $\lambda \in \mathbb{C}$ of the system (192), consider the following equation

$$\lambda + \alpha - \alpha\eta\xi_o \mathrm{e}^{-\lambda} = 0, \tag{193}$$

where ξ_o is an eigenvalue of C. Obviously, $\xi_o \neq 0$ (otherwise $\lambda = -\alpha \in \mathbb{R}$ cannot be purely imaginary). A similar reason forces

$$\alpha \neq 0. \tag{194}$$

By direct computation, $\lambda = \mathrm{i}\beta$ is a purely imaginary root of (193) if

$$\begin{cases} \cos\beta = \dfrac{1}{\eta\xi_o}, \\ \sin\beta = -\dfrac{1}{\alpha\eta\xi_o}\beta. \end{cases} \tag{195}$$

For the sake of simplicity, in our computations of the local bifurcation invariants, we take concrete values of the numbers c, d and η, involved in (195). For example, put

$$c = 2, \quad d = 3, \quad \eta = 6.$$

Then, clearly $|\frac{1}{\xi_i \eta}| < 1$ for $i = 0, 1, 2, 3, 4$, there exists $\beta_i \in (0, \pi]$ such that $\cos\beta_i = \frac{1}{\eta\xi_i}$, and, in addition, it is possible to find a unique $\alpha_i = -\beta_i \cot\beta_i$ (which clearly satisfies

conditions (A4) and (A5) from Subsection 6.1). The approximate values of the pairs of solutions (α_o, β_o) to (195) are:

$$
\begin{aligned}
(\alpha_0, \beta_0) &= (-0.03229787433, 1.549961486),\\
(\alpha_1, \beta_1) &= (-0.04123910887, 1.544095051),\\
(\alpha_2, \beta_2) &= (-0.1243798137, 1.487366240),\\
(\alpha_3, \beta_3) &= (0.1226042867, 1.645182058),\\
(\alpha_4, \beta_4) &= (0.06724485806, 1.612475059).
\end{aligned}
$$

Crossing numbers In order to determine the value of the crossing number associated with a purely imaginary characteristic root $\lambda_o = \mathrm{i}\beta_o$, compute (by implicit differentiation) $\frac{\mathrm{d}}{\mathrm{d}\alpha}u(\alpha)$. By following the same lines as in [19], we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}u|_{\alpha=\alpha_o} = \frac{\beta_o^2}{\alpha_o((\alpha_o+1)^2+\beta_o^2)},$$

thus

$$\operatorname{sign}\frac{\mathrm{d}}{\mathrm{d}\alpha}u|_{\alpha=\alpha_o} = \operatorname{sign}\alpha_o.$$

Consider an eigenvalue $\xi_k \in \sigma(C)$. Then, there is exists a purely imaginary characteristic root $\mathrm{i}\beta_k$, $\beta_k > 0$, of the characteristic equation (192) for $\alpha = \alpha_k$, where

$$\cos\beta_k = -\frac{1}{\eta\xi_k}, \qquad \alpha_k = \frac{\beta_k}{\eta\xi_k\sin\beta_k}$$

and consequently,

$$
\begin{aligned}
&\text{if } \alpha_k > 0, \qquad \text{then } \mathfrak{t}_{j,1}(\alpha_k, \beta_k) = -m_j(\mathrm{i}\beta_k) = -1;\\
&\text{if } \alpha_k < 0, \qquad \text{then } \mathfrak{t}_{j,1}(\alpha_k, \beta_k) = m_j(\mathrm{i}\beta_k) = 1.
\end{aligned}
$$

Notice that in our case, for the chosen values of c, d and η, all the crossing numbers $\mathfrak{t}_{j,l}(\alpha_k, \beta_k), l > 1$, are zero.

Negative spectrum σ_- We have $A := a_0(\alpha_k, \beta_k) = -\alpha_k\,\mathrm{Id} + \alpha_k h(0)g'(0)C = -\alpha_k\,\mathrm{Id} + \alpha_k\eta C$ (cf. (89), (90) and Subsection 5.2), so

$$\sigma(A) = \big\{\mu\colon\ \mu = -\alpha_k + \alpha_k\eta\xi,\ \xi \in \sigma(C)\big\}.$$

By direct computations, for the bifurcation points (α_k, β_k) we have the following sets σ_-:

$$
\begin{aligned}
&(\alpha_0, \beta_0): \quad \sigma_- = \{-\alpha_0 + \alpha_0\eta\xi_0, -\alpha_0 + \alpha_0\eta\xi_1, -\alpha_0 + \alpha_0\eta\xi_2\},\\
&(\alpha_1, \beta_1): \quad \sigma_- = \{-\alpha_1 + \alpha_1\eta\xi_0, -\alpha_1 + \alpha_1\eta\xi_1, -\alpha_1 + \alpha_1\eta\xi_2\},\\
&(\alpha_2, \beta_2): \quad \sigma_- = \{-\alpha_2 + \alpha_2\eta\xi_0, -\alpha_2 + \alpha_2\eta\xi_1, -\alpha_2 + \alpha_2\eta\xi_2\},\\
&(\alpha_3, \beta_3): \quad \sigma_- = \{-\alpha_3 + \alpha_3\eta\xi_3, -\alpha_3 + \alpha_3\eta\xi_4\},\\
&(\alpha_4, \beta_4): \quad \sigma_- = \{-\alpha_4 + \alpha_4\eta\xi_3, -\alpha_4 + \alpha_4\eta\xi_4\}.
\end{aligned}
$$

Maple© input data Consequently (for the chosen values of c, d and η), we obtain the following input data for the computation of the exact value of the local bifurcation invariants

(α_o, β_o)	$\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6$	$\mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5, \mathfrak{t}_6$
(α_0, β_0)	1,1,1,0,0,0,0	1,0,0,0,0,0,0
(α_1, β_1)	1,1,1,0,0,0,0	0,1,0,0,0,0,0
(α_2, β_2)	1,1,1,0,0,0,0	0,0,1,0,0,0,0
(α_3, β_3)	0,0,0,1,0,1,0	0,0,0,−1,0,0,0
(α_4, β_4)	0,0,0,1,0,1,0	0,0,0,0,0,−1,0

Table of results Finally, based on the fact that (D_8^d), (D_8), $(\widetilde{D}_4^d)$, (D_4^d), $(\mathbb{Z}_8^{t_1})$, $(\mathbb{Z}_8^{t_2})$, $(\mathbb{Z}_8^{t_3})$, $(\widetilde{D}_2^d)$ and (D_2^d) are the dominating orbit types in W, we are able to establish the following table of the local bifurcation invariants for each the bifurcation point (α_i, β_i), $i = 0, 1, 2, 3, 4$:

(α_i, β_i)	$\omega(\alpha_i, \beta_i)$	# of branches
(α_0, β_0)	$-(D_8) + (\widetilde{D}_2) + (D_2) - (\widetilde{D}_1) - (D_1) - (\mathbb{Z}_2) + (\mathbb{Z}_1)$	1
(α_1, β_1)	$-(\mathbb{Z}_8^{t_1}) + (\widetilde{D}_2^d) + (D_2^d) - (\widetilde{D}_1^z) - (\widetilde{D}_1) - (D_1^z) - (D_1) - (\mathbb{Z}_2^-) + 2(\mathbb{Z}_1)$	10
(α_2, β_2)	$-(\widetilde{D}_4^d) - (D_4^d) - (\mathbb{Z}_8^{t_2}) + (\widetilde{D}_2^z) + (\widetilde{D}_2) + (D_2^z) + (D_2) + (\mathbb{Z}_4^d)$ $- (\widetilde{D}_1^z) - (\widetilde{D}_1) - (D_1^z) - (D_1) - 2(\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	6
(α_3, β_3)	$(\mathbb{Z}_8^{t_3}) - (\widetilde{D}_2^d) - (D_2^d) + (\widetilde{D}_1^z) + (\widetilde{D}_1) + (D_1^z) + (D_1) + (\mathbb{Z}_2^-) - 2(\mathbb{Z}_1)$	10
(α_4, β_4)	$-(D_8^d) + (D_4) + (\widetilde{D}_1^z) - (D_1)$	1

8.2. *Hopf bifurcation in symmetric configuration of transmission lines*

In this subsection, we consider two simple types of symmetric configurations for the lossless transmission line models. For all details we refer to [19].
Configuration 1: Internal coupling Consider a cube of symmetrically coupled lossless transmission line networks between two recipients and two power stations. Assume all coupled networks are identical, each of which is a uniformly distributed lossless transmission line with the inductance L_s and parallel capacitance C_s per unit length. To derive

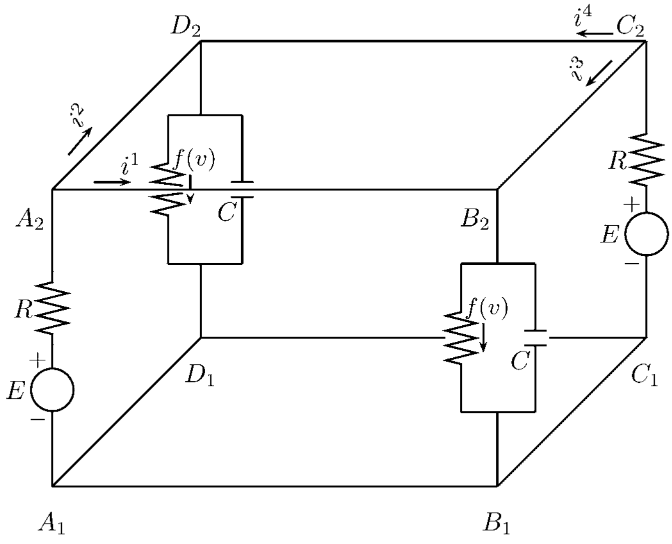

Fig. 6. Symmetric model of transmission lines: internal coupling.

the network equations, place the x-axis in the direction of the line with two ends of the normalized line at $x = 0$ and $x = 1$ (see Fig. 6).[5]

Denote by $i^j(x,t)$ the current flowing in the j-th line at time t and distance x down the line and $v^j(x,t)$ the voltage across the line at t and x, for $j = 1, 2, 3, 4$. It is well-known that (see, for instance, [109]) the functions $i^j := i^j(x,t)$ and $v^j := v^j(x,t)$ obey the *telegrapher's equation*

$$\begin{cases} \dfrac{\partial v^j}{\partial x} = -L_s \dfrac{\partial i^j}{\partial t}, \\ C_s \dfrac{\partial v^j}{\partial t} = -\dfrac{\partial i^j}{\partial x}, \end{cases} \tag{196}$$

with the boundary conditions determined by Kirchoff law

$$\begin{cases} E = v_0^1 + (i_0^1 + i_0^2)R, \\ i_1^1 + i_1^3 = f(v_1^1) + C\dfrac{\mathrm{d}v_1^1}{\mathrm{d}t}, \\ E = v_0^3 + (i_0^3 + i_0^4)R, \\ i_1^2 + i_1^4 = f(v_1^2) + C\dfrac{\mathrm{d}v_1^2}{\mathrm{d}t}, \\ v_0^1 = v_0^2, \quad v_0^3 = v_0^4, \\ v_1^1 = v_1^3, \quad v_1^2 = v_1^4, \end{cases} \tag{197}$$

[5]This example of internal coupling can be easily generalized to a coupling of N recipients and N power stations with $N > 2$.

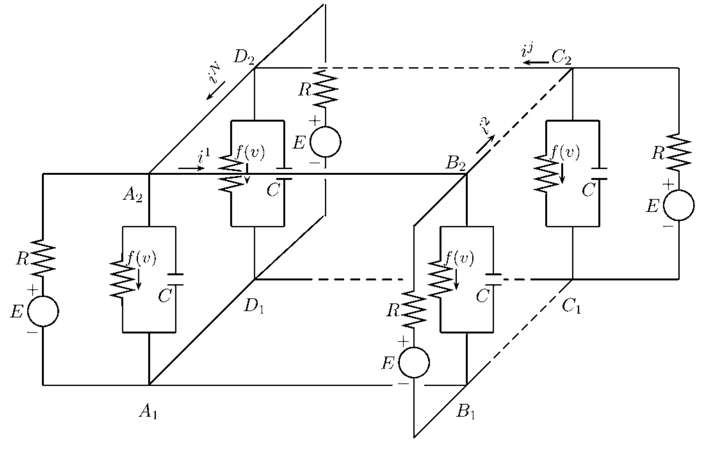

Fig. 7. Symmetric model of transmission lines: external coupling.

where $i^j_\delta = i^j_\delta(t) := i^j(\delta, t)$, $v^j_\delta = v^j_\delta(t) := v^j(\delta, t)$ for $\delta \in \{0, 1\}$, E is the constant direct current voltage and $f(v^j_1)$ is the current through the non-linear resistor in the direction shown in Fig. 6. Assume that

(E1) the boundary value problem (196)–(197) admits a unique solution $(v^j_*, i^j_*) := (v^j_*(x,t), i^j_*(x,t))$, for $j = 1, 2, 3, 4$ such that $\frac{\partial i^j_*}{\partial x} = \frac{\partial v^j_*}{\partial x} = 0$ (the so-called *equilibrium point*).

By applying standard transformations (cf. [19]) we obtain from (196) the following system of neutral functional differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t) - Qx(t-r)\big] = -\frac{2}{bC}x(t) - \frac{2}{bC}Qx(t-r) - \frac{1}{C}G\big(x(t)\big) + \frac{1}{C}QG\big(x(t-r)\big), \tag{198}$$

where

$$r = \frac{2}{a}, \quad x(t) = \begin{bmatrix} v^1_1(t) \\ v^2_1(t) \end{bmatrix}, \quad G\big(x(t)\big) = \begin{bmatrix} g(v^1_1(t)) \\ g(v^2_1(t)) \end{bmatrix},$$
$$Q = \begin{bmatrix} \frac{b}{2R+b} & -\frac{2R}{2R+b} \\ -\frac{2R}{2R+b} & \frac{b}{2R+b} \end{bmatrix}.$$

Configuration 2: External coupling A second example of symmetric coupling was considered in [145], where N recipients are mutually coupled via lossless transmission line network which are interconnected by a common resistor R_o between neighboring recipients, and extensively connected with N power stations (see Fig. 7).

Denote by $i^j(x, t)$ the current flowing in the j-th line at time t and distance x down the line and $v^j(x, t)$ the voltage across the line at t and x, for $j = 1, \dots, N$. The same teleg-

rapher's equation (196) holds for $i^j(x,t)$ and $v^j(x,t)$. However, the boundary conditions need to be modified for this external coupling. For $j = 1, \dots, N$,

$$\begin{cases} E = v_0^j + i_0^j R, \\ i_1^j = f(v_1^j) + C\dfrac{\mathrm{d}v_1^j}{\mathrm{d}t} - \big(I^{j-1}(t) - I^j(t)\big), \\ v_1^j - v_1^{j+1} = I^j(t) R_o, \end{cases} \tag{199}$$

where $I^0(t) := I^N(t)$, $v^{N+1} := v^1$, I^j's are the so-called coupling terms (see [145]).

For mathematical simplicity, assume that (cf. (E1))

(E2) the boundary value problem (196) and (199) admits a unique equilibrium point $(v_*^j, i_*^j) := (v_*^j(x,t), i_*^j(x,t))$, for $j = 1, \dots, N$.

By applying similar standard transformations, the system (196) with the boundary conditions (199) can be translated to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\big[x(t) - \alpha x(t-r)\big] = &-\frac{1}{bC}Px(t) - \frac{1}{bC}\alpha Px(t-r) - \frac{1}{C}G\big(x(t)\big) \\ &+ \frac{1}{C}\alpha G\big(x(t-r)\big), \end{aligned} \tag{200}$$

where

$$r = \frac{2}{a}, \quad x(t) = \begin{bmatrix} v_1^1(t) \\ \vdots \\ v_1^N(t) \end{bmatrix}, \quad G\big(x(t)\big) = \begin{bmatrix} g(v_1^1(t)) \\ \vdots \\ g(v_1^N(t)) \end{bmatrix},$$

$$\alpha = -\frac{R-b}{R+b}, \quad P = \begin{bmatrix} 1 + \frac{2b}{R_o} & -\frac{b}{R_o} & 0 & \dots & 0 & -\frac{b}{R_o} \\ -\frac{b}{R_o} & 1 + \frac{2b}{R_o} & -\frac{b}{R_o} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{b}{R_o} & 0 & 0 & \dots & -\frac{b}{R_o} & 1 + \frac{2b}{R_o} \end{bmatrix}.$$

Notice that (200) is a D_N-symmetric system in the following sense: consider D_N acting on $V := \mathbb{R}^N$ by permuting the coordinates of vectors

$$x = \begin{bmatrix} v^1 \\ \vdots \\ v^N \end{bmatrix} \in V,$$

then the system (200) is symmetric with respect to the D_N-action on V.

Symmetric configuration of transmission lines Being motivated by the above two generic models of symmetric couplings, we consider a general symmetric system of functional differential equations and discuss several crucial elements related to computations of its

associated bifurcation invariant, which are the prerequisite for the usage of our Maple© package.

Combine the two coupling models in the following symmetric neutral functional differential equation

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t)-\alpha Qx(t-r)\big]\\&\quad=-P_1x(t)-\alpha QP_2x(t-r)-aG\big(x(t)\big)+a\alpha QG\big(x(t-r)\big)\end{aligned}\tag{201}$$

where a and r are positive constants, $\alpha\in\mathbb{R}$ is the bifurcation parameter, and

$$x(t)=\begin{bmatrix}x^1(t)\\x^2(t)\\\vdots\\x^n(t)\end{bmatrix}\in\mathbb{R}^n,\qquad G\big(x(t)\big)=\begin{bmatrix}g(x^1(t))\\g(x^2(t))\\\vdots\\g(x^n(t))\end{bmatrix}\in\mathbb{R}^n.$$

For definitiveness, assume $n=12$, $r=1$, and

(H1) $g:\mathbb{R}\to\mathbb{R}$ is continuously differentiable, $g(0)=g'(0)=0$;

(H2) $\Gamma=D_{12}$ acts on $\mathbb{R}^{12}$ by permuting the coordinates of vectors $x\in V:=\mathbb{R}^{12}$, meaning geometrically it permutes the vertices of the 12-gon;

(H3)

(i) Q, P_1, P_2 are 12×12-matrices of the type:

$$C=\begin{bmatrix}c&d&0&0&0&0&0&0&0&0&0&d\\d&c&d&0&0&0&0&0&0&0&0&0\\0&d&c&d&0&0&0&0&0&0&0&0\\0&0&d&c&d&0&0&0&0&0&0&0\\0&0&0&d&c&d&0&0&0&0&0&0\\0&0&0&0&d&c&d&0&0&0&0&0\\0&0&0&0&0&d&c&d&0&0&0&0\\0&0&0&0&0&0&0&0&d&c&d&0\\0&0&0&0&0&0&0&d&c&d&0&0\\0&0&0&0&0&0&0&0&d&c&d&0\\0&0&0&0&0&0&0&0&0&d&c&d\\d&0&0&0&0&0&0&0&0&0&d&c\end{bmatrix}.\tag{202}$$

(ii) for the matrix Q, we choose $c=-\frac{1}{3}$, $d=\frac{1}{4}$; for P_1, $c=\frac{1}{2}$, $d=\frac{1}{3}$; and for P_2, $c=25$, $d=10$;

(H4) we will consider α satisfying $|\alpha|\cdot\|Q\|=\frac{5}{6}|\alpha|<1$.

REMARK 8.1.

(i) It is clear that the system (201) is symmetric with respect to the D_{12}-action on V and $(\alpha,0)$ is a stationary point for all α.

(ii) Q, P_1, P_2 are pairwisely commuting matrices. They can be diagonalized simultaneously. In other words, Q, P_1, P_2 share the same eigenspaces with respect to a certain choice of a basis of V. We use the symbols ξ, ζ and η to denote the eigenvalues of Q, P_1, and P_2 (respectively) corresponding to the same eigenvector $v \in V$.

(iii) One can verify that in the case $\zeta\eta > 0$, ζ and η satisfy $\sqrt{\zeta\eta} \neq \frac{2k+1}{2}\pi$ for any $k \in \mathbb{Z}$.

In this way, we are dealing here with a D_{12}-symmetric system of functional differential equations, and we are interested in studying a Hopf bifurcation phenomenon.

Let us organize all the eigenvalues of the matrices Q, P_1 and P_2 according to their D_{12}-isotypical type (see Appendix A):

k	ξ_k	ζ_k	η_k	Eigenspace
0	$\frac{1}{6}$	$\frac{7}{6}$	45	$\mathcal{V}_0$
1	$-\frac{1}{3}+\frac{1}{4}\sqrt{3}$	$\frac{1}{2}+\frac{1}{3}\sqrt{3}$	$25+10\sqrt{3}$	$\mathcal{V}_1$
2	$-\frac{1}{12}$	$\frac{5}{6}$	35	$\mathcal{V}_2$
3	$-\frac{1}{3}$	$\frac{1}{2}$	25	$\mathcal{V}_3$
4	$-\frac{7}{12}$	$\frac{1}{6}$	15	$\mathcal{V}_4$
5	$-\frac{1}{3}-\frac{1}{4}\sqrt{3}$	$\frac{1}{2}-\frac{1}{3}\sqrt{3}$	$25-10\sqrt{3}$	$\mathcal{V}_5$
6	$-\frac{5}{6}$	$-\frac{1}{6}$	5	$\mathcal{V}_7$

Characteristic equation and isolated centers By linearizing the system (201) at $x = 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t) - \alpha Q x(t-1)\big] = -P_1 x(t) - \alpha Q P_2 x(t-1),$$

and substituting $x = \mathrm{e}^{\lambda t} v$ for $\lambda \in \mathbb{C}$, $0 \neq v \in V$, leads to

$$\lambda \mathrm{e}^{\lambda t} v - \alpha Q \lambda \mathrm{e}^{\lambda(t-1)} v = -P_1 \mathrm{e}^{\lambda t} v - \alpha Q P_2 \mathrm{e}^{\lambda(t-1)} v$$

i.e.

$$[\lambda\,\mathrm{Id} - \alpha Q \lambda \mathrm{e}^{-\lambda} + P_1 + \alpha Q P_2 \mathrm{e}^{-\lambda}]v = 0.$$

Therefore, we have the following characteristic equation for the system (201)

$$\det\nolimits_{\mathbb{C}} \Delta_{(\alpha,0)}(\lambda) = 0, \tag{203}$$

where

$$\Delta_{(\alpha,0)}(\lambda) := (\lambda\,\mathrm{Id} - \alpha Q \lambda \mathrm{e}^{-\lambda}) + P_1 + \alpha Q P_2 \mathrm{e}^{-\lambda}.$$

Table 3
Some of the first bifurcation points (α_o, β_o), their isotypical types and the corresponding negative spectrum σ_-

k	$\mathcal{V}_k$	α_o	β_o	σ_-
0	$\mathcal{V}_0$	0.3134847613	2.044048292	$\mu_2, \mu_3, \mu_4, \mu_5, \mu_7$
1	$\mathcal{V}_1$	0.5409015727	2.014360183	$\mu_2, \mu_3, \mu_4, \mu_5, \mu_7$
2	$\mathcal{V}_2$	−0.7179480745	1.924503245	μ_0, μ_1
3	$\mathcal{V}_3$	−0.2206845553	1.774570762	μ_0
4	$\mathcal{V}_4$	−0.1796805216	1.572004370	μ_0
5	$\mathcal{V}_5$	−0.2247337034	1.340355130	μ_0
6	$\mathcal{V}_7$	−0.2821763604	1.197438321	μ_0, μ_1
0	$\mathcal{V}_0$	−0.6602221152	4.841664958	μ_0, μ_1
1	$\mathcal{V}_1$	−1.163028327	4.818957370	μ_0, μ_1
2	$\mathcal{V}_2$	1.638785602†	4.751099765	excluded
3	$\mathcal{V}_3$	0.5502165702	4.636440934	$\mu_2, \mu_3, \mu_4, \mu_5, \mu_7$
4	$\mathcal{V}_4$	0.4889847314	4.460684627	$\mu_2, \mu_3, \mu_4, \mu_5, \mu_7$
5	$\mathcal{V}_5$	0.6255618890	4.194073856	$\mu_2, \mu_3, \mu_4, \mu_5, \mu_7$
6	$\mathcal{V}_7$	0.7498744480	3.996406449	$\mu_2, \mu_3, \mu_4, \mu_5, \mu_7$
0	$\mathcal{V}_0$	1.039854303	7.829632575	$\mu_2, \mu_3, \mu_4, \mu_5, \mu_7$
1	$\mathcal{V}_1$	1.837568325†	7.808625800	excluded
2	$\mathcal{V}_2$	−2.607182120†	7.743452649	excluded
3	$\mathcal{V}_3$	−0.8769191325	7.623492141	μ_0, μ_1
4	$\mathcal{V}_4$	−0.7600522375	7.417224874	μ_0, μ_1
5	$\mathcal{V}_5$	−0.8856961981	7.097079227	μ_0, μ_1
6	$\mathcal{V}_7$	−0.9713506495	6.886959465	μ_0, μ_1

†These values of α_o have to be excluded because they are not in the required interval $(-\frac{6}{5}, \frac{6}{5})$.

Next, we need to find possible values of α such that (203) has a purely imaginary root $\mathrm{i}\beta$ for some $\beta > 0$, so we could detect potential bifurcation points $(\alpha, \beta, 0) \in (-\frac{6}{5}, \frac{6}{5}) \times \mathbb{R}_+ \times W$.

By Remark 8.1(ii), when restricted to the same eigenspace of Q, P_1 and P_2, the characteristic equation (203) reduces to the following algebraic equation

$$(\lambda + \zeta)\mathrm{e}^{\lambda} - \alpha\xi(\lambda - \eta) = 0. \tag{204}$$

By replacing in (204) λ with $\mathrm{i}\beta$ and separating the real and imaginary parts, we obtain

$$\tan(\beta) = \frac{\beta(\zeta + \eta)}{\beta^2 - \zeta\eta}, \qquad \alpha = \frac{\zeta \sin\beta + \beta\cos\beta}{\beta\xi}. \tag{205}$$

The system (205) has infinitely many solutions, which can be easily estimated. In Table 3, we present some of the first values of the bifurcation points (α_o, β_o) arranged according to the isotypical type $\mathcal{V}_k$ of the eigenspace $E(\xi_k)$.

Negative spectrum σ_- Consider a bifurcation point (α_o, β_o). The spectrum $\sigma(A(\alpha_o, \beta_o))$ of the linear operator $A(\alpha_o, \beta_o) := -\frac{1}{\beta}_o D_x f(\alpha_o, 0)|_V$ is given by

$$\sigma\big(A(\alpha_o, \beta_o)\big) = \left\{\mu_k = \frac{1}{\beta}_o (\zeta_k + \alpha_o \xi_k \eta_k)\colon\ k = 0, 1, 2, 3, 4, 5, 6\right\}.$$

Therefore, the negative spectrum σ_- of $A(\alpha_o, \beta_o)$ can be obtained easily by inspection (see Table 3). The $\mathcal{V}_i$-multiplicity $m_i(\mu)$ for each $\mu \in \sigma_-$ is 1.

Crossing numbers $\mathfrak{t}_{k,1}$ For a given bifurcation point (α_o, β_o), in order to evaluate the crossing number $\mathfrak{t}_{k,1}(\alpha_o, \beta_o)$, we substitute $\lambda = u + \mathrm{i}v$ in (204), separate the real and imaginary parts, and apply implicit differentiation with respect to α at α_o, $u = 0$, $v = \beta_o$. If $\beta_o > 1$, then one can show (see [19]) that

$$\operatorname{sign}\left(\frac{\mathrm{d}u}{\mathrm{d}\alpha}(\alpha_o)\right) = \operatorname{sign}(\alpha_o).$$

Consequently, if (α_o, β_o) satisfies (205) with $\xi = \xi_k$, $\zeta = \zeta_k$ and $\eta = \eta_k$, for some $k = 0, 1, 2, 3, 4, 5, 6$, then

$$\mathfrak{t}_{k,1}(\alpha_o, \beta_o) = -\operatorname{sign}(\alpha_o).$$

Table of results In Table 4 we list the exact values of the local bifurcation invariants $\omega(\alpha_o, \beta_o)$ for all bifurcation points indicated in Table 3. The dominating orbit types in W are (D_{12}), (D^d_{12}), $(\mathbb{Z}^{t_1}_{12})$, $(\mathbb{Z}^{t_2}_{12})$, $(\mathbb{Z}^{t_3}_{12})$, $(\mathbb{Z}^{t_4}_{12})$, $(\mathbb{Z}^{t_5}_{12})$, $(\widetilde{D}^d_6)$, and (D^d_6).

8.3. *Global continuation of bifurcating branches*

Here, we apply the general results presented Subsection 6.3 to study the global behavior of branches of periodic solutions to the system (187) (with the dihedral symmetries D_8) occurring as a result of the (local) Hopf bifurcation.

By applying the same setting as in Subsection 6.1, the problem of finding periodic solutions to (187) can be reformulated as an equation of type (155), where $\mathfrak{F}$ is given by

$$\mathfrak{F}(\alpha, \beta, u) = u - \mathcal{F}(\alpha, \beta, u), \tag{206}$$

and

$$\mathcal{F}(\alpha, \beta, u)(t) = (L + K)^{-1}\left[-\frac{\alpha}{\beta}\big(u(t) - H\big(u(t)\big)\big)\cdot CG\big(u(t - \beta)\big)\big) + Ku\right]. \tag{207}$$

Throughout this subsection, we assume that the condition (G1) of Subsection 8.1 is satisfied with $c = -3$, $d = 1$, and $\eta := h(0)g'(0) = 2$.

Table 4
Classification of the D_{12}-symmetric Hopf bifurcation in (201) for selected bifurcation points

k	(α_o, β_o)	σ_-	$\omega(\alpha_0, \beta_o)$	#
0	(0.3134847613,2.044048292)	$\mu_2, \mu_3, \mu_4, \mu_5, \mu_7$	$-(D_{12}) + (D_6) + 2(D_4) + (\widetilde{D}_3)$ $- (D_3) - (\widetilde{D}_2) - (D_2) - (\mathbb{Z}_4)$ $- (\widetilde{D}_1) + (D_1) + (\mathbb{Z}_2)$	1
1	(0.5409015727,2.014360183)	$\mu_2, \mu_3, \mu_4, \mu_5, \mu_7$	$-(\mathbb{Z}_{12}^{t_1}) + (\mathbb{Z}_6^{t_1}) - (\widetilde{D}_2^d) + (D_2^d)$	6
2	(−0.7179480745,1.924503245)	μ_0, μ_1	$-(\mathbb{Z}_{12}^{t_2}) - (D_4^d) - (D_4^{\hat{d}}) + (\mathbb{Z}_4^d) + (\widetilde{D}_1^z)$ $+ (D_1^z) + (\widetilde{D}_1) + (D_1) - 2(\mathbb{Z}_1)$	4
3	(−0.2206845553,1.774570762)	μ_0	$-(\widetilde{D}_6^d) - (D_6^d) - (\mathbb{Z}_{12}^{t_3}) + (\mathbb{Z}_6^d)$	6
4	(−0.1796805216,1.572004370)	μ_0	$-(\mathbb{Z}_{12}^{t_4}) - (D_4^z) - (D_4) + (\mathbb{Z}_4)$	4
5	(−0.2247337034,1.340355130)	μ_0	$-(\mathbb{Z}_{12}^{t_5}) - (\widetilde{D}_2^d) - (D_2^d) + (\mathbb{Z}_2^-)$	4
6	(−0.2821763604,1.197438321)	μ_0, μ_1	$-(D_{12}^d) + (\widetilde{D}_1^z) + (D_1) - (\mathbb{Z}_1)$	1
0	(−0.6602221152,4.841664958)	μ_0, μ_1	$-(D_{12}) + (\widetilde{D}_1) + (D_1) - (\mathbb{Z}_1)$	1
1	(−1.163028327,4.818957370)	μ_0, μ_1	$-(\mathbb{Z}_{12}^{t_1}) - (\widetilde{D}_2^d) - (D_2^d) + (\widetilde{D}_1^z)$ $+ (D_1^z) + (\widetilde{D}_1) + (D_1) + (\mathbb{Z}_2^-) - 2(\mathbb{Z}_1)$	6
3	(0.5502165702,4.636440934)	$\mu_2, \mu_3, \mu_4, \mu_5, \mu_7$	$-(\widetilde{D}_6^d) + (D_6^d) - (\mathbb{Z}_{12}^{t_3}) + (\widetilde{D}_3^z)$ $- (D_3^z) + (\widetilde{D}_3) - (D_3) + (\mathbb{Z}_6^d)$ $- (\widetilde{D}_1^z) + (D_1^z) - (\widetilde{D}_1) + (D_1)$	6
4	(0.4889847314,4.460684627)	$\mu_2, \mu_3, \mu_4, \mu_5, \mu_7$	$-(\mathbb{Z}_{12}^{t_4}) + (D_4^z) + (D_4) + (\mathbb{Z}_6^{t_2})$ $- (\widetilde{D}_2^z) - (\widetilde{D}_2) - (\mathbb{Z}_4) + (\mathbb{Z}_2)$	4
5	(0.6255618890,4.194073856)	$\mu_2, \mu_3, \mu_4, \mu_5, \mu_7$	$-(\mathbb{Z}_{12}^{t_5}) + (\mathbb{Z}_6^{t_1}) - (\widetilde{D}_2^d) + (D_2^d)$	4
6	(0.7498744480,3.996406449)	$\mu_2, \mu_3, \mu_4, \mu_5, \mu_7$	$(D_{12}^d) - (D_6) - 2(D_4^d) - (\widetilde{D}_3^z)$ $+ (D_3) + (\widetilde{D}_2^z) + (D_2) + (\mathbb{Z}_4^d)$ $+ (\widetilde{D}_1^z) - (D_1) - (\mathbb{Z}_2)$	1
0	(1.039854303,7.829632575)	$\mu_2, \mu_3, \mu_4, \mu_5, \mu_7$	$-(D_{12}) + (D_6) + 2(D_4) + (\widetilde{D}_3)$ $- (D_3) - (\widetilde{D}_2) - (D_2) - (\mathbb{Z}_4)$ $- (\widetilde{D}_1) + (D_1) + (\mathbb{Z}_2)$	1
3	(−0.8769191325,7.623492141)	μ_0, μ_1	$-(\widetilde{D}_6^d) - (D_6^d) - (\mathbb{Z}_{12}^{t_3}) + (\mathbb{Z}_6^d) + (\widetilde{D}_1^z)$ $+ (D_1^z) + (\widetilde{D}_1) + (D_1) - 2(\mathbb{Z}_1)$	6
4	(−0.7600522375,7.417224874)	μ_0, μ_1	$-(\mathbb{Z}_{12}^{t_4}) - (D_4^z) - (D_4) + (\mathbb{Z}_4) + (\widetilde{D}_1^z)$ $+ (D_1^z) + (\widetilde{D}_1) + (D_1) - 2(\mathbb{Z}_1)$	4
5	(−0.8856961981,7.097079227)	μ_0, μ_1	$-(\mathbb{Z}_{12}^{t_5}) - (\widetilde{D}_2^d) - (D_2^d) + (\widetilde{D}_1^z)$ $+ (D_1^z) + (\widetilde{D}_1) + (D_1) + (\mathbb{Z}_2^-) - 2(\mathbb{Z}_1)$	4
6	(−0.9713506495,6.886959465)	μ_0, μ_1	$-(D_{12}^z) + (\widetilde{D}_1^z) + (D_1^z) - (\mathbb{Z}_1)$	1

Global Hopf bifurcation in a system with dihedral symmetries D_8 The system (187) is symmetric with respect to the dihedral group $\Gamma = D_8$ acting on $V = \mathbb{R}^8$ by permuting the coordinates of vectors. In addition, $\sigma(C)$ is composed of negative numbers: $\xi_0 := -1$, $\xi_1 := -3 + \sqrt{2}$, $\xi_2 := -3$, $\xi_3 := -3 - \sqrt{2}$, $\xi_4 := -5$ and clearly $\eta\xi_o < -1$ for all $\xi_o \in \sigma(C)$. Therefore, for every $\xi_o \in \sigma(C)$, the system of Eqs. (195) has a unique solution (α_o, β_o) such that $\beta_o \in (\frac{\pi}{2}, \pi)$. We are interested in the point (α_4, β_4) (for $\xi_o = \xi_4$), which can be easily approximated, i.e. $(\alpha_4, \beta_4) \simeq (0.1679381755, 1.670963748)$. By applying the same steps as in Subsection 8.1, one can verify that the set σ_- for the point (α_4, β_4) is empty. Since the eigenspace $E(\xi_4)$ is equivalent to $\mathcal{V}_5$, we obtain that the $\mathcal{V}_5$-isotypical crossing number $\mathfrak{t}_{5,1}(\alpha_4, \beta_4) = -1$, which allows us to compute the exact value of the local bifurcation invariant

$$\omega(\alpha_4, \beta_4) = -(D_8^d).$$

Let us point out that (D_8^d) is a dominating orbit type in W. Therefore, by the local bifurcation theorem (cf. Theorem 5.4), there exists a branch $\mathcal{C} := \mathcal{C}_{(D_8^{d,l})}$ of non-constant periodic solutions of the orbit type $(D_8^{d,l})$ (for some $l \geqslant 1$), bifurcating from (α_4, β_4). Notice that $D_8^d \subset D_8^{d,l}$.

Assume in addition, the following conditions to be satisfied

(B1) $tg(t)/h(t) > 0$ for all $t \neq 0$; $\lim_{t\to\infty} tg(t)/h(t) = \infty$.

(B2) There exist constants $A, B > 0$ and $\delta, \gamma > 0$ with $1 > \delta + \gamma$ such that

$$\left|h(t)\right| \leqslant A + B|t|^{\delta}, \qquad \left|g(t)\right| \leqslant A + B|t|^{\gamma}.$$

Let us point out, that under the above assumptions one has (cf. Remark 6.11):

(a) if (α, x_o) is a stationary point for the system (187) and $\alpha \neq 0$, then $x_o = 0$;

(b) if a Hopf bifurcation occurs at (α_o, x_o), then $\alpha_o \neq 0$.

Suppose that (α, β, x) belongs to $\mathcal{C}$. Then, the D_8^d-symmetry properties of $x(t)$ can be translated as follows: $x(t) = (x^0(t), x^1(t), \ldots, x^7(t))^T$ is a $\frac{2\pi}{\beta}$-periodic solution such that

$$x^k(t) = x^{k+1}\left(t - \frac{\pi}{\beta}\right) \quad (\mathrm{mod}\ 8), \tag{208}$$

and

$$x^k(t) = x^{n-k-1}\left(t - \frac{\pi}{\beta}\right) \quad (\mathrm{mod}\ 8). \tag{209}$$

Combining (208), (209) with condition (B1) and applying the same argument as in [94], one can easily show that the periods $p = \frac{2\pi}{\beta}$ of solutions $(\alpha, \beta, x) \in \mathcal{C}$ satisfy the inequality $2 < p < 4$. This fact immediately implies $\mathcal{C} \cap \mathbb{R}^2 \times \{0\} = (\alpha_4, \beta_4, 0)$.

Consequently, $\mathcal{C} \subset \mathbb{R} \times (\pi/2, \pi) \times W$. Further, using the assumption (B2), one can easily show that there exists a constant $M > 0$ such that for every periodic solution $x(t)$ to (187) $\sup\{\|x(t)\|\colon t \in \mathbb{R}\} \leqslant M$ (see [19]). Thus,

$$\mathcal{C} \subset \mathbb{R} \times (\pi/2, \pi) \times \left\{x \in W\colon \|x\| \leqslant M\right\}.$$

Finally, the system (187) has no non-constant periodic solution for $\alpha = 0$, from which it follows

$$\mathcal{C} \subset (0, \infty) \times (\pi/2, \pi) \times \{x \in W\colon \|x\| \leqslant M\}.$$

However, by Corollary 6.10, the connected component $\mathcal{C}$ is unbounded, therefore

$$[\alpha_{n/2}, \infty) \subset \{\alpha\colon (\alpha, \beta, x) \in \mathcal{C}\}.$$

In this way we obtained the following

THEOREM 8.2. *Consider the system* (187) *and suppose the conditions* (G1) *from Subsection* 8.1 (*with* $c = -3$, $d = 1$ *and* $\eta = 2$), (B1) *and* (B2) *are satisfied. Then, the branch* $\mathcal{C} := \mathcal{C}_{D_8^{d,l}}$ *of periodic solutions bifurcating from* $(\alpha_4, \beta_4, 0)$ *is unbounded and satisfies*

$$[\alpha_4, \infty) \subset \{\alpha\colon (\alpha, \beta, x) \in \mathcal{C}\}.$$

8.4. *Symmetric system of Hutchinson model in population dynamics*

Hutchinson model of an n species ecosystem The standard model for the dynamics of a simple (single) population (in terms of its density) is the Verhulst equation (cf. [73,70])

$$\dot{v} = \alpha v\left(1 - \frac{v}{K}\right), \tag{210}$$

which is based on the idea that the population grows exponentially at low densities and saturates towards the carrying capacity K (of resources) at high densities. The Hutchinson model (of a single species) is obtained from (210) by taking into account a delayed response to the remaining resources,

$$\dot{v}(t) = \alpha v(t)\left(1 - \frac{v(t-\tau)}{K}\right), \tag{211}$$

where $\tau > 0$ is a presumed delay constant and α refers to the intrinsic growth rate.

Consider an ecosystem composed of n species interacting with each other (according to a certain symmetry) by competing (or cooperating) over shared resources such as food and habitats, while maintaining a self-inhibiting nature (meaning self-limiting in respond to rare resources and self-reproducing to abundant resources). A mathematical treatment for such a community model was developed by Levins in [104], where one attaches a loop diagram in order to carry out a loop analysis for this community type situation.

In such a system (which is illustrated on Fig. 8), the coefficient a_{jj} describes the self-inhibiting nature of the j-th species, and $a_{ij} < 0$ (resp. $a_{ij} > 0$) is the competing (resp. cooperating) coefficient between species i and j. Also, observe that $a_{ij} = a_{ji}$.

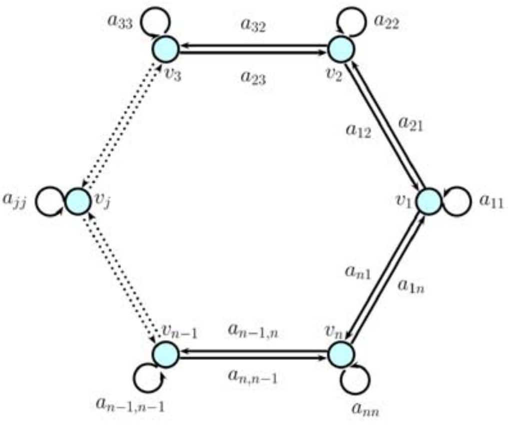

Fig. 8. System with dihedral symmetries.

Introduce

$$C = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{212}$$

and call it the *community matrix*. This community ecosystem can be described by the following equations,

$$\dot{v}(t) = \alpha C v(t) \cdot \left(1 - \frac{v(t-\tau)}{K}\right), \tag{213}$$

where '$\cdot$' stands for the component-wise "vector-multiplication" $u \cdot v = [u_1 v_1, \dots, u_n v_n]^T$ for $u = [u_1, \dots, u_n]^T$ and $v = [v_1, \dots, v_n]^T$ (cf. Subsection 8.1).

By applying the standard substitution, (213) is transformed to the equivalent system

$$\dot{u}(t) = -\alpha C u(t-\tau) \cdot \big[1 + u(t)\big], \tag{214}$$

where $u(t) = v(t)/K - 1$ is, in fact, a population saturation index with respect to the available resources. In order to study the system (214) in a heterogeneous environment, we add to (214) a spatial diffusion term, which leads to the following reaction–diffusion equations

$$\frac{\partial}{\partial t} u(x,t) = d \frac{\partial^2}{\partial x^2} u(x,t) - \alpha C u(x, t-1)\big[1 + u(x,t)\big], \tag{215}$$

where $d > 0$ is a spatial diffusion coefficient.

Symmetric system of the Hutchinson model Consider a symmetric system of n species Hutchinson model of the form (215) (for $t > 0$ and $x \in (0, \pi)$)

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = \dfrac{\partial^2}{\partial x^2} u(x,t) - \alpha C u(x, t-1) \cdot \big[1 + u(x,t)\big], \\ \dfrac{\partial}{\partial x} u(x,t) = 0, \quad x = 0, \pi, \end{cases} \tag{216}$$

where $u : [0, \pi] \times \mathbb{R} \to \mathbb{R}^n$ is a population saturation index (cf. (214)), $d > 0$ is a spatial diffusion coefficient and $\alpha \neq 0$ is the intrinsic growth rate (cf. (211)), which is considered as a bifurcation parameter, and C is a (symmetric) community matrix describing the interaction among the species. For simplicity, in what follows, we assume that $d = 1$.

Assume that

(A1) The geometrical configuration described by the system (216) has the icosahedral symmetry group A_5. The group A_5 permutes twenty vertices of the related dodecahedron, which means it acts on $\mathbb{R}^{20}$ by permuting the coordinates of the vectors $x \in \mathbb{R}^{20}$. The (symmetric) matrix C commuting with this A_5-action is of the type:

$$C = \begin{bmatrix}
1&2&0&0&2&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0\\
2&1&2&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0\\
0&2&1&2&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0\\
0&0&2&1&2&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0\\
2&0&0&2&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&1&2&0&0&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&2&1&2&0&0&0&0&0&0&0&0&2&0&0&0\\
2&0&0&0&0&0&2&1&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&1&2&0&0&0&0&0&0&0&2&0&0\\
0&2&0&0&0&0&0&0&2&1&2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&2&1&2&0&0&0&0&0&0&2&0\\
0&0&2&0&0&0&0&0&0&0&2&1&2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&1&2&0&0&0&0&0&2\\
0&0&0&2&0&0&0&0&0&0&0&0&2&1&2&0&0&0&0&0\\
0&0&0&0&0&2&0&0&0&0&0&0&0&2&1&2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&1&2&0&0&2\\
0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&2&1&2&0&0\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&2&1&2&0\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&2&1&2\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&2&0&0&2&1
\end{bmatrix}. \tag{217}$$

Under the assumption (A1), the space $V := \mathbb{R}^{20}$ becomes an orthogonal A_5-representation and the system (216) is symmetric with respect to the action of the group $\Gamma = A_5$.

Characteristic equation and isolated centers At the A_5-symmetric steady-state solution $(\alpha, 0)$ the system (216) has the linearization

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) - \alpha C u(x, t-1), \\ \frac{\partial}{\partial x} u(x,t) = 0, \quad x = 0, \pi. \end{cases} \tag{218}$$

Since the matrix C is symmetric, it is completely diagonalizable with respect to a basis composed of its eigenvectors. Consider the spectrum $\sigma(C)$ of the matrix C,

$$\sigma(C) = \{\xi_0 = 7, \xi_1 = 1, \xi_2 = -3, \xi_3 = 3, \xi_4 = 1 + 2\sqrt{5}, \xi_5 = 1 - 2\sqrt{5}\}$$

and denote by $\mathfrak{E}(\xi_k) \subset V$ the eigenspace of ξ_k. One can verify that

$$\begin{aligned} &\mathfrak{E}(\xi_0) = \mathcal{V}_0, \quad \mathfrak{E}(\xi_1) = \mathcal{V}_1, \quad \mathfrak{E}(\xi_2) = \mathcal{V}_1, \\ &\mathfrak{E}(\xi_3) = \mathcal{V}_2, \quad \mathfrak{E}(\xi_4) = \mathcal{V}_3, \quad \mathfrak{E}(\xi_5) = \mathcal{V}_4. \end{aligned}$$

Then,

$$L^2\big([0,\pi]; V\big) = \bigoplus_{k=0}^{5} L^2\big([0,\pi]; \mathfrak{E}(\xi_k)\big), \tag{219}$$

and $w \in L^2([0,\pi]; V)$ can be represented as $w(x) = \sum_k w_k(x)$, where $w_k \in L^2([0,\pi]; \mathfrak{E}(\xi_k))$. In a similar way, one also has

$$L^2\big([0,\pi]; V^c\big) = \bigoplus_{k=0}^{5} L^2\big([0,\pi]; \mathfrak{E}^c(\xi_k)\big), \tag{220}$$

where $\mathfrak{E}^c(\xi_k)$ denotes the complexification of the eigenspace $\mathfrak{E}(\xi_k)$.

Notice that $(\alpha, 0)$ is a Γ-symmetric steady-state solution to (216) for all (non-zero) α. Thus, we can take the set $(\alpha, \beta, 0)$, $\alpha \neq 0$, for the manifold $M \subset \mathcal{P} \times \mathbb{E}^G$ described in Subsection 7.1 (see also Subsection 7.2). Moreover, $(\alpha_o, 0)$ is non-singular if $0 \notin \sigma(\mathcal{L}_{\alpha_o})$, where $\mathcal{L}_{\alpha_o} := \frac{\partial^2}{\partial x^2} - \alpha_o C : H_0^2([0,\pi]; V) \to L^2([0,\pi); V]$ with $H_0^2([0,\pi]; V)$ being the subspace of $H^2([0,\pi]; V)$ consisting of functions u satisfying $u(0) = u(\pi) = 0$. One can easily verify that if

$$-\frac{\alpha_o \xi_k}{d} \neq m^2 \quad \text{for all } k = 0, 2, \dots, 5 \text{ and } m = 0, 1, \dots,$$

then $(\alpha_o, 0)$ is a non-singular Γ-symmetric steady-state solution, i.e. $(\alpha_o, 0)$ satisfies the condition (C2) from Subsection 7.2.

A number $\lambda \in \mathbb{C}$ is a characteristic root of the system (216) at a Γ-symmetric steady-state solution $(\alpha, 0) \in \mathbb{R} \oplus V$ if there exists a non-zero function $v \in L^2([0,\pi]; V^c)$ such that

$$\Delta_\alpha(\lambda)v(x) := \lambda v(x) - \frac{\partial^2}{\partial x^2}v(x) + \alpha \mathrm{e}^{-\lambda} C v(x) = 0, \tag{221}$$

where we put $\Delta_\alpha := \Delta_{\alpha;0}$ (cf. (174)).

By using the decomposition (220), v can be written as $v(x) = \sum_{k=0}^{5} v_k(x)$, for $v_k(x) \in E(\xi_k)$. Consequently, (221) yields

$$\Delta_\alpha(\lambda)v(x) = \sum_k \left(\lambda v_k(x) - \frac{\partial^2}{\partial x^2}v_k(x) + \alpha \mathrm{e}^{-\lambda}\xi_k v_k(x)\right) = 0. \tag{222}$$

Next, by using the point spectrum $\{\zeta_m := m^2\}_{m=0}^{\infty}$ of the (scalar-valued) Laplace operator $L := -\frac{\partial^2}{\partial x^2}$ and the corresponding eigenspaces $E(\zeta_m)$, we can write $v_k(x) = \sum_m v_{k,m}(x)$, for $v_{k,m} \in E(\zeta_m)$, thus

$$\Delta_\alpha(\lambda)v(x) = \sum_{k,m}\bigl(\lambda v_{k,m}(x) + m^2 v_{k,m}(x) + \alpha \mathrm{e}^{-\lambda}\xi_k v_{k,m}(x)\bigr) = 0. \tag{223}$$

Therefore, $\lambda \in \mathbb{C}$ is a characteristic root of (216) at the Γ-symmetric steady-state solution $(\alpha, 0)$, if

$$\lambda + m^2 + \alpha\xi_k \mathrm{e}^{-\lambda} = 0, \quad \text{for } k = 1, \ldots, 5 \text{ and } m = 0, 1, \ldots. \tag{224}$$

Computations for the local bifurcation $\Gamma \times S^1$*-invariant* In order to find the values α_o for which the condition (C3) from Subsection 7.2 holds, we need to find purely imaginary roots $\lambda = \mathrm{i}\beta$ $(\beta > 0)$ of (224). Assume that $(\alpha, 0)$ is a non-singular Γ-symmetric steady-state solution to (216) (in particular, $\alpha \neq 0$).

• **Computation for purely imaginary roots** $\lambda = \mathrm{i}\beta$ $(\beta > 0)$

By substituting $\lambda = \mathrm{i}\beta$ into (224),

$$\begin{cases} m^2 + \alpha\xi_k \cos\beta = 0, \\ \beta - \alpha\xi_k \sin\beta = 0, \end{cases} \quad \text{for } k = 0, \ldots, 5. \tag{225}$$

In the case $m = 0$,

$$\begin{cases} \beta := \beta_{\nu,0,k} = \dfrac{\pi}{2} + \nu\pi, \\ \alpha := \alpha_{\nu,0,k} = (-1)^\nu \dfrac{\beta}{\xi_k}, \end{cases}$$

for $k = 0, \ldots, 5$ and $\nu = 0, 1, \ldots$. Consequently,

$$\operatorname{sign}\alpha_{\nu,0,k} = (-1)^\nu \operatorname{sign}\xi_k. \tag{226}$$

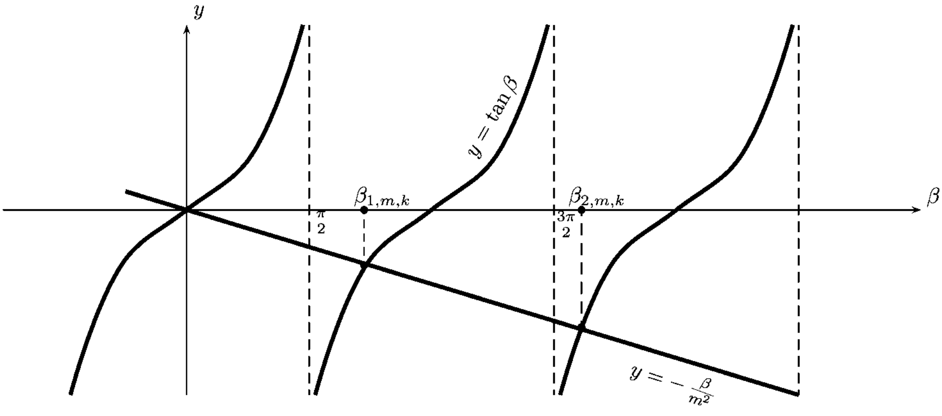

Fig. 9. Purely imaginary roots of the characteristic equation.

In the case $m \neq 0$ (thus $\cos\beta \neq 0$ by the first equation in (225)),

$$\tan\beta = -\frac{\beta}{m^2}, \tag{227}$$

$$\alpha = -\frac{m^2}{\xi_k \cos\beta}, \tag{228}$$

Eq. (227) has infinitely many positive solutions, which are denoted by $\{\beta_{\nu,m,k}\}_{\nu=1}^{\infty}$ (see Fig. 9). The corresponding solution α of (228) is denoted by $\alpha_{\nu,m,k}$. In Table 5 we list approximate values of the roots $(\alpha_{\nu,m,k}, \beta_{\nu,m,k})$ for $m = 0, 1, 2, 3$ and $\nu = 0, 1, 2, 3, 4$.

Also, notice that $\operatorname{sign}\cos\beta_{\nu,m,k} = (-1)^\nu$, thus by (228) for $m > 0$,

$$\operatorname{sign}\alpha_{\nu,m,k} = (-1)^{\nu+1}\operatorname{sign}\xi_k. \tag{229}$$

• **Computation for** $\operatorname{sign}\frac{\mathrm{d}}{\mathrm{d}\alpha}w(\alpha)|_{\alpha=\alpha_{\nu,m,k}}$

Put $\alpha_o := \alpha_{\nu,m,k}$ and $\beta_o := \beta_{\nu,m,k}$. In order to determine the value of the crossing number $t_{j,1}(\alpha_o, \beta_o, 0)$, compute $\frac{\mathrm{d}}{\mathrm{d}\alpha}w(\alpha)|_{\alpha=\alpha_o}$ by implicit differentiation.

By substituting $\lambda = w + \mathrm{i}v$ into (224), then, differentiating with respect to α, and substituting $\alpha = \alpha_o$, $w = 0$ and $v = \beta_o$, we obtain

$$\operatorname{sign}\frac{\mathrm{d}w}{\mathrm{d}\alpha}|_{\alpha=\alpha_o=\alpha_{\nu,m,k}} = \begin{cases} (-1)^\nu \operatorname{sign}\xi_k, & \text{if } m = 0, \\ (-1)^{\nu+1}\operatorname{sign}\xi_k, & \text{if } m = 1, 2, \ldots. \end{cases} \tag{230}$$

Therefore, for $m = 0$

$$\mathfrak{t}_{j,1}(\alpha_o, \beta_o) = \begin{cases} (-1)^{\nu+1}\operatorname{sign}\xi_k m_j(\mathrm{i}\beta_o), & j = j_{\beta_o}, \\ 0, & j \neq j_{\beta_o}, \end{cases} \tag{231}$$

Table 5
List of approximate values of several bifurcation points (α_o, β_o) for (216)

(ν, m)	$(\alpha_{\nu,m,0}, \beta_{\nu,m,0})$	$(\alpha_{\nu,m,1}, \beta_{\nu,m,1})$	$(\alpha_{\nu,m,2}, \beta_{\nu,m,2})$	$(\alpha_{\nu,m,3}, \beta_{\nu,m,3})$	$(\alpha_{\nu,m,4}, \beta_{\nu,m,4})$	$(\alpha_{\nu,m,5}, \beta_{\nu,m,5})$
(0,0)	(0.224, 1.57)	(1.57, 1.57)	(−0.523, 1.57)	(0.523, 1.57)	(0.286, 1.57)	(−0.451, 1.57)
(1,0)	(−0.673, 4.71)	(−4.71, 4.71)	(1.57, 4.71)	(−1.57, 4.71)	(−0.857, 4.71)	(1.35, 4.71)
(2,0)	(1.12, 7.85)	(7.85, 7.85)	(−2.62, 7.85)	(2.62, 7.85)	(1.43, 7.85)	(−2.25, 7.85)
(3,0)	(−1.57, 11.0)	(−11.0, 11.0)	(3.67, 11.0)	(−3.67, 11.0)	(−2.00, 11.0)	(3.16, 11.0)
(4,0)	(2.01, 14.1)	(14.1, 14.1)	(−4.70, 14.1)	(4.70, 14.1)	(2.57, 14.1)	(−4.05, 14.1)
(1,1)	(0.323, 2.03)	(2.26, 2.03)	(−0.752, 2.03)	(0.752, 2.03)	(0.411, 2.03)	(−0.649, 2.03)
(2,1)	(−0.730, 4.91)	(−5.10, 4.91)	(1.70, 4.91)	(−1.70, 4.91)	(−0.928, 4.91)	(1.46, 4.91)
(3,1)	(1.13, 7.98)	(7.94, 7.98)	(−2.64, 7.98)	(2.64, 7.98)	(1.45, 7.98)	(−2.28, 7.98)
(4,1)	(−1.38, 11.1)	(−9.62, 11.1)	(3.20, 11.1)	(−3.20, 11.1)	(−1.75, 11.1)	(2.76, 11.1)
(1,2)	(0.679, 2.57)	(4.76, 2.57)	(−1.58, 2.57)	(1.58, 2.57)	(0.866, 2.57)	(−1.37, 2.57)
(2,2)	(−0.960, 5.35)	(−6.72, 5.35)	(2.24, 5.35)	(−2.24, 5.35)	(−1.22, 5.35)	(1.93, 5.35)
(3,2)	(1.32, 8.30)	(9.28, 8.30)	(−3.09, 8.30)	(3.09, 8.30)	(1.69, 8.30)	(−2.66, 8.30)
(4,2)	(−1.90, 11.3)	(−13.3, 11.3)	(4.43, 11.3)	(−4.43, 11.3)	(−2.42, 11.3)	(3.82, 11.3)
(1,3)	(1.35, 2.84)	(9.42, 2.84)	(−3.14, 2.84)	(3.14, 2.84)	(1.71, 2.84)	(−2.70, 2.84)
(2,3)	(−1.52, 5.72)	(−10.6, 5.72)	(3.55, 5.72)	(−3.55, 5.72)	(−1.93, 5.72)	(3.04, 5.72)
(3,3)	(1.79, 8.66)	(12.5, 8.66)	(−4.16, 8.66)	(4.16, 8.66)	(2.28, 8.66)	(−3.59, 8.66)
(4,3)	(−1.99, 11.7)	(−13.9, 11.7)	(4.63, 11.7)	(−4.63, 11.7)	(−2.53, 11.7)	(3.99, 11.7)

and for $m \neq 0$

$$\mathfrak{t}_{j,1}(\alpha_o, \beta_o) = \begin{cases} (-1)^\nu \operatorname{sign} \xi_k \, m_j(\mathrm{i}\beta_o), & j = j_{\beta_o}, \\ 0, & j \neq j_{\beta_o}. \end{cases} \tag{232}$$

On the other hand, we have

$$m_j(\mathrm{i}\beta_{\nu,m,k}) = m_j(\xi_k) = \begin{cases} 1 & \text{for } j = k,\ k \neq 1, \\ 2 & \text{for } j = k,\ k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Negative spectrum σ_- For a given bifurcation point $(\alpha_o, \beta_o, 0)$, we need to determine the negative spectrum σ_- of the operator $\mathcal{A}$ (given by (180) with $u_o = 0$). One can easily verify that the spectrum $\sigma(\mathcal{A})$ of the operator $\mathcal{A}$ is given by

$$\sigma(\mathcal{A}) = \left\{ \zeta_{m,k} := \frac{m^2 + \alpha_o \xi_k}{m^2 + 1} \colon m = 0, 1, 2, 3, \dots; k = 0, 1, \dots, 5 \right\}$$

where the isotypical multiplicity of $\zeta_{m,k}$ is the same as that of ξ_k. Consequently, we can define

$$m_i := \sum_{\mu \in \sigma_-} m_i(\mu), \quad i = 0, 1, 2, 3, 4, 5.$$

For the considered bifurcation points, the values $(m_0, m_1, m_2, m_3, m_4, m_5)$ are listed in Table 6.

Table 6
Numbers $(m_0, m_1, m_2, m_3, m_4)$ for the bifurcation points $(\alpha_{\nu,m,k}, \beta_{\nu,m,k})$

(ν, m, k)	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$(0, 0, k)$	[0, 1, 0, 0, 1]	[0, 3, 0, 0, 3]	[2, 1, 2, 2, 0]	[0, 2, 0, 0, 2]	[0, 1, 0, 0, 1]	[2, 1, 2, 2, 0]
$(1, 0, k)$	[3, 1, 2, 2, 0]	[6, 3, 4, 6, 0]	[0, 3, 0, 0, 3]	[4, 2, 3, 3, 0]	[3, 1, 2, 3, 0]	[0, 3, 0, 0, 3]
$(2, 0, k)$	[0, 2, 0, 0, 2]	[0, 5, 0, 0, 6]	[5, 2, 3, 4, 0]	[0, 3, 0, 0, 4]	[0, 3, 0, 0, 3]	[4, 2, 3, 4, 0]
$(3, 0, k)$	[4, 2, 3, 3, 0]	[8, 4, 6, 8, 0]	[0, 4, 0, 0, 4]	[6, 2, 4, 5, 0]	[4, 2, 3, 4, 0]	[0, 4, 0, 0, 4]
$(4, 0, k)$	[0, 3, 0, 0, 3]	[0, 7, 0, 0, 8]	[6, 3, 4, 6, 0]	[0, 4, 0, 0, 5]	[0, 3, 0, 0, 3]	[6, 3, 4, 5, 0]
$(1, 1, k)$	[0, 1, 0, 0, 2]	[0, 3, 0, 0, 3]	[3, 1, 2, 3, 0]	[0, 2, 0, 0, 2]	[0, 2, 0, 0, 2]	[3, 1, 2, 2, 0]
$(2, 1, k)$	[3, 1, 2, 2, 0]	[6, 3, 4, 6, 0]	[0, 3, 0, 0, 3]	[4, 2, 3, 4, 0]	[3, 1, 2, 3, 0]	[0, 3, 0, 0, 3]
$(3, 1, k)$	[0, 2, 0, 0, 2]	[0, 5, 0, 0, 6]	[5, 2, 3, 4, 0]	[0, 3, 0, 0, 4]	[0, 3, 0, 0, 3]	[4, 2, 3, 4, 0]
$(4, 1, k)$	[4, 2, 3, 3, 0]	[8, 4, 6, 8, 0]	[0, 4, 0, 0, 4]	[5, 2, 4, 5, 0]	[4, 2, 3, 4, 0]	[0, 3, 0, 0, 4]
$(1, 2, k)$	[0, 2, 0, 0, 2]	[0, 4, 0, 0, 5]	[4, 2, 3, 3, 0]	[0, 3, 0, 0, 3]	[0, 2, 0, 0, 2]	[4, 2, 3, 3, 0]
$(2, 2, k)$	[3, 1, 2, 3, 0]	[7, 3, 5, 7, 0]	[0, 3, 0, 0, 3]	[4, 2, 3, 4, 0]	[3, 2, 2, 3, 0]	[0, 3, 0, 0, 3]
$(3, 2, k)$	[0, 2, 0, 0, 3]	[0, 6, 0, 0, 6]	[5, 0, 4, 5, 0]	[0, 4, 0, 0, 4]	[0, 3, 0, 0, 3]	[5, 2, 3, 4, 0]
$(4, 2, k)$	[4, 2, 3, 4, 0]	[8, 4, 7, 8, 0]	[0, 4, 0, 0, 4]	[6, 3, 4, 5, 0]	[5, 2, 3, 4, 0]	[0, 4, 0, 0, 4]
$(1, 3, k)$	[0, 3, 0, 0, 3]	[0, 6, 0, 0, 6]	[5, 2, 4, 5, 0]	[0, 4, 0, 0, 4]	[0, 3, 0, 0, 3]	[5, 2, 3, 4, 0]
$(2, 3, 0)$	[4, 2, 3, 3, 0]	[8, 4, 6, 8, 0]	[0, 4, 0, 0, 4]	[5, 2, 4, 5, 0]	[4, 2, 3, 4, 0]	[0, 4, 0, 0, 4]
$(3, 3, 0)$	[0, 3, 0, 0, 3]	[0, 7, 0, 0, 7]	[6, 3, 4, 5, 0]	[0, 4, 0, 0, 4]	[0, 3, 0, 0, 3]	[6, 2, 4, 5, 0]
$(4, 3, 0)$	[4, 2, 3, 4, 0]	[8, 4, 7, 8, 0]	[0, 4, 0, 0, 5]	[6, 3, 4, 6, 0]	[5, 2, 3, 4, 0]	[0, 4, 0, 0, 4]

Consequently, in order to compute the value of the local bifurcation invariant $\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k})$ we need to use the following Maple© procedure

$$\text{showdegree[A5]}(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5)$$

where $\varepsilon_i = m_i \bmod 2$, and (since only ξ_2 and ξ_5 are negative)

$$\mathfrak{t}_j := \mathfrak{t}_j(\alpha_{\nu,m,k}, \beta_{\nu,m,k}) = \begin{cases} (-1)^\nu & \text{if } j = k,\ k = 0, 3, 4, \\ (-1)^{\nu+1} & \text{if } j = k,\ k = 2, 5, \\ (-1)^\nu 2 & \text{if } j = k,\ k = 1. \end{cases}$$

Preparation of input data for the Maple© routines Based on the above computations we are now in a position to prepare the input data for the Maple© routines in order to evaluate (for selected bifurcation points $(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)$) the exact value of the bifurcation invariant

$$\begin{aligned} &\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}) \\ &\quad = \texttt{showdegree[A5]}(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5), \end{aligned}$$

which is presented in Table 7.

Table of results In Table 8, we present the classification of the A_5-symmetric Hopf bifurcation in (216) for selected bifurcation points $(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)$.

Table 7
Input data $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5)$ for selected bifurcation points $(\alpha_{\nu,m,k}, \beta_{\nu,m,k})$

$(\alpha_{\nu,m,k}, \beta_{\nu,m,k})$	(ν, m, k)	$\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$	$\mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4$
(0.224, 1.57)	(0, 0, 0)	0, 1, 0, 0, 1	−1, 0, 0, 0, 0
(0.286, 1.57)	(0, 0, 4)	0, 1, 0, 0, 1	0, 0, 0, −1, 0
(−0.673, 4.71)	(1, 0, 0)	1, 1, 0, 0, 0	−1, 0, 0, 0, 0
(1.57, 4.71)	(1, 0, 2)	0, 0, 1, 1, 0	0, 1, 0, 0, 0
(1.35, 4.71)	(1, 0, 5)	0, 1, 0, 0, 1	0, 0, 0, 0, 1
(7.85, 7.85)	(2, 0, 1)	0, 1, 0, 0, 0	0, −1, 0, 0, 0
(1.43, 7.85)	(2, 0, 4)	0, 1, 0, 0, 1	0, 0, 0, −1, 0
(−11.0, 11.0)	(3, 0, 1)	0, 0, 0, 0, 0	0, 1, 0, 0, 0
(−2.00, 11.0)	(3, 0, 4)	0, 0, 1, 0, 0	0, 0, 0, 1, 0
(2.01, 14.1)	(4, 0, 0)	0, 1, 0, 0, 1	−1, 0, 0, 0, 0
(4.70, 14.1)	(4, 0, 3)	0, 0, 0, 0, 1	0, 0, −1, 0, 0
(0.323, 2.03)	(1, 1, 0)	0, 1, 0, 0, 0	0, 0, −1, 0, 0
(−0.649, 2.03)	(1, 1, 5)	1, 1, 0, 0, 0	0, 0, 0, 0, 1
(1.70, 4.91)	(2, 1, 2)	0, 1, 0, 0, 1	0, 1, 0, 0, 0
(−0.928, 4.91)	(2, 1, 4)	1, 1, 0, 1, 0	0, 0, 0, −1, 0
(7.94, 7.98)	(3, 1, 1)	0, 1, 0, 0, 0	0, 1, 0, 0, 0
(1.45, 7.98)	(3, 1, 4)	0, 1, 0, 0, 1	0, 0, 0, 1, 0
(−3.20, 11.1)	(4, 1, 3)	0, 0, 0, 0, 0	0, 0, −1, 0, 0
(2.76, 11.1)	(4, 1, 5)	0, 1, 0, 0, 0	0, 0, 0, 0, 1
(4.76, 2.57)	(1, 2, 1)	0, 0, 1, 1, 0	0, 1, 0, 0, 0
(−1.37, 2.57)	(1, 2, 5)	0, 0, 1, 1, 0	0, 0, 0, 0, 1
(2.24, 5.35)	(2, 2, 2)	0, 1, 0, 0, 1	0, 1, 0, 0, 0
(1.93, 5.35)	(2, 2, 5)	0, 1, 0, 0, 1	0, 0, 0, 0, 1
(−3.09, 8.30)	(3, 2, 2)	1, 0, 0, 1, 0	0, −1, 0, 0, 0
(1.69, 8.30)	(3, 2, 4)	0, 1, 0, 0, 1	0, 0, 0, 1, 0
(−2.42, 11.3)	(4, 2, 4)	1, 0, 1, 0, 0	0, 0, 0, −1, 0
(3.82, 11.3)	(4, 2, 5)	0, 0, 0, 0, 0	0, 0, 0, 0, 1
(−3.14, 2.84)	(1, 3, 2)	1, 0, 0, 1, 0	0, −1, 0, 0, 0
(−2.70, 2.84)	(1, 3, 5)	1, 0, 1, 0, 0	0, 0, 0, 0, −1
(3.55, 5.72)	(2, 3, 2)	0, 0, 0, 0, 0	0, 1, 0, 0, 0
(−1.93, 5.72)	(2, 3, 4)	0, 0, 1, 0, 0	0, 0, 0, −1, 0
(4.16, 8.66)	(3, 3, 3)	0, 0, 0, 0, 0	0, 0, 1, 0, 0
(−3.59, 8.66)	(3, 3, 5)	0, 0, 0, 1, 0	0, 0, 0, 0, −1
(−4.63, 11.7)	(4, 3, 3)	0, 1, 0, 0, 0	0, 0, −1, 0, 0
(−2.53, 11.7)	(4, 3, 4)	1, 0, 1, 0, 0	0, 0, 0, −1, 0

8.5. *Bibliographical remarks*

The model presented in Subsection 8.1 was rigorously studied in [11] . Symmetric configuration of transmission lines presented in Subsection 8.2, was analyzed in details in [15] The global bifurcation results given in Subsection 8.3 were published in [13]. For more details we refer to [19]. Similar systems of symmetric DEs and FDEs were also considered in [67,65,66,68,92,94,95,129,140,145,146,148]. In the non-symmetric case, local and global Hopf bifurcation problems in FDEs were studied in [36,110,111,124,125,122,123].

The results presented in Subsection 8.4 were taken from [16]. For population ecology background, which was used in this section, we refer to [73,136,56,70]. An equation, sim-

Table 8
Classification of the A_5-symmetric Hopf bifurcation in (216) for selected bifurcation points $(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)$

$(\alpha_{\nu,m,k}, \beta_{\nu,m,k})$	(ν, m, k)	$\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k})$	# branches
(0.224, 1.57)	(0, 0, 0)	$-(A_5) + 2(A_4) + 2(D_3) + (\mathbb{Z}_5) - 2(\mathbb{Z}_3) - 2(\mathbb{Z}_2) + (\mathbb{Z}_1)$	1
(0.286, 1.57)	(0, 0, 4)	$-(D_5^z) + (D_3^z) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5) + (V_4^-) + (\mathbb{Z}_3^t) - (\mathbb{Z}_1)$	48
(−0.673, 4.71)	(1, 0, 0)	$(A_5) - 2(A_4) - 2(D_3) + 3(\mathbb{Z}_3) + 3(\mathbb{Z}_2) - 2(\mathbb{Z}_1)$	1
(1.57, 4.71)	(1, 0, 2)	$(A_4) - (D_3^z) - (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3)$ $+ (\mathbb{Z}_2^-) + (\mathbb{Z}_2) - 2(\mathbb{Z}_1)$	55
(1.35, 4.71)	(1, 0, 5)	$(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_2}) - (\mathbb{Z}_5) - (V_4^-) - (\mathbb{Z}_3^t) + (\mathbb{Z}_1)$	48
(7.85, 7.85)	(2, 0, 1)	$(A_4) + (D_3^z) + (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) + (V_4^-) - (\mathbb{Z}_3^t)$ $- 3(\mathbb{Z}_3) - 3(\mathbb{Z}_2^-) - 3(\mathbb{Z}_2) + 4(\mathbb{Z}_1)$	55
(1.43, 7.85)	(2, 0, 4)	$-(D_5^z) + (D_3^z) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5) + (V_4^-) + (\mathbb{Z}_3^t) - (\mathbb{Z}_1)$	48
(−11.0, 11.0)	(3, 0, 1)	$(A_4) + (D_3^z) + (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (V_4^-) + (\mathbb{Z}_3^t)$ $- (\mathbb{Z}_3) - (\mathbb{Z}_2^-) - (\mathbb{Z}_2)$	55
(−2.00, 11.0)	(3, 0, 4)	$-(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_1}) + (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_2) + (\mathbb{Z}_1)$	48
(2.01, 14.1)	(4, 0, 0)	$-(A_5) + 2(A_4) + 2(D_3) + (\mathbb{Z}_5) - 2(\mathbb{Z}_3) - 2(\mathbb{Z}_2) + (\mathbb{Z}_1)$	1
(4.70, 14.1)	(4, 0, 3)	$-(A_4^{t_1}) - (A_4^{t_2}) - (D_5) - (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2})$ $+ (\mathbb{Z}_5) - (V_4^-) + 4(\mathbb{Z}_3^t) + (\mathbb{Z}_3) + 2(\mathbb{Z}_2^-) + 5(\mathbb{Z}_2) - 5(\mathbb{Z}_1)$	50
(0.323, 2.03)	(1, 1, 0)	$(A_4^{t_1}) + (A_4^{t_2}) - (D_5) + (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2})$ $+ (V_4^-) - 4(\mathbb{Z}_3^t) - (\mathbb{Z}_3) - 2(\mathbb{Z}_2^-) - 3(\mathbb{Z}_2) + 4(\mathbb{Z}_1)$	50
(−0.649, 2.03)	(1, 1, 5)	$-(D_5^z) + (D_3^z) - (\mathbb{Z}_5^{t_2}) + (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3) - 2(\mathbb{Z}_2^-)$ $- (\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	48
(1.70, 4.91)	(2, 1, 2)	$-(A_4) - (D_3^z) - (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (V_4^-) - (\mathbb{Z}_3^t)$ $+ (\mathbb{Z}_3) + (\mathbb{Z}_2^-) + (\mathbb{Z}_2)$	55
(−0.928, 4.91)	(2, 1, 4)	$(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5) - (V_4^-) - (\mathbb{Z}_3^t) + (\mathbb{Z}_1)$	48
(7.94, 7.98)	(3, 1, 1)	$-(A_4) - (D_3^z) - (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) - (V_4^-) + (\mathbb{Z}_3^t)$ $+ 3(\mathbb{Z}_3) + 3(\mathbb{Z}_2^-) + 3(\mathbb{Z}_2) - 4(\mathbb{Z}_1)$	55
(1.45, 7.98)	(3, 1, 4)	$(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5) - (V_4^-) - (\mathbb{Z}_3^t) + (\mathbb{Z}_1)$	48
(−3.20, 11.1)	(4, 1, 3)	$-(A_4^{t_1}) - (A_4^{t_2}) - (D_5) - (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2})$ $- (V_4^-) + 2(\mathbb{Z}_2)$	50
(2.76, 11.1)	(4, 1, 5)	$(D_5^z) - (D_3^z) + (\mathbb{Z}_5^{t_2}) - (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3) + 2(\mathbb{Z}_2^-)$ $+ (\mathbb{Z}_2) - 2(\mathbb{Z}_1)$	48
(4.76, 2.57)	(1, 2, 1)	$(A_4) - (D_3^z) - (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (V_4^-) + (\mathbb{Z}_3^t)$ $+ (\mathbb{Z}_3) + (\mathbb{Z}_2^-) + (\mathbb{Z}_2) - 2(\mathbb{Z}_1)$	55
(−1.37, 2.57)	(1, 2, 5)	$-(D_5^z) - (D_3^z) + (\mathbb{Z}_5^{t_2}) + (\mathbb{Z}_5) + (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3)$ $+ 2(\mathbb{Z}_2^-) - 2(\mathbb{Z}_1)$	48
(2.24, 5.35)	(2, 2, 2)	$-(A_4) - (D_3^z) - (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (V_4^-) - (\mathbb{Z}_3^t)$ $+ (\mathbb{Z}_3) + (\mathbb{Z}_2^-) + (\mathbb{Z}_2)$	55
(1.93, 5.35)	(2, 2, 5)	$(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_2}) - (\mathbb{Z}_5) - (V_4^-) - (\mathbb{Z}_3^t) + (\mathbb{Z}_1)$	48
(−3.09, 8.30)	(3, 2, 2)	$(A_4) + (D_3^z) + (D_3)_-(\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) + (V_4^-) - (\mathbb{Z}_3^t)$ $- 3(\mathbb{Z}_3) - 3(\mathbb{Z}_2^-) - 3(\mathbb{Z}_2) + 4(\mathbb{Z}_1)$	55
(1.69, 8.30)	(3, 2, 4)	$(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5) - (V_4^-) - (\mathbb{Z}_3^t) + (\mathbb{Z}_1)$	48
(−2.42, 11.3)	(4, 2, 4)	$-(D_5^z - (D_3^z) - (\mathbb{Z}_5^{t_1}) + (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_2) + (\mathbb{Z}_1)$	48
(3.82, 11.3)	(4, 2, 5)	$(D_5^z) + (D_3^z) + (\mathbb{Z}_5^{t_2}) + (V_4^-) + (\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-)$	48

(*continued on next page*)

Table 8
(*continued*)

$(\alpha_{\nu,m,k}, \beta_{\nu,m,k})$	(ν, m, k)	$\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k})$	# branches
(−3.14, 2.84)	(1, 3, 2)	$(A_4) + (D_3^z) + (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) + (V_4^-) - (\mathbb{Z}_3^t)$ $- 3(\mathbb{Z}_3) - 3(\mathbb{Z}_2^-) - 3(\mathbb{Z}_2) + 4(\mathbb{Z}_1)$	55
(−2.70, 2.84)	(1, 3, 5)	$-(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_2}) + (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_2) + (\mathbb{Z}_1)$	48
(3.55, 5.72)	(2, 3, 2)	$(A_4) + (D_3^z) + (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (V_4^-) + (\mathbb{Z}_3^t)$ $- (\mathbb{Z}_3) - (\mathbb{Z}_2^-) - (\mathbb{Z}_2)$	55
(−1.93, 5.72)	(2, 3, 4)	$(D_5^z) + (D_3^z) + (\mathbb{Z}_5^{t_1}) - (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_2) - (\mathbb{Z}_1)$	48
(4.16, 8.66)	(3, 3, 3)	$(A_4^{t_1}) + (A_4^{t_2}) + (D_5) + (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (V_4^-) - 2(\mathbb{Z}_2)$	50
(−3.59, 8.66)	(3, 3, 5)	$-(D_5^z) - (D_3^z) + (\mathbb{Z}_5^{t_2}) + (\mathbb{Z}_5) - (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3)$ $+ 4(\mathbb{Z}_2^-) + (\mathbb{Z}_2) - 3(\mathbb{Z}_1)$	48
(−4.63, 11.7)	(4, 3, 3)	$(A_4^{t_1}) + (A_4^{t_2}) - (D_5) + (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2})$ $+ (V_4^-) - 4(\mathbb{Z}_3^t) - (\mathbb{Z}_3) - 2(\mathbb{Z}_2^-) - 3(\mathbb{Z}_2) + 4(\mathbb{Z}_1)$	50
(−2.53, 11.7)	(4, 3, 4)	$-(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_1}) + (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_2) + (\mathbb{Z}_1)$	48

ilar to (215) (for a single species in one dimensional space), was also studied in [145] for the stability property of the periodic solutions bifurcating from 0.

Appendix A

Dihedral group D_N

Represent the dihedral group D_N of order $2N$ as the group of rotations $1, \xi, \xi^2, \dots, \xi^{N-1}$ of the complex plane (where ξ is the multiplication by $e^{\frac{2\pi i}{N}}$) plus the reflections κ, $\kappa\xi$, $\kappa\xi^2, \dots, \kappa\xi^{N-1}$ with κ being the operator of complex conjugation described by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Let us introduce some special subgroups of D_N and of $D_N \times S^1$. For a positive integer k with $k|N$, and $\gamma := e^{\frac{2\pi i}{k}}$, we have:

(i) the subgroups $D_k = \{1, \gamma, \gamma^2, \dots, \gamma^{k-1}, \kappa, \kappa\gamma, \dots, \kappa\gamma^{k-1}\}$ and their isomorphic copies

$$D_{k,j} = \{1, \gamma, \gamma^2, \dots, \gamma^{k-1}, \kappa\xi^j, \kappa\xi^j\gamma, \dots, \kappa\xi^j\gamma^{k-1}\} \subset D_N,$$

$j = 0, 1, \dots, \frac{N}{k} - 1$, which are all conjugate if $\frac{N}{k}$ is odd, but split into two conjugacy classes (D_k) and $(\widetilde{D}_k)$, where $\widetilde{D}_k := D_{k,1}$, if $\frac{N}{k}$ is even;

(ii) the twisted subgroups of $\mathbb{Z}_k = \{1, \gamma, \gamma^2, \dots, \gamma^{k-1}\}$, which are the subgroups

$$\mathbb{Z}_k^{t_r} := \big\{(1,1), (\gamma, \gamma^r), (\gamma^2, \gamma^{2r}), \dots, (\gamma^{k-1}, \gamma^{(k-1)r})\big\},$$

with $r \in \{0, 1, \dots, k-1\}$. Obviously, for $0 < r < \frac{k}{2}$, $\kappa \mathbb{Z}_k^{t_r} \kappa = \mathbb{Z}_k^{t_{k-r}}$, i.e. $\mathbb{Z}_k^{t_r}$ and $\mathbb{Z}_k^{t_{k-r}}$ are conjugate;

(iii) the twisted subgroups

$$D_k^z = \big\{(1,1), (\gamma,1), \dots, (\gamma^{k-1},1), (\kappa,-1), (\kappa\gamma,-1), \dots, (\kappa\gamma^{k-1},-1)\big\},$$

of D_k (and similarly for the $D_{k,j}$ and, in particular, for $\widetilde{D}_k$);

(iv) another set of twisted subgroups in the case $k = 2m$:

First,

$$D_{2m}^d = \big\{(1,1), (\gamma,-1), (\gamma^2,1), \dots, (\gamma^{k-1},-1), (\kappa,1), (\kappa\gamma,-1), \dots, (\kappa\gamma^{k-1},-1)\big\}$$

and similarly

$$\widetilde{D}_{2m}^d = \big\{(1,1), (\gamma,-1), (\gamma^2,1), \dots, (\gamma^{k-1},-1), (\kappa\xi,1), (\kappa\xi\gamma,-1), \dots, (\kappa\xi\gamma^{k-1},-1)\big\}.$$

If $\frac{N}{2m} =: l$ is even, then

$$\begin{aligned} D_{2m}^{\hat{d}} &:= \big\{(1,1), (\gamma,-1), (\gamma^2,1), \dots, (\gamma^{k-1},-1), (\kappa,-1), (\kappa\gamma,1), \\ &\quad \dots, (\kappa\gamma^{k-1},1)\big\} \\ &= \mathrm{e}^{-\frac{2\pi\mathrm{i}}{4m}} D_{2m}^d \mathrm{e}^{\frac{2\pi\mathrm{i}}{4m}} = \xi^{-\frac{l}{2}} D_{2m}^d \xi^{\frac{l}{2}}, \end{aligned}$$

i.e. $D_{2m}^{\hat{d}}$ and D_{2m}^d are conjugate, but not in the case of $l := \frac{N}{2m}$ being odd. Observe that for $\frac{N}{2m} \in \{3, 5, 7, \dots\}$,

$$\xi^{-\frac{1}{2}(\frac{N}{2m}+1)} D_{2m}^{\hat{d}} \xi^{\frac{1}{2}(\frac{N}{2m}+1)} = \widetilde{D}_{2m}^d, \quad \text{i.e.} \quad D_{2m}^{\hat{d}} \sim \widetilde{D}_{2m}^d;$$

(v) the subgroup, for $k = 2m$,

$$\mathbb{Z}_{2m}^d := \mathbb{Z}_{2m}^{t_m} = \big\{(1,1), (\gamma,-1), (\gamma^2,1), \dots, (\gamma^{2m-1},-1)\big\}.$$

In the case $m = 1$ we will write $\mathbb{Z}_2^-$ instead of $\mathbb{Z}_2^d$.

Irreducible representations of dihedral groups

Let us describe the list of all real irreducible representations of D_n.

(a0) Clearly, there is a one-dimensional trivial representation $\mathcal{V}_0$. In this case,

$$\deg_{\mathcal{V}_0} = -(D_n).$$

(a1) For every integer number $1 \leqslant j < n/2$, there is an orthogonal representation $\mathcal{V}_j$ of D_n on $\mathbb{C}$ given by

$$\begin{aligned} &\gamma z := \gamma^j \cdot z, \quad \text{for } \gamma \in \mathbb{Z}_n \text{ and } z \in \mathbb{C};\\ &\kappa z := \bar{z}, \end{aligned}$$

where $\gamma^j \cdot z$ denotes the usual complex multiplication. Put

$$h := \gcd(j, n) \quad \text{and} \quad m := n/h. \tag{233}$$

Put $j_n := [(n+1)/2]$.

(a2) There is a representation $\mathcal{V}_{j_n}$ given by the homomorphism $c : D_n \to \mathbb{Z}_2$, such that $\ker c = \mathbb{Z}_n$.

(a3) For n even, there is an irreducible representation $\mathcal{V}_{j_n+1}$ given by $d : D_n \to \mathbb{Z}_2$ such that $\ker d = D_{n/2}$.

(a4) Also, for n even, there is an irreducible representation $\mathcal{V}_{j_n+2}$ given by $\hat{d} : D_n \to \mathbb{Z}_2$, where $\ker \hat{d} = \widetilde{D}_{n/2}$.

Icosahedral group A_5

Consider the icosahedral group $\mathbb{I}$ consisting of symmetries (preserving orientation) of a regular icosahedron. The group $\mathbb{I}$ is isomorphic to the alternating group of five elements A_5 and it has 60 elements. Let us list (nontrivial) representatives of the conjugacy classes of the subgroups in A_5:

$$\begin{aligned} \mathbb{Z}_2 &= \{(1), (12)(34)\},\\ \mathbb{Z}_3 &= \{(1), (123), (132)\},\\ V_4 &= \{(1), (12)(34), (13)(24), (23)(14)\},\\ \mathbb{Z}_5 &= \{(1), (12345), (13524), (14253), (15432)\},\\ D_3 &= \{(1), (123), (132), (12)(45), (13)(45), (23)(45)\},\\ A_4 &= \{(1), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134),\\ &\qquad (143), (234), (243)\},\\ D_5 &= \{(1), (12345), (13524), (14253), (15432), (12)(35), (13)(45), (14)(23),\\ &\qquad (15)(24), (25)(34)\}. \end{aligned}$$

We have the following list, up to conjugacy, of the twisted (one-folded) subgroups H^{φ} of $A_5 \times S^1$, where H is a subgroup of A_5, $\varphi : H \to S^1$ a group homomorphism, and $H^{\varphi} = \{(h, z) \in H \times S^1\colon \varphi(h) = z\}$:

$$
\begin{aligned}
\mathbb{Z}_2^- &= \big\{\big((1), 1\big), \big((12)(34), -1\big)\big\},\\
V_4^- &= \big\{\big((1), 1\big), \big((12)(34), -1\big), \big((13)(24), -1\big), \big((23)(14), 1\big)\big\},\\
\mathbb{Z}_5^{t_1} &= \big\{\big((1), 1\big), \big((12345), \xi\big), \big((13524), \xi^2\big), \big((14253), \xi^3\big), \big((15432), \xi^4\big)\big\},\\
\mathbb{Z}_5^{t_2} &= \big\{\big((1), 1\big), \big((12345), \xi^2\big), \big((13524), \xi^4\big), \big((14253), \xi\big), \big((15432), \xi^3\big)\big\},\\
\mathbb{Z}_3^{t} &= \big\{\big((1), 1\big), \big((123), \gamma\big), \big((132), \gamma^2\big)\big\},\\
D_3^{z} &= \big\{\big((1), 1\big), \big((123), 1\big), \big((132), 1\big), \big((12)(45), -1\big), \big((13)(45), -1\big),\\
&\qquad \big((23)(45), -1\big)\big\},\\
A_4^{t_1} &= \big\{\big((1), 1\big), \big((12)(34), 1\big), \big((13)(24), 1\big), \big((14)(23), 1\big), \big((123), \gamma\big),\\
&\qquad \big((132), \gamma^2\big), \big((124), \gamma^2\big), \big((142), \gamma\big), \big((134), \gamma\big), \big((143), \gamma^2\big),\\
&\qquad \big((234), \gamma^2\big), \big((243), \gamma\big)\big\},\\
A_4^{t_2} &= \big\{\big((1), 1\big), \big((12)(34), 1\big), \big((13)(24), 1\big), \big((14)(23), 1\big), \big((123), \gamma^2\big),\\
&\qquad \big((132), \gamma\big), \big((124), \gamma\big), \big((142), \gamma^2\big), \big((134), \gamma^2\big), \big((143), \gamma\big), \big((234), \gamma\big),\\
&\qquad \big((243), \gamma^2\big)\big\},\\
D_5^{z} &= \big\{\big((1), 1\big), \big((12345), 1\big), \big((13524), 1\big), \big((14253), 1\big), \big((15432), 1\big),\\
&\qquad \big((12)(35), -1\big), \big((13)(45), -1\big), \big((14)(23), -1\big), \big((15)(24), -1\big),\\
&\qquad \big((25)(34), -1\big)\big\},
\end{aligned}
$$

where $\xi = e^{\frac{2\pi}{5}i}$, $\gamma = e^{\frac{2\pi i}{3}}$.

Irreducible representations of A_5

There are exactly 5 irreducible representations of A_5: $\mathcal{V}_0$ – the trivial representation, $\mathcal{V}_1$ – the natural 4-dimensional representation of A_5, $\mathcal{V}_2$ – the 5-dimensional representation of A_5, and two 3-dimensional representations $\mathcal{V}_3$ and $\mathcal{V}_4$. We distinguish these two representations by their characters χ_3 and χ_4, i.e. we assume that

$$
\chi_3\big((12345)\big) = \frac{1+\sqrt{5}}{2}, \qquad \chi_4\big((12345)\big) = \frac{1-\sqrt{5}}{2}.
$$

Notice that all the irreducible Γ-representations considered in this appendix are of real type. Therefore, the corresponding to $\mathcal{V}_j$ irreducible $\Gamma \times S^1$-representation $\mathcal{V}_{j,l}$ can be easily constructed from the formula $\mathcal{V}_{j,l} = {}^{l}\mathcal{V}_j^{c}$.

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CHAPTER 2

Nonautonomous Differential Systems in Two Dimensions

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1. Introduction

In the last four decades, the field of nonautonomous dynamical systems has developed from a loose collection of results and techniques into a coherent area with a recognizable and characteristic set of methods and problems. Among the more classical areas of pure and applied mathematics which it has influenced are spectral theory, control theory, bifurcation theory, functional differential equations, monotone systems, and others as well.

There is a certain amount of conceptual overlap between the fields of nonautonomous dynamical systems and stochastic dynamical systems. In the latter area, one places emphasis on the use of dynamical techniques to study stochastic differential equations. This point of view has been applied systematically and with particular success by L. Arnold and his school. It is natural to think of the two fields – nonautonomous dynamics and stochastic dynamics – as occupying complementary parts of the theory of differential equations with time-varying coefficients. It may not be unreasonable to use a term such as "random dynamics" to cover their union.

Be that as it may, the following considerations allow one to distinguish the two fields.

First of all, a nonautonomous dynamical system is frequently defined in terms of the solutions of a differential equation whose coefficients vary with time, but in a nonstochastic way. The limiting case is that of equations with periodic time dependence; however, one is also interested in equations whose coefficients satisfy Bohr almost periodicity, Birkhoff recurrence, or still weaker recurrence properties. One can express the same idea in a different way by saying that the time dependence of the coefficients exhibits randomness which is "between" periodicity and stochasticity.

A second distinction occurs on the methodological level. Ergodic techniques, for example the Oseledets theory, have an important place in the study of stochastic dynamical systems. This is true in the study of the nonautonomous ones, as well. However, in the latter area, techniques of topological dynamics are of equal importance. There is a reason for this. Namely, a Bebutov-type construction can often be applied to the right-hand side of a nonautonomous differential equation, giving rise to a compact hull. The solutions of the original equation together with those of the equations in the hull then form a continuous dynamical system. Of course one has a vast array of results available to study this system; they are associated with the names of Bronstein, Ellis, Furstenberg, Sell, and many other researchers. These results, however, cannot usually be applied to a stochastic dynamical system since, for such a system, the equivalent of the "hull" has only a measurable structure (but see [69]).

We will explain the Bebutov construction later on. Before doing so, we make some simple observations concerning *autonomous* differential equations, with the purpose of putting the succeeding discussion into perspective.

It is well known that the solutions of an autonomous differential equation

$$x' = f(x), \quad x \in \mathbb{R}^n \tag{1.1}$$

define a local flow on $\mathbb{R}^n$ if f is Lipschitz continuous. They define a global flow if f is complete; that is, if for each $x_0 \in \mathbb{R}^n$, the solution $x(t)$ of (1.1) satisfying $x(0) = x_0$ is defined on the whole real line $-\infty < t < \infty$. This simple observation is very useful if one

wishes to study the asymptotic behavior of the solutions of Eq. (1.1). In particular, a coherent and articulated answer to the question "where do the solutions go?" can be given using the concepts of α- and ω-limit sets, attractors and repellers, basins of attraction/repulsion, etc.

The situation is quite different if f is time dependent: $f = f(t,x)$, $(t \in \mathbb{R}, x \in \mathbb{R}^n)$. It is of course true that, if f is T-periodic for some $T > 0$, then one can introduce the period map defined by solving

$$x' = f(t,x) \tag{1.2}$$

for initial conditions $x(0) \in \mathbb{R}^n$ and continuing them for time T. One thus obtains a discrete flow or discrete dynamical system, at least if all solutions of (1.2) exist for all $t \in \mathbb{R}$. The study of such a system is in many (but not all) ways analogous to that of the continuous dynamical system defined by the solutions of the autonomous equation (1.1).

If f is aperiodic, however, the period map is not available. The fact is that such objects as ω-limit sets do not in general provide much information about the asymptotic behavior of the solutions of (1.2). One can adjoin the equation $t' = 1$ to (1.2), thereby obtaining an autonomous equation on $\mathbb{R} \times \mathbb{R}^n$ which defines a dynamical system on that space. However, all α- and ω-limit sets are empty. Moreover, the t-recurrence properties of f are not reflected in the extended vector field. The addition of the equation $t' = 1$ to (1.2) therefore is unsatisfying if one wishes to use dynamical methods to study Eq. (1.2).

Fortunately, it turns out that very often a simple and direct construction – first studied systematically by Bebutov ([12], also [143]) – permits one to use the full panoply of methods of dynamical systems theory to study the solutions of a nonautonomous differential equation (1.2). See, e.g., [33,94,143]. Let us illustrate the Bebutov construction in the case of a linear system

$$x' = A(t)x, \quad x \in \mathbb{R}^n, t \in \mathbb{R} \tag{1.3}$$

where A is a bounded, uniformly continuous function with values in the space $\mathbb{M}_n$ of $n \times n$ real matrices. Introduce the space $\mathcal{C} = C(\mathbb{R}, \mathbb{M}_n) = \{C : \mathbb{R} \to \mathbb{M}_n \mid C \text{ is continuous and bounded}\}$, equipped with the topology of uniform convergence on compact subsets of $\mathbb{R}$. For each $t \in \mathbb{R}$, let $\tau_t : \mathcal{C} \to \mathcal{C}$ be the translation:

$$\tau_t(C)(\cdot) = C(t + \cdot) \quad (C \in \mathcal{C}).$$

Then $\{\tau_t \mid t \in \mathbb{R}\}$ defines a flow on $\mathcal{C}$. That is $\tau_0 = \mathrm{i\,d}y$; $\tau_t \circ \tau_s = \tau_{t+s}$ for all $t, \in \mathbb{R}$; and $\tau : \mathcal{C} \times \mathbb{R} \to \mathcal{C}$: $\tau(C,t) = \tau_t(C)$ is continuous. We call $\{\tau_t\}$ the *Bebutov flow* on $\mathcal{C}$.

Now view $A(\cdot)$ as an element of $\mathcal{C}$. Let $\Omega = \mathrm{cls}\{\tau_t(A) \mid t \in \mathbb{R}\}$. Then Ω is compact since A is uniformly continuous. Let us outline a proof of this fact. Let $\{t_n\} \subset \mathbb{R}$ be a sequence. We first show that the sequence $\{\tau_{t_n}(A)\}$ contains a subsequence which converges in $\mathcal{C}$. For, let $k \geqslant 1$, let $I_k = [-k, k] \subset \mathbb{R}$. By the Ascoli–Arzelá theorem, there is a subsequence $t_{n_1}, \ldots, t_{n_j}, \ldots$ of $\{t_n\}$ such that $\{t_{n_j}(A)\}$ converges uniformly on I_k. We label this sequence t_j^k $(j = 1, 2, \ldots)$, then set $s_k = t_k^k$ $(k = 1, 2, \ldots)$. Clearly $\{\tau_{s_k}(A)\}$ converges in $\mathcal{C}$. If $\{\omega_n\} \subset \Omega$ is a sequence, we approximate $\omega_n \in \mathcal{C}$ by $\tau_{t_n}(A)$ for an appropriate $t_n \in \mathbb{R}$, then extract

a convergent subsequence of $\{\omega_n\}$ by applying the Ascoli–Arzelá theorem and a Cantor diagonal argument to the sequence $\{\tau_{t_n}(A)\}$.

Note further that Ω admits a dense orbit. Also Ω is invariant with respect to the Bebutov flow. If $\omega \in \Omega$, then it makes sense to speak of the differential equation $x' = \omega(t)x$. A notational device is convenient. Define $\tilde{A} : \Omega \to \mathbb{M}_n$: $\tilde{A}(\omega) = \omega(0)$. Then $\tilde{A}(\tau_t(\omega)) = \omega(t)$ for all $\omega \in \Omega$, $t \in \mathbb{R}$. Moreover if we write $\omega_0 = A \in \Omega$, then $\tilde{A}(\tau_t(\omega_0)) = A(t)$, and so the differential equation $x' = \tilde{A}(\tau_t(\omega_0))x$ coincides with (1.3).

Let us abuse notation and write $A : \Omega \to \mathbb{M}_n$: $A(\omega) = \omega(0)$. We consider the family of equations

$$x' = A\big(\tau_t(\omega)\big)x \quad (x \in \mathbb{R}^n, \omega \in \Omega). \tag{1.3_ω}$$

This family is indexed by the compact, Bebutov-invariant set $\Omega \subset \mathcal{C}$, and contains the initial equation (1.3) = (1.3_{ω_0}).

Let $\omega \in \Omega$, and let $\Phi_\omega(t)$ be the fundamental matrix solution of (1.3_ω). The collection $\{\Phi_\omega(t) \mid \omega \in \Omega\}$ defines a map $\Phi : \Omega \times \mathbb{R} \to \mathrm{GL}(n, \mathbb{R}) : (\omega, t) \mapsto \Phi_\omega(t)$. The map Φ is a *cocycle*; i.e., it satisfies the relation

$$\Phi_{\omega \cdot t}(s)\Phi_\omega(t) = \Phi_\omega(t + s) \quad (\omega \in \Omega; t, s \in \mathbb{R}).$$

The cocycle relation follows from uniqueness of solutions, which holds for Eqs. (1.3_ω). Now, using the cocycle relation, one verifies the following assertion.

For each $t \in \mathbb{R}$ define $\tilde{\tau} : \Omega \times \mathbb{R}^n \to \Omega \times \mathbb{R}^n$ via the formula

$$\tilde{\tau}_t(\omega, x) = \big(\tau_t(\omega), \Phi_\omega(t)x\big);$$

then $(\Omega \times \mathbb{R}^n, \{\tilde{\tau}_t\})$ is a flow. Note that the first coordinate on the right does not depend on x, and that the second coordinate on the right is linear in x.

The flow $\{\tilde{\tau}_t\}$ is a nonautonomous analogue of the one-parameter group $\{\hat{\tau}_t = \mathrm{e}^{At}\}$ generated by the linear autonomous equation $x' = Ax$. The systematic study of the flow $\{\tilde{\tau}_t\}$ on $\Omega \times \mathbb{R}^n$ has led to new insights concerning the properties of solutions of linear nonautonomous ODEs. Experience has shown that, although some problems involving linear systems (1.3) can be treated without introducing the hull Ω and the flow $\{\tilde{\tau}_t\}$, others seem to require it for effective study.

Next, consider a nonlinear nonautonomous equation (1.2). Suppose that, for each compact set $K \subset \mathbb{R}^n$, f is uniformly continuous and uniformly bounded on $\mathbb{R} \times K$. For simplicity, suppose also that f is uniformly Lipschitz on each set $\mathbb{R} \times K$, in the sense that there is a constant M_K such that

$$\big\| f(t, x) - f(t, y)\big\| \leqslant M_K \|x - y\|$$

for all $t \in \mathbb{R}$, $x, y \in K$.

In this case, one can carry out a Bebutov-type construction to obtain a compact, translation-invariant set $\Omega = \Omega_f$ of functions $\omega(t, x)$, each having the properties satisfied by f which are listed above. The flow is defined by $\tau_t(\omega)(\cdot, x) = \omega(\cdot + t, x)$. Furthermore,

if $\omega \in \Omega, x_0 \in \mathbb{R}$, and $x(t)$ is the solution of the equation $x' = \omega(t, x)$ which satisfies $x(0) = x_0$, then the locally defined mappings given by $\tilde{\tau}_t(\omega, x_0) = (\tau_t(\omega), x(t))$ define a local flow on $\Omega \times \mathbb{R}^n$. This flow is global if all solutions of all equations $x' = \omega(t, x)$ exist on $(-\infty, \infty)$. In this case, one obtains a dynamical system on $\Omega \times \mathbb{R}^n$, the behavior of whose orbits reflects the behavior of the solutions of the various equations $x' = \omega(t, x)$, and also reflects the t-recurrence properties of f. These last properties are encoded in the translation flow $\{\tau_t\}$ on Ω.

One can now systematically apply the methods of topological dynamics, ergodic theory etc. to study the solutions of the family of equations $x' = \omega(t, x)$ $(\omega \in \Omega)$. This procedure not infrequently leads to insights concerning the original equation which might not otherwise have come to light. Of course the remark made above for linear systems applies to the nonlinear ones, as well.

Constructions of Bebutov-type can also be carried out for wide classes of parabolic or hyperbolic partial differential equations and for functional differential equations with time-varying coefficients. As a consequence, one can effectively use "dynamical" methods to study such equations. See, e.g., [4,119,124,125] for examples illustrating this approach. We will not give more details here because our attention in this paper will be focused on linear ordinary differential equations, usually in dimension 2.

We turn to an outline of the contents of this paper. Consider the linear, nonautonomous, two-dimensional differential equation

$$x' = A(t)x, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ t \in \mathbb{R}. \tag{1.4}$$

We are interested in functions $A(\cdot)$ which depend aperiodically on time; on the other hand we will impose certain recurrence properties on A. In fact, we will always assume that A is stationary and ergodic in a sense to be defined later (Section 2). Often we will assume that A is a Birkhoff recurrent function of t, or even that A is Bohr almost periodic. Even under such restrictive hypotheses, we will see that the solutions of (1.4) exhibit remarkable properties which have no analogues in the case of periodic systems.

In Section 3, we will study the minimal subsets of the projective flow defined by Eq. (1.4). We will always assume that $A(\cdot)$ is Birkhoff recurrent and usually that it is Bohr almost periodic. A characterization of the minimal subsets of the projective flow constitutes one approach to generalizing to recurrent systems (1.4) the standard Floquet theory for periodic linear systems.

In more detail, let $\mathcal{C} = C(\mathbb{R}, \mathbb{M}_2)$ with the compact-open topology, and let $\Omega \subset \mathcal{C}$ be the result of the Bebutov construction applied to $A(\cdot)$. By carrying out the various steps discussed previously, we obtain a family of differential systems

$$x' = A\bigl(\tau_t(\omega)\bigr)x; \tag{1.4_ω}$$

the original equation (1.4) is in this family.

Let $\mathbb{P}$ be the set of lines through the origin in $\mathbb{R}^2$. Thus $\mathbb{P}$ is the one-dimensional real projective space and is topologically a circle. The family (1.4_ω) induces a flow on $\Sigma =$

$\Omega \times \mathbb{P}$, in the following way. Let $\Phi_\omega(t)$ be the fundamental matrix solution of (1.4_ω). For each $t \in \mathbb{R}$, set

$$\hat{\tau}_t(\omega, l) = \big(\tau_t(\omega), \Phi_\omega(t)l\big) \quad (\omega \in \Omega, l \in \mathbb{P}).$$

It is easy to verify that $(\Sigma, \{\hat{\tau}_t\})$ is indeed a flow. We set the goal of classifying those compact, nonempty, $\{\hat{\tau}_t\}$ – invariant subsets $M \subset \Sigma$ which are minimal in the sense that, for each $m \in M$, the orbit $\{\hat{\tau}_t(m) \mid t \in \mathbb{R}\}$ is dense in M.

Such minimal sets are quite easy to understand if A is periodic. In the case when A is Bohr almost periodic, however, a minimal subset $M \subset \Sigma$ may have properties quite unlike those occurring when A is periodic. In the recurrent case, there are even more possibilities for M. To simplify the discussion, we will assume, with few exceptions, that A is almost periodic.

One striking phenomenon which may be present in the projective flow of an almost periodic family (1.4_ω) is that of almost automorphy in the sense of Bochner and Veech [20,150,151]. We will see that various types of almost automorphic, non-almost periodic minimal sets can be realized as invariant subsets of the projective flow defined by an almost periodic family (1.4_ω). Moreover, we will see that these almost automorphic minimal sets illustrate in a concrete way such phenomena as Marcus–Moore disconjugacy, unbounded mean motion, and strange nonchaotic attractors. Examples illustrating these phenomena are of "M-V type", where the letters indicate the names of Millionscikov and Vinograd. They constructed the first examples of almost periodic families (1.4_ω) containing equations which are not almost reducible [108,152]. It later turned out that these and related examples give rise to almost automorphic subsets of Σ as mentioned above.

We note further that the projective flow $(\Sigma, \{\hat{\tau}_t\})$ of an almost periodic family (1.4_ω) may contain an almost automorphic minimal set which is not of M-V type. For instance, there are families (1.4_ω) for which the Favard separation property is violated and for which each Eq. (1.4_ω) admits a bounded solution. In these circumstances, it may happen that Σ contains an almost automorphic, non-almost periodic minimal set M; for reasons which we will discuss, M cannot be of M-V type. Examples illustrating this phenomenon were constructed in [158] and, later and independently, in [73]. The phenomenon itself was nicely clarified by Ortega and Tarallo [126].

There are other classes of almost periodic families (1.4_ω) whose projective flows admit minimal sets with no analogue in the periodic case. For example, Σ may admit a unique "invariant two-sheet". Examples of this sort were predicted in [83] and were written down in [88,123].

Furthermore, there are situations in which the projective flow Σ is itself minimal but is quite unlike those arising when Σ is minimal and A is periodic (these last are all of Kronecker type). We will consider several classes of examples. One consists of families (1.4_ω) for which Σ is minimal, uniquely ergodic, and is a proximal extension of Ω ([61], also [87]). Another class is illustrated by a recent example of Bjerklov [13,14], for which Σ is minimal and the family (1.4_ω) is weakly hyperbolic. Such examples exhibit the property of "Li–Yorke chaos". See [17] for more information on this topic.

We close this introductory discussion of the contents of Section 3 by noting that the measure-theoretic (ergodic) structure of the projective flow of Eqs. (1.4_ω) is more com-

pletely understood than is the topologico-dynamical structure. See [122,121] for a discussion and classification of the ergodic measures of the projective flow when $n = 2$. See also [9] and references therein for a complete theory of the ergodic measures of the projective flow for n-dimensional linear systems.

In Section 4, we discuss some recent developments in the spectral theory and inverse spectral theory of the classical Sturm–Liouville operator. The material reviewed here is found in the Ph.D. thesis of one of us [157] and in [96,95]. The methods of nonautonomous differential equations are systematically exploited both in the formulation of and in the proofs of the results we consider. Other techniques, in particular certain constructions from the classical theory of algebraic curves, are also found to be very useful.

The theory we present is motivated in part by the progress in the inverse spectral theory of the one-dimensional Schrödinger operator which took place in the 1970s. In fact, it was discovered that there are basic and nonobvious relations between the inverse spectral theory of the Schrödinger operator, the theory of algebraic curves, and the solutions of a certain hierarchy of nonlinear partial differential equations. One of the equations of this hierarchy is the K-dV (Korteweg–de Vries) equation

$$\frac{\partial u}{\partial t} = 3u\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial^3 u}{\partial x^3}$$

of shallow water-wave theory.

It turns out that the inverse spectral theory of the Sturm–Liouville operator also can be studied using the theory of algebraic curves, in particular certain facts concerning generalized Jacobians. To work out the relation between these a priori separate fields, it is convenient to first develop certain aspects of the *direct* spectral theory of the Sturm–Liouville operator. To this end, one uses facts from the theory of nonautonomous differential equations. Among the "nonautonomous" facts which are found useful are the following:

– The equivalence between the resolvent of a Sturm–Liouville operator and the existence of an exponential dichotomy for the associated nonautonomous linear system.
– The relation between the spectrum of a Sturm–Liouville operator and the rotation number of the associated nonautonomous linear system.
– The Kotani theory for Sturm–Liouville type operators.

We will discuss these matters. We will in addition give a brief discussion of the Gilbert–Pearson theory [60] in the context of the Sturm–Liouville operator. This theory has been found to be of basic importance in the study of the one-dimensional Schrödinger operator and we think it is important to indicate how the Gilbert–Pearson theory can be extended to the more general Sturm–Liouville operator.

Our inverse spectral theory takes these and other facts as a starting point. Making use of the Weyl m-functions, one then can make a connection between a Sturm–Liouville operator having certain spectral properties, and a generalized Jacobian. One is able to give a concrete description of the operators having these spectral properties. We will discuss all these matters in Section 4.

In analogy to the relation between the Schrödinger operator and the K-dV equation, there is a close connection between the special Sturm–Liouville operator defined by the equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+1\right)\varphi = Ey(x)\varphi$$

and the Camassa–Holm equation

$$4\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^2 \partial t} + 2u\frac{\partial^3 u}{\partial x^3} + 4\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - 24u\frac{\partial u}{\partial x}.$$

Using our methods, it is possible to work out an ample family of quasi periodic solutions of the Camassa–Holm equation and of other equations in the Camassa–Holm hierarchy. These facts are discussed in [157,156]; for earlier work see [2,3,58]. For reasons of space we will not take up this theme in the present paper.

In the final Section 5 of this paper, we consider the following basic question. Let $(\Omega,\mathbb{R})$ be a fixed minimal flow. Let $\mathrm{sl}(2,\mathbb{R})$ be the set of all 2×2 real matrices with trace zero. Let $\mathcal{C}_0 = C(\Omega,\mathrm{sl}(2,\mathbb{R}))$ be the set of continuous functions $A:\Omega\to\mathrm{sl}(2,\mathbb{R})$ with the uniform topology induced by some fixed norm on $\mathrm{sl}(2,\mathbb{R})$. Each $A\in\mathcal{C}_0$ defines a family of two-dimensional differential systems of type (1.4_ω):

$$x' = A\big(\tau_t(\omega)\big)x, \quad x\in\mathbb{R}^2$$

where $\mathrm{tr}\,A(\cdot)=0$. We ask "how many" matrix functions $A(\cdot)$ give rise to a family (1.4_ω) which exhibits an exponential dichotomy. In particular, we ask if the set of $\mathrm{sl}(2,\mathbb{R})$-valued functions $\{A\}$ for which the family (1.4_ω) has an exponential dichotomy is open and dense in $\mathcal{C}_0$.

Related questions have been considered by Millionscikov [109], Novikov [118], Cong [116,117] and many other authors. We will not give an encyclopedia of the results which have been obtained. Rather, we will consider in some detail a particular result concerning quasi-periodic minimal flows, in the C^0-category.

It is known that, for certain minimal flows $(\Omega,\mathbb{R})$ with compact metric phase space Ω, the set $\mathcal{C}_{\mathrm{ED}} = \{A\in\mathcal{C}_0 \mid \text{the family } (1.4_\omega) \text{ admits an exponential dichotomy}\}$ is *not* dense in $\mathcal{C}_0$. In fact, this is true if Ω is the circle and the flow $\{\tau_t\}$ is defined by a rigid rotation; this case occurs exactly when (1.4_ω) is a family of periodic equations with fixed period and the coefficient matrices are related one to another by translations. It also occurs whenever the image of the so-called Schwarzmann homomorphism is isomorphic to $\{0\}$ or $\mathbb{Z}$. This condition in ensured if the first integer Čech cohomology group $H^1(\Omega,\mathbb{Z})$ is, modulo torsion, equal to $\{0\}$ or $\mathbb{Z}$.

However, if Ω is the phase space of a nonperiodic, almost periodic minimal flow, then the image of the Schwarzmann homeomorphism is dense in $\mathbb{R}$ (it coincides with the so-called frequency module $\mathcal{M}$ of $(\Omega,\mathbb{R})$). In this case, the above criterion breaks down. If $(\Omega,\mathbb{R})$ is a "limit-periodic" flow, then it can be shown that $\mathcal{C}_{\mathrm{ED}}\subset\mathcal{C}_0$ is open and dense. This is not a completely straightforward matter, but arguments of Moser [110] can be adapted to do the job.

In the case when $(\Omega, \mathbb{R})$ is quasi-periodic – that is, when Ω is a k-dimensional torus and the flow is determined by a Kronecker winding – we consider the question as to whether $\mathcal{C}_{\mathrm{ED}}$ is dense in $\mathcal{C}_0$ (we note that $\mathcal{C}_{\mathrm{ED}}$ is open in $\mathcal{C}_0$ by standard perturbation results for exponential dichotomies; see, e.g., [29]). We will discuss a result to the effect that, if the Kronecker winding is defined by a Liouville frequency vector, then $\mathcal{C}_{\mathrm{ED}}$ is indeed dense in $\mathcal{C}_0$. This result is proved using methods worked out in [77,44,50].

This result can be viewed as included in a line of research which aims at determining the nature of the set of elements $A \in \mathcal{C}_0$ for which the family (1.4_ω) has a positive Lyapunov exponent. See, e.g., [117,100,15], and for definitive information [19]. We note, however, that positivity of the Lyapunov exponent is much weaker than the dichotomy property.

On the other hand, the result we discuss should be viewed as in some sense orthogonal to the impressive body of theorems worked out in, e.g., [98,39,102,135], and papers by other authors. These theorems concern the case when A is an analytic (or at least highly regular) function on the torus $\Omega = \mathbb{T}^k$, and the frequency vector satisfies a Diophantine condition. Several of these results consist of statements to the following effect: in a suitable parameterized family of analytic quasi-periodic differential systems with Diophantine frequency vector, there is a measure-theoretically large set of parameter values for which the system is reducible. That is, there is an analytic quasi-periodic change of variables $x = B(\omega \cdot t)y$ such that $x' = A(\omega \cdot t)x$ is transformed into $y' = A_0 y$ where A_0 is a constant matrix. Interestingly enough, there is often a topologically large set of parameter values for which reducibility does *not* hold. Instead, an almost reducibility property holds; moreover the projective flow corresponding to such an almost reducible system seems to have interesting properties.

Our results concerning density of $\mathcal{C}_{\mathrm{ED}} \subset \mathcal{C}_0$ would appear to break down when one passes from the C^0-category to the C^1-category, at least when $k \geqslant 3$. (Our methods may produce a density result in the C^r-category if r is less than 1.) The question of C^1-density of the dichotomy property (for any kind of frequency vector, Liouville or Diophantine or "other") has not been clarified, and only nondefinitive indications concerning the real facts of the case are at this point available [23,26,30]. We mention that, very recently, A. Avila, J. Bochi and D. Damanik have proved the density result for all frequencies in the C^0 category. Their methods seem specific to this category, though they can also deal with certain non-almost periodic base flows. We thank Prof. David Damanik for this information.

We finish the introduction by listing some notation which will be used in this paper.

If $k \geqslant 1$, let $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ denote the standard k-torus. Thus $\mathbb{T}^1$ is the circle, which can be identified with $\{e^{2\pi i\psi} \mid 0 \leqslant \psi < 1\}$.

We use $|\cdot|$ to indicate all norms which arise, and $\langle , \rangle$ to indicate all inner products.

2. Preliminaries

We review basic concepts and fix notation which will be used in the succeeding sections.

Let us first recall the definition of a (continuous, real) *flow*. Let Ω be a metric space. For each $t \in \mathbb{R}$, let $\tau_t : \Omega \to \Omega$ be a homeomorphism. Suppose that the following conditions are satisfied:

– $\tau_0(\omega) = \omega$, for all $\omega \in \Omega$;

– $\tau_{t+s}(\omega) = \tau_t \circ \tau_s(\omega)$ for all $t, s \in \mathbb{R}$, $\omega \in \Omega$;
– the map $\tau : \Omega \times \mathbb{R} \to \Omega : (\omega, t) \mapsto \tau_t(\omega)$ is continuous.

We call the pair $(\Omega, \{\tau_t\})$ a *flow* or *dynamical system* on Ω. We will also use the notation $(\Omega, \mathbb{R})$ to denote a flow, or Ω resp. $\{\tau_t\}$ if there is no doubt concerning the identity of $\{\tau_t\}$ resp. Ω. We will often write $\omega \cdot t$ in place of $\tau_t(\omega)$.

Here is some standard terminology concerning flows. If $\omega \in \Omega$, then the *orbit* through ω is $\{\omega \cdot t \mid t \in \mathbb{R}\}$. If $\Omega_* \subset \Omega$, then Ω_* is *invariant* if for every $\omega \in \Omega_*$, the orbit through ω is contained in Ω_*. Let $(\tilde{\Omega}, \{\tilde{\tau}_t\})$ be another flow. A continuous map $\pi : \tilde{\Omega} \to \Omega$ is a *flow homomorphism* if $\pi \circ \tilde{\tau}_t = \tau_t \circ \pi$ for all $t \in \mathbb{R}$. If π is also homeomorphism of $\tilde{\Omega}$ onto Ω, then π is a *flow isomorphism*.

Some flows of interest to us arise via the Bebutov construction, which we described in the Introduction. We repeat it briefly. Let $n \geqslant 1$, and let $\mathcal{C} = \{C\colon\ \mathbb{R} \to \mathbb{M}_n \mid C \text{is continuous and bounded}\}$. Endow $\mathcal{C}$ with the compact-open topology. For each $t \in \mathbb{R}$ and $C \in \mathcal{C}$, set $\tau_t(C)(\cdot) = C(t + \cdot)$; then τ_t is the t-translation. Then $(\mathcal{C}, \{\tau_t\})$ is a flow. If $A \in \mathcal{C}$ is uniformly continuous, define $\Omega = \mathrm{cls}\{\tau_t(A) \mid t \in \mathbb{R}\}$. Then Ω is compact and is invariant with respect to the Bebutov flow $\{\tau_t\}$. We call Ω the *hull* of A. As we pointed out in the Introduction, the map $\tilde{A} : \Omega \to \mathbb{M}_n\colon\ \tilde{A}(\omega) = \omega(0)$ is continuous and has the property that, if $\omega_0 = A$, then $\tilde{A}(\omega_0 \cdot t) = A(t)$ for all $t \in \mathbb{R}$. In this case it is convenient to abuse notation and write A instead of $\tilde{A}$.

Certain specific types of flow will be important in what follows; we review the relevant definitions. Let Ω be a *compact* metric space, and let $(\Omega, \{\tau_t\})$ be a flow.

First of all, the flow $(\Omega, \{\tau_t\})$ is called *minimal* or *Birkhoff recurrent* if for each $\omega \in \Omega$, the orbit $\{\omega \cdot t \mid t \in \mathbb{R}\}$ is dense in Ω. If Ω is the hull of a uniformly continuous function $A : \mathbb{R} \to \mathbb{M}_n$, then Ω is minimal if and only if A satisfies the Birkhoff recurrence condition, namely if to each compact set $K \subset \mathbb{R}$ and each $\varepsilon > 0$ there corresponds a relatively dense set $\{t_k \mid k \in \mathbb{Z}\} \subset \mathbb{R}$ such that, if $k \in \mathbb{Z}$, then $|A(t + t_k) - A(t)| \leqslant \varepsilon$ for all $t \in K$.

A special class of minimal flows is determined by those which are *Bohr almost periodic* or simply *almost periodic*. By definition, a flow $(\Omega, \mathbb{R})$ satisfies this condition if it is minimal and if there is a metric d on Ω (which is compatible with the topology on Ω) such that, if $\omega_1, \omega_2 \in \Omega$, then $d(\omega_1 \cdot t, \omega_2 \cdot t) = d(\omega_1, \omega_2)$ for all $t \in \mathbb{R}$. If Ω is the hull of a uniformly continuous function $A \in \mathcal{C}$, then $(\Omega, \mathbb{R})$ is Bohr almost periodic if and only if for each $\varepsilon > 0$ there is a relatively dense set $\{t_k \mid k \in \mathbb{Z}\}$ such that, if $k \in \mathbb{Z}$, then $|A(t + t_k) - A(t)| \leqslant \varepsilon$ for all $t \in \mathbb{R}$. One sometimes calls $\{t_k\}$ a set of almost periods of A. See, e.g., [5,21,54].

It turns out that, if $(\Omega, \mathbb{R})$ is minimal and Bohr almost periodic, then Ω may be given the structure of a compact Abelian topological group with dense subgroup $\mathbb{R}$ in such a way that the group structure and the flow structure are compatible. In fact, if $\omega_0 \in \Omega$ and $t_1, t_2 \in \mathbb{R}$, define $(\omega_0 \cdot t_1) * (\omega_0 \cdot t_2) = \omega_0 \cdot (t_1 + t_2)$. It is an exercise to show that $*$ extends to a group operation on Ω with the stated properties. Moreover ω_0 is the identity with respect to this group structure.

To each minimal almost periodic flow $(\Omega, \mathbb{R})$ there corresponds the so-called frequency module $\mathcal{M} = \mathcal{M}_\Omega$. This object is constructed in the following way. View Ω as a compact Abelian topological group with dense subgroup $\mathbb{R}$. Let $\hat{\Omega}$ be the group of continuous homomorphisms (*characters*) $\chi : \Omega \to \mathbb{T}^1$. Since $\mathbb{R} \subset \Omega$ is a dense subgroup, a character χ is determined by its restriction $\chi_{\mathbb{R}}$ to $\mathbb{R}$, and this restriction has the form $\chi_{\mathbb{R}}(t) = \mathrm{e}^{2\pi \mathrm{i} \lambda t}$ for

some $\lambda \in \mathbb{R}$. The smallest $\mathbb{Z}$-submodule of $\mathbb{R}$ containing all the numbers λ arising in the this way is the *frequency module* of Ω. Due to the fact that Ω is a compact metric Abelian group, the frequency module is a countable subgroup of $\mathbb{R}$.

We mention the following examples. Let $\Omega = \mathbb{T}^k$ be the k-torus. Let $\gamma_1, \dots \gamma_k$ be real numbers which are independent over the rational field $\mathbb{Q}$. If $\omega \in \Omega$, write $\omega = (\exp 2\pi \mathrm{i}\psi_1, \dots, \exp 2\pi \mathrm{i}\psi_k)$, then define for each $t \in \mathbb{R}$

$$\tau_t(\omega) = \big(\exp 2\pi \mathrm{i}(\psi_1 + \gamma_1 t), \dots, \exp 2\pi \mathrm{i}(\psi_k + \gamma_k t)\big).$$

Below we will sometimes write $\psi = (\psi_1, \dots, \psi_k)$ and also write $\tau_t(\psi) = \psi + \gamma t = (\psi_1 + \gamma_1 t, \dots, \psi_k + \gamma_k t)$. One says that $(\Omega, \{\tau_t\})$ is a *quasi-periodic* flow or a *Kronecker* flow. One also speaks of a Kronecker winding. It is minimal and almost periodic. The numbers $\gamma_1, \dots, \gamma_g$ are called the *frequencies* of the flow.

Second, set $T_0 = 1, T_1 = n_1, \dots, T_k = n_k T_{k-1}, \dots$ where $n_1 > 1, \dots, n_k > 1, \dots$ are integers. Consider the subgroup $\mathcal{M}$ of $\mathbb{R}$ generated by $T_0^{-1}, \dots, T_k^{-1}, \dots$

$$\mathcal{M} = \left\{ \frac{l_0}{T_0} + \frac{l_1}{T_1} + \dots + \frac{l_r}{T_r} \;\Big|\; r \geqslant 1; l_0, \dots, l_r \in \mathbb{Z} \right\}.$$

Then $\mathcal{M}$ is the frequency module of an almost periodic minimal flow. Such a flow is called *limit periodic*. If Ω is the hull of an almost periodic function $A : \mathbb{R} \to \mathbb{M}_n$, then $(\Omega, \mathbb{R})$ is limit periodic if and only if A is the uniform limit of a sequence of periodic functions.

Still another type of flow is distinguished by the property of *almost automorphy*. Let $(\Omega_0, \mathbb{R})$ be an almost periodic minimal flow. Let $(\Omega, \mathbb{R})$ be another minimal flow, and let $\pi : \Omega \to \Omega_0$ be a flow homomorphism. Note that π is necessarily surjective since Ω_0 is minimal. Suppose that there is a point $\omega_0 \in \Omega_0$ such that the inverse image $\pi^{-1}(\omega_0)$ reduces to a single point in Ω. Then $(\Omega, \mathbb{R})$ is called an almost automorphic extension of $(\Omega_0, \mathbb{R})$, or an *almost automorphic* flow. If Ω is the hull of an uniformly continuous bounded function $A : \mathbb{R} \to \mathbb{M}_n$, then $(\Omega, \mathbb{R})$ is almost automorphic if (but not only if) the function A satisfies the Bochner condition. Namely, if $\{t_k\} \subset \mathbb{R}$ is a sequence such that $|t_k| \to \infty$ and $A(t_k + \cdot) \to A(\cdot)$ in $\mathcal{C}$, then $A(-t_k + \cdot) \to A(\cdot)$ in $\mathcal{C}$. If the Bochner condition is satisfied, then there is an almost periodic minimal flow $(\Omega_0, \mathbb{R})$ together with a flow homomorphism $\pi : \Omega \to \Omega_0$ such that $\pi^{-1}\pi(A) = \{A\}$ [151]. A function $A(\cdot)$ which satisfies the Bochner condition is said to be an *almost automorphic function*.

Almost automorphic minimal sets were studied by Veech [150,151]. See [144,145,154] for a review of their properties and a discussion of their presence in various applied situations.

Next we review some basic definitions from ergodic theory. Let Ω be a compact metric space and let $(\Omega, \{\tau_t\})$ be a flow.

A regular Borel probability measure ν on Ω is called $\{\tau_t\}$-*invariant*, or simply *invariant* if there is no doubt concerning the identity of $\{\tau_t\}$, if for each Borel subset $B \subset \Omega$ and each $t \in \mathbb{R}$, there holds $\nu(\tau_t(B)) = \nu(B)$. An invariant measure ν is called *ergodic* if it is indecomposable in the sense that, whenever $B \subset \Omega$ is a Borel set such that $\mu(\tau_t(B) \triangle B) = 0$ for all $t \in \mathbb{R}$ ($\triangle =$ symmetric difference of sets), then either $\nu(B) = 0$ or $\nu(B) = 1$.

One can show that, if Ω is a compact metric space and $(\Omega, \mathbb{R})$ is a flow, then there exists an ergodic measure ν on Ω. In fact, one can first use a classical construction of Krylov and

Bogoliubov to determine an *invariant* measure ν on Ω; see, e.g., [55,113]. We will make use of this construction below. Second, one notes that the set of invariant measures forms a weak-$*$ compact, convex set of the vector space of all Borel regular signed measures on Ω. By the Krein–Mil'man Theorem this set admits an extreme point. It is easy to see that such an extreme point is ergodic.

If Ω is a compact metric space and ν is a Borel regular measure on Ω, then the *topological support* $\operatorname{Supp}\nu$ of ν is by definition the complement in Ω of the largest open set $U \subset \Omega$ which has zero ν-measure. This concept is well defined and $\operatorname{Supp}\nu$ is compact. If ν is $\{\tau_t\}$-invariant, then $\operatorname{Supp}\nu$ is invariant. Observe that if $U \subset \operatorname{Supp}\nu$ is relatively open then $\nu(U) > 0$. If $(\Omega, \mathbb{R})$ is minimal then $\operatorname{Supp}\nu = \Omega$ for every invariant measure ν on Ω.

It may happen that a flow $(\Omega, \mathbb{R})$ admits a unique invariant measure ν. This is the case, for example, if $(\Omega, \mathbb{R})$ is minimal and almost periodic; in fact, the normalized Haar measure ν on Ω is the unique invariant measure. If $(\Omega, \mathbb{R})$ admits a unique invariant measure ν, then ν is ergodic. If $(\Omega, \mathbb{R})$ is minimal and admits a unique invariant (ergodic) measure ν, then $(\Omega, \mathbb{R})$ is said to be *strictly ergodic*.

We state a version of the Birkhoff Ergodic Theorem which is adequate for our purposes. For a formulation and proof of the general Birkhoff theorem see [113]. We include a refinement of the usual Birkhoff theorem in point (ii) of Theorem 2.1; for a proof of this statement see, e.g., [55].

THEOREM 2.1. *Let Ω be a compact metric space with flow $\{\tau_t\}$, and let ν be an ergodic measure on Ω.*

(i) *Let $f \in L^1(\Omega, \nu)$, then for ν-a.a. $\omega \in \Omega$ the time averages $\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\omega \cdot s)\,\mathrm{d}s$ and $\lim_{t\to-\infty} \frac{1}{t}\int_0^t f(\omega \cdot s)\,\mathrm{d}s$ exist and equal $\int_\Omega f\,\mathrm{d}\nu$.*

(ii) *If Ω admits a unique ergodic measure ν and if f is continuous, then*

$$\lim_{|t|\to\infty} \frac{1}{t}\int_0^t f(\omega \cdot s)\,\mathrm{d}s = \int_\Omega f\,\mathrm{d}\nu$$

for all $\omega \in \Omega$, and the limit is uniform in Ω.

Next we state some facts concerning linear nonautonomous differential systems. We adopt the following point of view. Let Ω be a compact metric space, and let $(\Omega, \mathbb{R})$ be a flow. Let $A : \Omega \to \mathbb{M}_n$ be a continuous function. Consider the family of differential systems

$$x' = A(\omega \cdot t)x, \quad x \in \mathbb{R}^n,\ \omega \in \Omega. \tag{2.1$_\omega$}$$

We have already seen that, via the Bebutov construction, a linear differential equation $x' = A(t)x$ with uniformly continuous coefficient matrix $A(\cdot)$ gives rise to a family of this type. We will study the solutions of Eqs. (2.1$_\omega$) using the methods of topological dynamics and ergodic theory.

Let $\omega \in \Omega$, and let $\Phi_\omega(t)$ be the fundamental matrix solution of (2.1$_\omega$). The collection $\{\Phi_\omega(t) \mid \omega \in \Omega\}$ defines a map $\Phi : \Omega \times \mathbb{R} \to \mathrm{GL}(n, \mathbb{R}) : (\omega, t) \mapsto \Phi_\omega(t)$. We noted in Section 1 that Φ is a cocycle and that the maps $\tilde{\tau}_t : \Omega \times \mathbb{R}^n \to \Omega \times \mathbb{R}^n : \tilde{\tau}_t(\omega, x) = (\tau_t(\omega), \Phi_\omega(t)x)$ define a flow on $\Omega \times \mathbb{R}^n (t \in \mathbb{R})$.

Let $\mathbb{P}^{n-1}$ be the $(n-1)$ dimensional space of lines through the origin in $\mathbb{R}^n$. For each $t \in \mathbb{R}$, define $\hat{\tau} : \Omega \times \mathbb{P}^{n-1} \to \Omega \times \mathbb{P}^{n-1}$: $\hat{\tau}_t(\omega, l) = (\tau_t(\omega), \Phi_\omega(t)l)$. Here $l \in \mathbb{P}^{n-1}$ is acted on in the natural way by $\Phi_\omega(t) \in GL(n, \mathbb{R})$. Using the cocycle identity again, one can show that $(\Omega \times \mathbb{P}^{n-1}, \{\hat{\tau}_t\})$ is a flow. This is the *projective flow* defined by Eqs. (2.1_ω). If one wishes to study the "angular behavior" of the solutions of Eqs. (2.1_ω), the projective flow is one natural object to study. We will write

$$\Sigma = \Omega \times \mathbb{P}^{n-1}, \quad \pi : \Sigma \to \Omega : (\omega, l) \mapsto \omega$$

for $\omega \in \Omega$, $l \in \mathbb{P}^{n-1}$. The projection π is a flow homomorphism.

Let us now briefly discuss some key concepts concerning the family (2.1_ω). These concepts will later be specialized to the case of interest in this paper, namely that when $n = 2$.

Fix $\omega \in \Omega$. A number $\beta \in \mathbb{R}$ is called a *Lyapunov exponent* of Eq. (2.1_ω) if there is a nonzero vector $x_0 \in \mathbb{R}^n$ such that

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \ln\left|\Phi_\omega(t)x_0\right| = \beta.$$

It is well-known that each fixed Eq. (2.1_ω) admits finitely many Lyapunov exponents $\beta_1 < \cdots < \beta_k$ where $1 \leqslant k \leqslant n$. Moreover, there is a filtration of $\mathbb{R}^n$ consisting of subspaces $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_k = \mathbb{R}^n$ such that

$$V_r = \{0\} \cup \left\{ 0 \neq x_0 \in \mathbb{R}^n \;\middle|\; \overline{\lim_{t\to\infty}} \frac{1}{t} \ln\left|\Phi_\omega(t)x_0\right| \leqslant \beta_r \right\} \quad (1 \leqslant r \leqslant k).$$

See, e.g., [8]. Difficulties are caused by the fact that neither the Lyapunov exponents nor the filtration need to be continuous with respect to variation of ω.

The Oseledets theory furnishes information concerning this point and others. We do not give the most general version of the Oseledets theorem; the one we do give is, however, more than sufficient for our purposes. Let ν be an ergodic measure on Ω.

THEOREM 2.2 (Oseledets). *One can find a ν-measurable, invariant subset $\Omega_* \subset \Omega$ with $\nu(\Omega_*) = 1$, and real numbers $\beta_1 < \cdots < \beta_k$ with $1 \leqslant k \leqslant n$, with the following properties.*

(i) *If $\omega \in \Omega_*$ then for each $r = 1, 2, \ldots, k$ there is a nonzero vector $x_0 \in \mathbb{R}^n$ such that the limits $\lim_{t\to\infty} \frac{1}{t} \ln|\Phi_\omega(t)x_0|$ and $\lim_{t\to-\infty} \frac{1}{t} \ln|\Phi_\omega(t)x_0|$ both exist and equal β_r.*

(ii) *For each $\omega \in \Omega_*$ and each $r = 1, 2, \ldots, k$, the set*

$$W_r(\omega) = \{0\} \cup \left\{ 0 \neq x_0 \in \mathbb{R}^n \;\middle|\; \lim_{|t|\to\infty} \frac{1}{t} \ln\left|\Phi_\omega(t)x_0\right| = \beta_r \right\}$$

is a vector subspace of $\mathbb{R}^n$, of dimension $d_r \in \{1, 2, \ldots, n\}$. One has $d_1 + \cdots + d_k = n$. The mapping $\omega \mapsto W_r(\omega)$ is measurable as a function from Ω_* to the Grassmann manifold $\mathrm{Gr}(n, d_r)$ of d_r-dimensional subspaces of $\mathbb{R}^n$.

(iii) *For each $r = 1, 2, \dots, k$, the "measurable subbundle"*

$$W_r = \bigcup \{(\omega, x_0) \mid x_0 \in W_r(\omega), \omega \in \Omega_*\}$$

is invariant in the sense that $\Phi_\omega(t)W_r(\omega) = W_r(\omega \cdot t)$ for all $\omega \in \Omega_*, t \in \mathbb{R}$.

(iv) *For each $\omega \in \Omega_*$, there holds $\beta_1 + \cdots + \beta_k = \lim_{|t|\to\infty} \frac{1}{t} \ln \det \Phi_\omega(t)$.*

Point (i) defines the "Oseledets spectrum" of the family (2.1_ω). Point (iv) states that ν-a.a. systems (2.1_ω) are both forward and backward Lyapunov regular. For proofs of the Oseledets theorem see [129,8,94]. When $n = 2$ it is possible to give a quite simple proof of a strengthened version of the Oseledets Theorem. We will discuss this point later on.

Next we introduce the concept of *exponential dichotomy* [29,137]. Again let Ω be a compact metric space, and let $(\Omega, \mathbb{R})$ be a real flow. Let $A : \Omega \to \mathbb{M}_n$ be a continuous map, and consider the corresponding family of Eqs. (2.1_ω).

DEFINITIONS 2.3.

(a) The family $\{(2.1_\omega) \mid \omega \in \Omega\}$ is said to have an exponential dichotomy (ED) over Ω if there are positive constants η, δ together with a continuous, projection-valued function $P : \Omega \to \mathbb{M}_n$ (thus $P(\omega)^2 = P(\omega)$ for all $\omega \in \Omega$) such that the following estimates hold:
 (i) $|\Phi_\omega(t)P(\omega)\Phi_\omega(s)^{-1}| \leqslant \eta e^{-\delta(t-s)}$, $t \geqslant s$,
 (ii) $|\Phi_\omega(t)(I - P(\omega))\Phi_\omega(s)^{-1}| \leqslant \eta e^{\delta(t-s)}$, $t \leqslant s$.

(b) The *dynamical* or *Sacker–Sell spectrum* of the family (2.1_ω) is $\{\lambda \in \mathbb{R} \mid$ the translated equations $x' = [-\lambda I + A(\omega \cdot t)]x$ do *not* admit an exponential dichotomy over $\Omega\}$.

If Eqs. (2.1_ω) admit an exponential dichotomy over Ω, then it is easy to see that $\Omega \times \mathbb{R}^n$ splits into a Whitney sum of the vector subbundles $W_1 = \{(\omega, x) \in \Omega \times \mathbb{R}^n \mid x \in \operatorname{Im} P(\omega)\}$ and $W_2 = \{(\omega, x) \in \Omega \times \mathbb{R}^n \mid x \in \operatorname{Ker} P(\omega)\}$. Of course it can happen that $P(\omega) = 0$ or $P(\omega) = I$. If Ω is connected then the dimension of $\operatorname{Im} P(\omega)$ does not depend on Ω.

The following theorem is due to Sacker and Sell [137,138]; see also Selgrade [142].

THEOREM 2.4. *Let Ω be a compact connected metric space, and let $(\Omega, \mathbb{R})$ be a flow. Let $A : \Omega \to \mathbb{M}_n$ be a continuous map.*

(a) *The dynamical spectrum of the family (2.1_ω) is a finite union $[a_1, b_1] \cup \cdots \cup [a_k, b_k]$ of compact subintervals of $\mathbb{R}$, where $1 \leqslant k \leqslant n$. Some or all of these intervals may degenerate to points.*

(b) *For each $r = 1, 2, \dots, k$ set*

$$W_r = \left\{(\omega, x) \in \Omega \times \mathbb{R}^n \;\middle|\; x_0 = 0 \textit{ or } \left(\limsup_{t\to\infty}, \liminf_{t\to\infty}\right) \frac{1}{t} \ln\left|\Phi_\omega(t)x_0\right| \in [a_r, b_r] \right.$$
$$\left. \textit{and } \left(\limsup_{t\to-\infty}, \liminf_{t\to-\infty}\right) \frac{1}{t} \ln\left|\Phi_\omega(t)x_0\right| \in [a_r, b_r]\right\}.$$

Then W_r is a topological vector subbundle of $\Omega \times \mathbb{R}^n$. One has

$$\Omega \times \mathbb{R}^n = W_1 \oplus W_2 \oplus \cdots \oplus W_k \quad (\textit{Whitney sum}).$$

There are various relations between the "Oseledets bundles" and the "Sacker–Sell bundles", and between the Oseledets spectrum and the dynamical spectrum. See [94] for a discussion of this issue.

Let us now consider the case $n = 2$. As we will see, there is no loss of generality for present purposes in assuming that the trace $\operatorname{tr} A(\omega) = 0$ for all $\omega \in \Omega$. Thus we assume that

$$A : \Omega \to \mathrm{sl}(2, \mathbb{R})$$

is a continuous function. Let us introduce the basis $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of $\mathrm{sl}(2, \mathbb{R})$. We expand $A(\cdot)$ with respect to this basis:

$$A = \begin{pmatrix} a & -b + c \\ b + c & -a \end{pmatrix}$$

where $a, b, c : \Omega \to \mathbb{R}$ are continuous functions.

Consider the family of equations

$$x' = A(\omega \cdot t)x, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \omega \in \Omega. \tag{2.2$_\omega$}$$

Let us express these equations in polar coordinates $r^2 = x_1^2 + x_2^2$, $\theta = \arctan \frac{x_2}{x_1}$:

$$\frac{r'}{r} = a(\omega \cdot t) \cos 2\theta + c(\omega \cdot t) \sin 2\theta,$$

$$\theta' = b(\omega \cdot t) - a(\omega \cdot t) \sin 2\theta + c(\omega \cdot t) \cos 2\theta.$$

If we think of θ as a π-periodic angular variable on the real projective space $\mathbb{P}$, then the θ-equation gives a concrete way of studying the projective flow defined by Eqs. (2.1$_\omega$). In fact, let $l_0 \in \mathbb{P}$, $\omega_0 \in \Omega$, and suppose that l_0 is parameterized by $\theta_0 \in \mathbb{R}$. Let $\theta(t)$ satisfy the θ-equation with $\theta(0) = \theta_0$ and $\omega = \omega_0$. Then $\hat{\tau}_t(\omega_0, l_0) = \hat{\tau}_t(\omega_0, \theta_0) = (\tau_t(\omega), \theta(t))$. It is understood that $\theta(t)$ is identified with the line through the origin in $\mathbb{R}^2$ which it parametrizes.

Next let us write

$$f : \Sigma \to \mathbb{R}: \ f(\omega, \theta) = a(\omega) \cos 2\theta + c(\omega) \sin 2\theta,$$

$$g : \Sigma \to \mathbb{R}: \ g(\omega, \theta) = b(\omega) - a(\omega) \sin 2\theta + c(\omega) \cos 2\theta.$$

Then Eqs. (2.2$_\omega$) are expressed in polar coordinates r, θ as follows:

$$\frac{r'}{r} = f\big(\hat{\tau}_t(\omega, \theta)\big), \tag{2.3$_\omega$}$$

$$\theta' = g\big(\hat{\tau}_t(\omega, \theta)\big). \tag{2.4$_\omega$}$$

Let $0 \neq x_0 \in \mathbb{R}^2$ have polar angle θ_0, and let $\theta(t)$ be the solution of (2.4$_\omega$) satisfying $\theta(0) = \theta_0$. Let $r(t) = |x(t)|$, where $x(t)$ satisfies (2.2$_\omega$) with $x(0) = x_0$. Then

$$\frac{1}{t}\ln r(t) = \frac{1}{t}\ln r(0) + \frac{1}{t}\int_0^t f\big(\hat{\tau}_s(\omega, \theta_0)\big)\,\mathrm{d}s. \tag{2.5}$$

Thus there is a close connection between the Lyapunov exponents of Eq. (2.2$_\omega$) and the time averages of f. Using the Birkhoff Theorem, we can thus relate the Lyapunov exponents of Eq. (2.2$_\omega$) to the averages of f with respect to the $\{\tau_t\}$-ergodic measures μ on Σ.

Following [78], we indicate how an analysis of the ergodic averages of f permits one to prove the Oseledets Theorem 2.2 and the Sacker–Sell Theorem 2.4 in the case $n = 2$. We sketch the necessary arguments in the case when $(\Omega, \mathbb{R})$ admits a unique invariant measure ν (which is then ergodic). As mentioned earlier, this condition is valid if Ω is minimal and almost periodic. As we go along, we will illustrate the Krylov–Bogoliubov construction, together with some simple but useful techniques.

Note first that there exists an ergodic measure μ on Σ. The image of μ under the projection $\pi : \Sigma \to \Omega$ is an invariant measure, hence equals ν. We distinguish two possibilities:

(a) $\int_\Sigma f\,\mathrm{d}\mu = 0$,

(b) $\int_\Sigma f\,\mathrm{d}\mu \neq 0$.

Here f is the function occurring in (2.3$_\omega$).

Suppose that (a) holds. We will prove that all Lyapunov exponents of all Eqs. (2.2$_\omega$) are zero.

PROPOSITION 2.5. *Suppose that there is an ergodic measure μ on Σ such that $\int_\Sigma f\,\mathrm{d}\mu = 0$. Let $\varepsilon > 0$ be given. Then there exists $T > 0$ such that, if $\omega \in \Omega$ and if $x_0 \in \mathbb{R}^2$ satisfies $|x_0| = 1$, then*

$$-\varepsilon \leqslant \frac{1}{t}\ln\big|\Phi_\omega(t)x_0\big| \leqslant \varepsilon \quad \textit{whenever } |t| > T.$$

PROOF. We make systematic use of Eq. (2.5), which relates the asymptotic behavior of the quantities $\frac{1}{t}\ln|\Phi_\omega(t)x_0|$ to the time averages of f. Suppose for contradiction that there are sequences $(\omega_k) \subset \Omega$, $(x_k) \subset \mathbb{R}^2$, and $(t_k) \subset \mathbb{R}$ such that $|x_k| = 1$, $|t_k| \to \infty$, and

$$\left|\frac{1}{t_k}\int_0^{t_k} f\big(\hat{\tau}_s(\omega_k, l_k)\big)\,\mathrm{d}s\right| \geqslant \varepsilon.$$

Here $l_k \in \mathbb{P}$ is the line through the origin in $\mathbb{R}^2$ containing x_k. Passing to subsequences if necessary, we can assume that $\lim_{k\to\infty} \frac{1}{t_k}\int_0^{t_k} f(\hat{\tau}_s(\omega_k, l_k))\,\mathrm{d}s$ exists. Call the limit $\Lambda(f)$. Clearly $\Lambda(f) \neq 0$.

Let $C(\Sigma) = \{h : \Sigma \to \mathbb{R} \mid h \text{is continuous}\}$. Let $\mathcal{H}$ be a countable dense $\mathbb{Q}$-vector subspace of $C(\Sigma)$. Using a Cantor diagonal argument, one can determine a subsequence (t_m) of (t_k) such that for each $h \in \mathcal{H}$ the limit

$$\lim_{m\to\infty} \frac{1}{t_m} \int_0^{t_m} h\big(\hat{\tau}_s(\omega_m, l_m)\big)\,\mathrm{d}s$$

exists. Call the limit $\Lambda(h)$ ($h \in \mathcal{H}$). It is easy to see that Λ defines a $\mathbb{Q}$-linear, bounded functional on $\mathcal{H}$, and that $|\Lambda| = 1$. This functional extends uniquely to a $\mathbb{R}$-linear functional – which we also call Λ – on $C(\Sigma)$. One sees that $|\Lambda| \leqslant 1$, and in fact $|\Lambda| = 1$ because, if we let h be the function on Σ whose only value is 1, then $\Lambda(h) = 1$. Moreover, Λ is invariant in the sense that, if $h \in C(\Sigma)$, $t \in \mathbb{R}$, and $(\hat{\tau}_t h)(\sigma) =: h(\hat{\tau}_{-t}(\sigma))$ for each $\sigma \in \Sigma$, then

$$\Lambda(\hat{\tau}_t h) = \Lambda(h).$$

By the Riesz representation theorem, there is a unique Borel regular probability measure μ_Λ on Σ such that $\Lambda(h) = \int_\Sigma h\,\mathrm{d}\mu_\Lambda$ for all $h \in C(\Sigma)$. One checks that μ_Λ is invariant. Clearly $\int_\Sigma f\,\mathrm{d}\mu_\Lambda \neq 0$.

Now let I be the weak-$*$ compact, convex subset of the dual space $C^*(\Sigma)$ which consists of the set of $\{\hat{\tau}_t\}$-invariant measures on Σ (or rather, the linear functionals which correspond to them under the Riesz theorem). Applying the Choquet theory [131] to the set I, and taking account of the fact that ergodic measures on Σ correspond to extreme points of I, one sees that there exists an ergodic measure μ_1 on Σ such that $\int_\Sigma f\,\mathrm{d}\mu_1 \neq 0$. Write $\beta = \int_\Sigma f\,\mathrm{d}\mu_1$.

Next, use the Birkhoff ergodic theorem to find a μ_1-measurable, invariant set $\Sigma_1 \subset \Sigma$ with $\mu_1(\Sigma_1) = 1$, such that, if $\sigma \in \Sigma_1$, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f\big(\hat{\tau}_s(\sigma)\big)\,\mathrm{d}s = \lim_{t\to-\infty} \frac{1}{t} \int_0^t f\big(\hat{\tau}_s(\sigma)\big)\,\mathrm{d}s = \beta.$$

Let $\Omega_1 = \pi(\Sigma_1)$; then Ω_1 is ν-measurable, invariant and $\nu(\Omega_1) = 1$. In a similar way, we can find a μ-measurable, invariant set $\Sigma_* \subset \Sigma$ with $\mu(\Sigma_*) = 1$, such that, if $\sigma \in \Sigma_*$, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f\big(\hat{\tau}_s(\sigma)\big)\,\mathrm{d}s = \lim_{t\to-\infty} \frac{1}{t} \int_0^t f\big(\hat{\tau}_s(\sigma)\big)\,\mathrm{d}s = 0.$$

Let $\Omega_* = \pi(\Sigma_*)$; then Ω_* is also ν-measurable with $\nu(\Omega_*) = 1$.

Choose $\omega \in \Omega_1 \cap \Omega_*$, then pick points $(\omega, l_1) \in \Sigma_1$ and $(\omega, l_*) \in \Sigma_*$. Clearly $l_1 \neq l_*$. Let $x_1 \in l_1$ and $x_* \in l_*$ have norm 1; then x_1 and x_* are linearly independent. Let $x_1(t) = \Phi_\omega(t)x_1$, $x_*(t) = \Phi_\omega(t)x_*$. Let $\Psi(t)$ be the matrix solution of Eq. (2.2$_\omega$) whose columns are $x_1(t)$ and $x_*(t)$. By Liouville's formula $\det\Psi(t) = \det\Psi(0) = const.$ for all $t \in \mathbb{R}$. However, using relation (2.5), we see that $\lim_{t\to\infty} \det\Psi(t) = 0$ if $\beta < 0$ and that $\lim_{t\to-\infty} \det\Psi(t) = 0$ if $\beta > 0$. This is a contradiction. The proof of Proposition 2.5 is complete. $\square$

Proposition 2.5 states that, if there is an ergodic measure μ on Σ for which $\int_\Sigma f\,\mathrm{d}\mu = 0$, then $\lim_{|t|\to\infty} \frac{1}{t} \ln|\Phi_\omega(t)x| = 0$, uniformly in $\omega \in \Omega$ and $x \in \mathbb{R}^2$ with $|x| = 1$. Thus all Lyapunov exponents of all Eqs. (2.2_ω) are zero, and this holds in a uniform way. Now let us describe what happens in the case when there is an ergodic measure μ_1 on Σ for which $\int_\Sigma f\,\mathrm{d}\mu_1 = \beta \neq 0$, i.e., the case (b).

First of all, using the Birkhoff ergodic theorem, we can find a μ_1-measurable, invariant set $\Sigma_1 \subset \Sigma$ with $\mu_1(\Sigma_1) = 1$ such that $\lim_{|t|\to\infty} \frac{1}{t} \int_0^t f(\hat{\tau}_s(\sigma))\,\mathrm{d}s = \beta$ for all $\sigma \in \Sigma_1$. Let $\Omega_1 = \pi(\Sigma_1)$, so that Ω_1 is ν-measurable, invariant, and $\nu(\Omega_1) = 1$. We claim that, for each $\omega_1 \in \Omega_1$, the fiber $\pi^{-1}(\omega_1) \cap \Sigma_1$ contains exactly one point. For, suppose that $(\omega_1, l_1) \in \Sigma_1$ and $(\omega_1, l_*) \in \Sigma_1$, where $l \neq l_*$. Then, using Liouville's formula and relation (2.5) as was done in the last part of the proof of Proposition 2.5, we argue to a contradiction. We thus conclude that Σ_1 is a ν-measurable invariant section in Σ when Σ is viewed as a fiber bundle over Ω.

Further reasoning of this elementary sort can be used to prove the following facts; see [78]. First, there is exactly one other ergodic measure μ_2 on Σ. Second, $\int_\Sigma f\,\mathrm{d}\mu_2 = -\beta$. Third, there is a ν-measurable, invariant section $\Sigma_2 \subset \Sigma$ such that $\mu_2(\Sigma_2) = 1$. The measurable invariant sections define measurable subbundles $W_{1,2} \subset \Omega \times \mathbb{R}^2$ of fiber dimension one.

There are now just two possibilities.

(i) The invariant measurable sections Σ_1, Σ_2 extend to continuous invariant sections of Σ. In this case, W_1 and W_2 are clearly topological subbundles of $\Omega \times \mathbb{R}^2$ of fiber dimension one. One can show that $\Omega \times \mathbb{R}^2 = W_1 \oplus W_2$ (Whitney sum). It is now easily verified that the dynamical spectrum of the family (2.4_ω) coincides with the two point set $\{-\beta, \beta\}$. See [78] for details.

(ii) The invariant sections Σ_1, Σ_2 do not extend to continuous sections of Σ. In this case, one can show that the dynamical spectrum of the family (2.4_ω) coincides with the interval $[-\beta, \beta]$. On the other hand, the Oseledets spectrum is the two point set $\{-\beta, \beta\}$. The Oseledets bundles are (ν-a.e.) the bundles W_1 and W_2. See again [78] for details.

Examples of type (ii) are particularly interesting and invariably give rise to projective flows with interesting structure. As noted in the Introduction, Millionščikov [108] and Vinograd [152] constructed examples of families (2.4_ω) for which $(\Omega, \mathbb{R})$ is an almost periodic minimal set and for which the conditions in (ii) turned out to be true. They were motivated by the search for almost periodic linear differential systems which are not almost reducible, and which are therefore not regular in the sense of Lyapunov. In what follows, we will refer to the number β as the Lyapunov exponent of the family (2.1_ω). The above discussion shows that, if $\beta = 0$, then $\lim_{|t|\to\infty} \frac{1}{t} \ln|\Phi_\omega(t)x_0| = 0$, uniformly with respect to $\omega \in \Omega$ and $x_0 \in \mathbb{R}^2$ satisfying $|x_0| = 1$.

Let us finish this section by defining and briefly discussing the rotation number for Eqs. (2.2_ω). This quantity was defined, and certain of its properties were worked out, in [89]. See also [81] for a further discussion of properties of the rotation number and for a generalization of this concept. It is related to the so-called integrated density of states [147]. A discrete version of the rotation number – the fiber rotation number – was worked out in [68].

The rotation number is most easily defined and analyzed using Eqs. (2.4_ω) for the polar angle θ. Let Ω be a compact metric space, let $(\Omega, \mathbb{R})$ be a flow, and let ν be an ergodic measure on Ω. Consider the function g in (2.4_ω): if $\omega \in \Omega$, $\theta_0 \in \mathbb{R}$, and if $\theta(t)$ is the solution of Eq. (2.4_ω) with $\theta(0) = \theta_0$, then

$$\theta(t) = \theta_0 + \int_0^t g\big(\hat{\tau}_s(\omega_0)\big)\,\mathrm{d}s. \tag{2.6}$$

We state the

DEFINITION 2.6. The *rotation number* α_ν of Eqs. (2.2_ω) with respect to the ergodic measure ν on Σ is

$$\alpha_\nu = \lim_{t\to\infty} \frac{\theta(t)}{t}.$$

Of course one has to prove that this definition makes sense, i.e. that the limit exists in an appropriate sense. It turns out that there is a ν-measurable, invariant set $\Omega_\nu \subset \Omega$ with $\nu(\Omega_\nu) = 1$ such that, if $\omega \in \Omega_\nu$, $\theta_0 \in \mathbb{P} = \mathbb{R}/\pi\mathbb{Z}$ and $\theta(t)$ is the solution of (2.4_ω) with $\theta(0) = \theta_0$, then $\lim_{|t|\to\infty} \frac{\theta(t)}{t}$ exists and does not depend on the choice of $(\omega, \theta_0) \in \Omega_\nu \times \mathbb{P}$. In fact, for each ergodic measure μ on Σ which is a lift of ν in the sense that $\pi(\mu) = \nu$, one has

$$\lim_{|t|\to\infty} \frac{\theta(t)}{t} = \int_\Sigma g\,\mathrm{d}\mu. \tag{2.7}$$

In particular, the right-hand side of (2.7) does not depend on the choice of the ergodic lift μ of ν. The proofs of these statements use (2.6), the Birkhoff ergodic theorem, and certain special arguments. See [89] for details.

It turns out that α has good continuity properties. We discuss one which turns out to be significant. A proof of the following statement can be based on an argument of Krylov–Bogoliubov type; see [89] for arguments of the appropriate type. There are various other continuity results concerning the rotation number. Some of these will be stated later when they are needed.

PROPOSITION 2.7. *Let Ω be a compact metric space, let $(\Omega, \{\tau_t\})$ be a flow, and let ν_k be a sequence of measures which are ergodic with respect to $\{\tau_t\}$. Suppose that ν is a $\{\tau_t\}$-ergodic measure on Ω, and suppose that $\nu_k \to \nu$ in the weak-$*$ sense. Then $\alpha_{\nu_k} \to \alpha_\nu$.*

3. The projective flow

Let us first recall some basic facts concerning a linear periodic system

$$x' = A(t)x, \quad x \in \mathbb{R}^n \tag{3.1}$$

where $A(t) = A(t + 2\pi)$ is a continuous function with values in $\mathbb{M}_n$. Let $\Phi(t)$ be the fundamental matrix solution of (3.1). The asymptotic behavior of the solution of (3.1) is determined by the Floquet matrix $\Phi(2\pi)$ and its iterates. For example, knowledge of the real parts of the logarithms of the eigenvalues and of the generalized eigenspaces of $\Phi(2\pi)$ permit one to determine which solutions of (3.1) exhibit exponential growth or decay, and with which rates.

We wish to understand the directional variation of the solutions of (3.1). Though the concept of "directional variation" does not have immediate significance, it can be rendered meaningful using the imaginary parts of the logarithms of the eigenvalues of $\Phi(2\pi)$ together with its generalized eigenspaces. Alternatively, one can use the Floquet "theory" to determine an invertible, 4π-periodic matrix function $L = L(t)$ such that the change of variables $x = L(t)y$ transforms (3.1) into

$$y' = Ry$$

for a constant real matrix R. Using the generalized eigenspaces and the imaginary parts of the eigenvalues of R, one can develop a theory concerning the rates with which solutions of (3.1) change direction with time, and related matters.

Suppose now that $A(\cdot)$ is nonperiodic. We seek information concerning the asymptotic behavior of the solutions of (3.1). We have seen in Section 2 that the concepts of Lyapunov exponent and exponential dichotomy allow one to develop profound theories concerning the exponential growth of the solutions of (3.1). However it is not clear how one might develop a reasonable theory of the directional variation of those solutions. In this regard, one might for example ask for an analogue of the Floquet theory in the context of nonperiodic systems (3.1). However, it is well known that one cannot hope in general to reduce (3.1) to a system with constant coefficients by mean of a bounded and boundedly invertible change of variables $x = L(t)y$. The search for an analogue of the period matrix $\Phi(2\pi)$ is even more problematic. In fact, it seems difficult to elaborate this concept in a useful way even in the case when $A(\cdot)$ is almost periodic and there exist "almost periods" with a definite meaning.

We therefore try to formulate a theory of the directional variation of the solutions of (3.1) in the nonperiodic case from another point of view. Suppose that a Bebutov-type construction has been carried out, with the following results: a compact metric space Ω, a flow $\{\tau_t\}$ on Ω, a continuous function $A : \Omega \to \mathbb{M}_n$, and a family of equations

$$x' = A\big(\tau_t(\omega)\big)x \tag{3.1_ω}$$

which contains (3.1). Let $\mathbb{P}$ be the $(n-1)$-dimensional projective space of lines through the origin in $\mathbb{R}^n$. Let $\Sigma = \Omega \times \mathbb{P}$. We define a flow $\{\hat{\tau}_t\}$ on Σ as was done in Section 2. Namely, let $\Phi_\omega(t)$ be the fundamental matrix solution of (3.1_ω). Let $\omega \in \Omega$, $l \in \mathbb{P}$ and set

$$\hat{\tau}_t(\omega, l) = \big(\tau_t(\omega), \Phi_\omega(t)l\big)$$

where $\Phi_\omega(t)$ acts linearly on l for each $t \in \mathbb{R}$.

We propose to study the directional variation of the solutions of Eqs. (3.1_ω) by collecting information about the compact invariant subsets of Σ. In particular, we would like to classify the minimal subsets of $(\Sigma, \mathbb{R})$.

When $n \geqslant 3$, the study of the projective flow $(\Sigma, \mathbb{R})$ is still in its infancy, though there is a substantial body of particular results. We remark that there is no reason to restrict attention to the projective flow. It is of interest to substitute $\mathbb{P}$ with other compact homogeneous spaces H of the general linear group or of a Lie subgroup thereof, and study the flow on $\Omega \times H$ induced by Eqs. (3.1_ω). See [46–49,81,90–93,120].

Let us now consider the case when $n = 2$. Then $\mathbb{P}$ is the projective circle, and Σ is a trivial circle bundle over Ω with projection $\pi : \Sigma \to \Omega$: $(\omega, l) \mapsto \omega$. Even in this relatively simple case, we will find diverse phenomena which illustrate the complexity of the directional variation of the solutions of linear differential systems with nonperiodic coefficients.

It is easy to classify the possible projective flows $(\Sigma, \mathbb{R})$ when $n = 2$ and when Ω is the Bebutov hull of a periodic function $A(\cdot)$. This classification can serve as a guide to the study of $(\Sigma, \mathbb{R})$ in more general cases. So return to Eq. (3.1) for a moment, and assume that $A(t + 2\pi) = A(t)$ for all $t \in \mathbb{R}$. We view A as an element of $\mathcal{C} = \{C : \mathbb{R} \to \mathbb{M}_2 \mid C \text{ is continuous and bounded}\}$ with the compact-open topology. We carry out the Bebutov construction beginning with A and obtain a compact hull Ω which is homeomorphic to a circle. In fact, we can think of $\omega \in \Omega$ as an angular coordinate on this circle, so that a generic element of Ω has the form A_ω where $A_\omega(\cdot) = A(\omega + \cdot)$. The space $\Sigma = \Omega \times \mathbb{P}$ is a two-torus.

Let us consider the following model equations:

(α) $x' = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} x$, $a > 0$;

(β) $x' = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} x$, $b \in \mathbb{R}$;

(γ) $x' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$. We view the coefficients matrices as constant functions defined on Ω. Let us examine the corresponding projective flows.

Case (α) There are exactly two minimal subsets $M_1, M_2 \subset \Sigma$; these are flow isomorphic to Ω via the projection $\pi : \Sigma \to \Omega$. In fact, if $l_1 \subset \mathbb{R}^2$ denotes the x_1-axis and $l_2 \subset \mathbb{R}^2$ denotes the x_2-axis, then $M_i = \{(\omega, l_i) \mid \omega \in \Omega\}$, $i = 1, 2$. The orbits in Σ which lie off $M_1 \cup M_2$ exhibit an hyperbolic structure.

Case (β) If $\frac{b}{2\pi}$ is rational, then the flow on Σ laminates in an obvious sense into uncountably many minimal subsets. If $b = 0$ these are defined by conditions of the type $l = constant$. If $\frac{b}{2\pi} = \frac{p}{q}$ where $q \geqslant 1$ and the fraction is expressed in lowest terms, then each minimal set defines a closed curve in Σ which winds q times around Ω as one traverses the circle $\mathbb{P}$. On the other hand, if $\frac{b}{2\pi}$ is irrational, then Σ itself is a minimal set, and the flow on Σ is a Kronecker winding. The orbits in Σ exhibit an elliptic structure.

Case (γ) Let $l_1 \in \mathbb{R}^2$ be the x_1-axis, and let $M = \{(\omega, l_1) \mid \omega \in \Omega\}$. Then $M \subset \Sigma$ is the unique minimal subset of Σ. The orbits of Σ which lie off M exhibit a parabolic structure.

It is rather easy to show that, when $A(\cdot)$ is 2π-periodic, then there is a flow isomorphism between the projective flow defined by Eqs. (3.1_ω) and one of the flows considered in the cases (α), (β) and (γ). To see this, note first that the projective flow $(\Sigma, \mathbb{R})$ does not change if a 2π-periodic, continuous function of the form $\delta(t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is added to $A(t)$, so choosing $\delta(t) = -\frac{1}{2} \operatorname{tr} A(t)$, we can arrange that the trace $\operatorname{tr} A(t)$ is zero of all $t \in \mathbb{R}$.

Assuming this, the Floquet theory assures the existence of a 4π-periodic change of variables $x = L(t)y$ which brings (3.1) to the form

$$y' = Ry$$

where R has one of the forms (α), (β), (γ) (this uses the fact that $L(\cdot)$ can be chosen to have values in $\mathrm{SL}(2, \mathbb{R})$). Now, the projective flow is insensitive to the 2π – respectively 4π – periodicity of the function L. We take advantage of this fact as follows: viewing ω as an angular coordinate on the circle Ω, we define $h : \Sigma \to \Sigma$: $h(\omega, l) = (\omega, L(\omega)l)$. One can check that h defines a flow isomorphism between the flow $(\Sigma, \mathbb{R})$ defined by the appropriate model equation, and the $(\Sigma, \mathbb{R})$ defined by the family (3.1_ω).

Let us now turn to a discussion of the projective flow in the nonperiodic case. For the rest of Section 3, we will assume that $(\Sigma, \mathbb{R})$ is a minimal flow. Frequently we will assume in addition that it is Bohr almost periodic. Let $A : \Omega \to \mathbb{M}_2$ be a continuous function. The family of Eqs. (3.1_ω):

$$x' = A(\omega \cdot t)x$$

defines a flow $(\Sigma, \mathbb{R})$.

One might ask for a classification up to flow isomorphism of all possible projective flows, for all such continuous maps A. This seems to be a completely hopeless endeavor. We set a more modest goal, and ask for a description of the possible minimal subsets of Σ. Progress along these lines has been made. We will describe this progress, and will encounter along the way a rich collection of interesting minimal sets which supplies examples illustrating various phenomena which often are not related in an a priori way to properties of two-dimensional linear differential systems.

Before beginning, let us observe that the projective flow $(\Sigma, \mathbb{R})$ is unchanged if the diagonal matrix function $-\frac{1}{2} \operatorname{tr} A(\cdot)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is added to $A(\cdot)$. Hence for present purposes, there is no loss of generality in assuming that the trace $\operatorname{tr} A(\cdot) = 0$, that is, that A takes values is the Lie algebra $\mathrm{sl}(2, \mathbb{R})$. For the rest of Section 3, we will assume that A is a continuous, $\mathrm{sl}(2, \mathbb{R})$-valued function on Ω.

We begin the discussion with a lemma which will be applied several times in what follows. We give a proof which uses some basic techniques in the theory of nonautonomous differential systems.

LEMMA 3.1. *Let $(\Omega, \mathbb{R})$ be a minimal flow, and let $f : \Omega \to \mathbb{R}$ be a continuous function. Suppose that there is an invariant measure ν_0 on Ω such that $\int_\Omega f \, d\nu_0 = 0$. Then one of the following conditions holds.*

(i) *There is a continuous function $F : \Omega \to \mathbb{R}$ such that $F(\omega \cdot t) - F(\omega) = \int_0^t f(\omega \cdot s) \, ds$ $(\omega \in \Omega, t \in \mathbb{R})$.*

(ii) *There is a residual subset $\Omega_* \subset \Omega$ such that, if $\omega_* \in \Omega_*$, then*

$$\limsup_{t \to \infty} \int_0^t f(\omega_* \cdot s) \, ds = \infty, \qquad \liminf_{t \to \infty} \int_0^t f(\omega_* \cdot s) \, ds = -\infty,$$

$$\limsup_{t\to-\infty}\int_0^t f(\omega_*\cdot s)\,\mathrm{d}s=\infty,\qquad \liminf_{t\to-\infty}\int_0^t f(\omega_*\cdot s)\,\mathrm{d}s=-\infty.$$

Condition (i) *holds if and only if there is a point* $\omega_0\in\Omega$ *such that* $\int_0^t f(\omega_0\cdot s)\,\mathrm{d}s$ *is bounded. Furthermore, if* (i) *holds, then* $\int_\Omega f\,d\nu=0$ *for every invariant measure* ν *on* Ω.

PROOF. We define a flow $\{\tilde\tau_t\mid t\in\mathbb{R}\}$ on $\Omega\times\mathbb{R}$ as follows: $\tilde\tau_t(\omega,x)=(\tau_t(\omega),x+\int_0^t f(\omega\cdot s)\,\mathrm{d}s)$; it is easy to check that $\{\tilde\tau_t\}$ is indeed a flow. Let $\tilde\pi:\Omega\times\mathbb{R}\to\Omega$ be the projection; it is a flow homomorphism.

Let us now first assume that there is a point $\omega_0\in\Omega$ such that $\int_0^t f(\omega_0\cdot s)\,\mathrm{d}s$ is bounded. The orbit $\{\tilde\tau_t(\omega_0,0)\mid t\in\mathbb{R}\}\subset\Omega\times\mathbb{R}$ is bounded, hence its closure D is compact and $\{\tilde\tau_t\}$-invariant. Let $M\subset D$ be a minimal subset. We claim that M defines a section of $\Omega\times\mathbb{R}$; that is, there exists a continuous function $\varphi:\Omega\to\mathbb{R}$ such that $M=\{(\omega,\varphi(\omega))\mid\omega\in\Omega\}$.

To see this, suppose that there exist a point $\omega\in\Omega$ and points $x_1,x_2\in\mathbb{R}$ such that $(\omega,x_i)\in M$ $(i=1,2)$. Let $\delta=|x_2-x_1|$, and let $T_\delta:\Omega\times\mathbb{R}\to\Omega\times\mathbb{R}$ be the map $T_\delta(\omega,x)=(\omega,x+\delta)$. Then $T_\delta(M)=M$ by invariance and minimality of M. Suppose that $\delta\neq 0$. Then $T_{n\delta}(M)=M$ $(n=0,\pm1,\pm2,\ldots)$. But then M is not compact. This contradiction proves that $\delta=0$. It follows that there is a function $\varphi:\Omega\to\mathbb{R}$ such that $M=\{(\omega,\varphi(\omega))\mid\omega\in\Omega\}$. The continuity of φ follows from the compactness of M.

Let us now define $F(\omega)=\varphi(\omega)$ $(\omega\in\Omega)$. Then for each $\omega\in\Omega$ and $t\in\mathbb{R}$, we have $F(\omega\cdot t)-F(\omega)=\int_0^t f(\omega\cdot s)\,\mathrm{d}s$. That is, F satisfies the condition (i).

It is clear that, if (i) holds, then $\int_0^t f(\omega\cdot s)\,\mathrm{d}s$ is bounded for every $\omega\in\Omega$. So we have verified the equivalence of (i) with the boundedness of the integrals $\int_0^t f(\omega\cdot s)\,\mathrm{d}s$.

Next, suppose that condition (i) does not hold. We will prove that the oscillation conditions (ii) are valid. To do so, we will adapt to the present (simple) situation techniques which are used by Sacker and Sell to prove a basic result in the theory of exponential dichotomies ([137]; see also Selgrade [142] and Mañé [105]).

To facilitate comparison with [137], we introduce the family of one-dimensional linear systems

$$x'=f(\omega\cdot t)x,\quad x\in\mathbb{R}\tag{3.2$_\omega$}$$

Let $\varphi(\omega,t)=\exp\int_0^t f(\omega\cdot s)\,\mathrm{d}s$ be the solution of (3.2$_\omega$) satisfying $\varphi(\omega,0)=1$. Equations (3.2$_\omega$) define a flow $\{\tilde\tau_t\mid t\in\mathbb{R}\}$ on $\Omega\times\mathbb{R}$, as follows: $\tilde\tau_t(\omega,x)=(\tau_t(\omega),\varphi(\omega,t)x)$. Here $\{\tilde\tau_t\}$ has a different meaning as compared to the preceding lines.

Let us assume that there is a constant c such that $\int_0^t f(\omega\cdot s)\,\mathrm{d}s\leqslant c$ for all $\omega\in\Omega$ and $t\geqslant 0$; we will presently see that this assumption leads to a contradiction. One checks that if our assumption holds, then $\int_0^t f(\omega\cdot s)\,\mathrm{d}s\geqslant -c$ for all $\omega\in\Omega$, $t\leqslant 0$. Since condition (i) does not hold, we have that $\int_0^t f(\omega\cdot s)\,\mathrm{d}s$ is unbounded above as $t\to-\infty$, for each $\omega\in\Omega$. This means that, for each $\omega\in\Omega$, the only solution $x(t)$ of (3.2$_\omega$) which is bounded on all of $\mathbb{R}$ is the zero solution $x(t)\equiv 0$.

We continue to assume that $\int_0^t f(\omega\cdot s)\,\mathrm{d}s\leqslant c$ for all $\omega\in\Omega$, $t\geqslant 0$. We claim that $\varphi(\omega,t)$ decays to zero in a uniform exponential manner as $t\to\infty$. To see this, we first show

that, for each $\omega \in \Omega$, $\varphi(\omega, t) \to 0$ as $t \to \infty$. For, if were not the case, there would exist $\delta > 0$, $\omega \in \Omega$, and a sequence $t_k \to \infty$ such that $\varphi(\omega, t_k) \geqslant \delta$ $(k = 1, 2, \ldots)$. Passing to a subsequence if necessary, we can assume that $\tau_{t_k}(\omega) \to \bar{\omega}$ and $\varphi(\omega, t_k) \to \bar{x}$ where $\bar{x} \neq 0$. Consider the omega-limit set with respect to the flow $\{\tilde{\tau}_t\}$ of the point $(\omega, 1) \in \Omega \times \mathbb{R}$. This set is compact and invariant, and contains the point $(\bar{\omega}, \bar{x})$. It follows that $\varphi(\bar{\omega}, t)\bar{x}$ is bounded for all $t \in \mathbb{R}$; in fact, $\varphi(\bar{\omega}, t)\bar{x} \leqslant \frac{c}{\delta}$ $(t \in \mathbb{R})$. However, the only bounded solution of $(2_{\bar{\omega}})$ is the zero solution, hence $\bar{x} = 0$. This contradiction shows that $\varphi(\omega, t) \to 0$ as $t \to \infty$.

Next note that, if $\omega \in \Omega$, then there exists t_ω such that, if $t \geqslant t_\omega$, then $\varphi(\omega, t) < 1/2$. There is a neighborhood U_ω of ω in Ω such that, if $\bar{\omega} \in U_\omega$, then $\varphi(\bar{\omega}, t_\omega) < 1/2$. Choose $\omega_1, \ldots, \omega_r \in \Omega$ so that $\Omega = U_{\omega_1} \cup \cdots \cup U_{\omega_r}$. Write $U_i = U_{\omega_i}$, $t_i = t_{\omega_i}$ $(1 \leqslant i \leqslant r)$, and set $T = \max\{t_1, \ldots, t_r\}$. If $\bar{\omega} \in \Omega$ and $\bar{\omega} \in U_k$, then $\varphi(\bar{\omega}, t_j + t_k) = \varphi(\tau_{t_j}(\bar{\omega}), t_k)\varphi(\bar{\omega}, t_j) < (\frac{1}{2})^2$. This argument can be iterated to obtain the following: let $\alpha = \frac{\ln 2}{T}$ and $K = 2\sup\{\varphi(\omega, t) \mid \omega \in \Omega,\ 0 \leqslant t \leqslant T\}$; then $\varphi(\omega, t) \leqslant K\mathrm{e}^{-\alpha t}$ $(t \geqslant 0)$. We have shown that $\varphi(\omega, t)$ decays to zero in a uniform exponential way as $t \to \infty$. Now, the exponential decay has the consequence that, if $\omega \in \Omega$, then

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t f(\omega \cdot s)\,\mathrm{d}s \leqslant -\alpha. \tag{$*$}$$

We claim that condition $(*)$ is inconsistent with the hypothesis that $\int_\Omega f\,\mathrm{d}\nu_0 = 0$.

To see this, observe first that there exist ergodic measures ν_1, ν_2 on Ω such that $\int_\Omega f\,\mathrm{d}\nu_1 \leqslant 0$ and $\int_\Omega f\,\mathrm{d}\nu_2 \geqslant 0$ (these ergodic measures may coincide: $\nu_1 = \nu_2 = \nu$, in which case $\int_\Omega f\,\mathrm{d}\nu = 0$). This follows from the Choquet theory [131] applied to the weak-$*$ compact convex subset I of invariant measures on Ω. In fact, the invariant measure ν_0 can be expressed as an integral over I with respect to a measure H concentrated on the set E of ergodic measures, in such a way that $\int_\Omega f\,\mathrm{d}\nu_0 = \int_E(\int_\Omega f\,\mathrm{d}\nu)\,\mathrm{d}H(\nu)$. So if, say, $\int_\Omega f\,\mathrm{d}\nu > 0$ for all ergodic measures ν, then $\int_\Omega f\,\mathrm{d}\nu_0 > 0$, and similarly if $\int_\Omega f\,\mathrm{d}\nu < 0$ for all ergodic measures ν. This proves the existence of the measures ν_1, ν_2.

Now, if $\int_\Omega f\,\mathrm{d}\nu_2 \geqslant 0$, then there exists a set $\Omega_2 \subset \Omega$ with $\nu_2(\Omega_2) = 1$ such that, if $\omega_2 \in \Omega_2$, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\omega_2 \cdot s)\,\mathrm{d}s \geqslant 0,$$

in contradiction with $(*)$.

Recapitulating, we have shown that, if condition (i) does not hold, then there does not exist a constant c such that $\int_0^t f(\omega \cdot s)\,\mathrm{d}s \leqslant c$ for all $\omega \in \Omega$, $t \geqslant 0$. We now set $\Omega_N = \{\omega \in \Omega \mid \int_0^t f(\omega \cdot s)\,\mathrm{d}s \leqslant N$ for all $t \geqslant 0\}$. Then Ω_N is closed $(N = 1, 2, \ldots)$. If for some N, Ω_N contains a set U which is open in Ω, then there are times $t_1 > 0, \ldots, t_r > 0$ such that $\tau_{t_1}(U) \cup \cdots \cup \tau_{t_r}(U) = \Omega$. It follows that $\int_0^t f(\omega \cdot s)\,\mathrm{d}s$ is bounded above by a uniform constant for all $\omega \in \Omega$, $t \geqslant 0$. But we know that this is not possible. We conclude that $\bigcup_{N=1}^\infty \Omega_N$ has first category in Ω.

Let $\Omega_1 = \Omega \setminus \bigcup_{N=1}^\infty \Omega_N$. If ω is in the residual set Ω_1, then $\limsup_{t\to\infty} \int_0^t f(\omega \cdot s)\,\mathrm{d}s = \infty$. We apply reasoning similar to the above to obtain residual sets $\Omega_2, \Omega_3, \Omega_4 \subset \Omega$ for

which the remaining oscillation conditions hold. Then the residual set $\Omega_* = \bigcap_{i=1}^4 \Omega_i$ satisfies the oscillation conditions (ii). □

One of the first proofs of the equivalence of condition (i) and the existence of a point ω_0 for which $\int_0^t f(\omega \cdot s)\,\mathrm{d}s$ is bounded was published in ([63], Theorem 14.11); we have repeated this proof. The fact that condition (ii) follows from the hypotheses of Lemma 3.1 was proved in [82]. There the condition that $\int_\Omega f\,\mathrm{d}\nu_0 = 0$ for some invariant measure ν_0 is substituted by the equivalent condition that, for some point $\omega \in \Omega$ and some sequence t_k such that $|t_k| \to \infty$, one has $\lim_{k\to\infty} \frac{1}{t_k} \int_0^{t_k} f(\omega \cdot s)\,\mathrm{d}s = 0$.

At several points in the discussion, we will apply a result proved by Cameron [25] in 1936 for almost periodic linear differential systems. It was later generalized to linear differential system with Birkhoff recurrent coefficients [41]. More recently, special cases of Cameron's theorem have been rediscovered by several authors.

THEOREM 3.2 (Cameron). *Suppose that* $(\Omega, \mathbb{R})$ *is minimal and that* $n \geqslant 1$. *Let* $A : \Omega \to \mathbb{M}_n$ *be a continuous function. Suppose that each* $\omega \in \Omega$ *has the property that all solutions of Eq.* (3.1_ω):

$$x' = A(\omega \cdot t)x$$

are bounded. Then there is a continuous and continuously invertible map $L : \Omega \to \mathrm{GL}(n, \mathbb{R})$ *with the following properties.*

- *L has symmetric values*: $L(\omega)^T = L(\omega)$ *for all* $\omega \in \Omega$.
- *The map* $\omega \to \frac{\mathrm{d}}{\mathrm{d}t} L(\omega \cdot t)|_{t=0}$ *is well-defined and continuous.*
- *The change of variables* $x = L(\omega \cdot t)y$ *takes* (3.1_ω) *to the form*

  $$y' = B(\omega \cdot t)y$$

  where $B : \Omega \to \mathbb{M}_n$ *is continuous and has antisymmetric values*: $B(\omega)^T = -B(\omega)$ *for all* $\omega \in \Omega$.
- *If A takes values in* $\mathrm{sl}(2, \mathbb{R})$, *then L takes values in* $\mathrm{SL}(2, \mathbb{R})$.

For a proof of this theorem see [41]. It makes use of Cameron's original idea and some results of Ellis concerning the structure of distal extensions of minimal flows.

We return to the case $n = 2$. Let $(\Omega, \mathbb{R})$ be a minimal flow. In what follows we will usually assume in addition that $(\Omega, \mathbb{R})$ is almost periodic. Let $A : \Omega \to \mathrm{sl}(2, \mathbb{R})$ be a continuous map. Our goal is to analyze the minimal subsets of the projective flow $(\Sigma, \mathbb{R})$ defined by Eqs. (3.1_ω). The first step is to count them. We begin with a strengthened version of a theorem of Sacker and Sell [139].

PROPOSITION 3.3. *Let* $(\Omega, \mathbb{R})$ *be a minimal almost periodic flow, let* $A : \Omega \to \mathrm{sl}(2, \mathbb{R})$ *be continuous, and let* $(\Sigma, \mathbb{R})$ *be the projective flow defined by Eqs.* (3.1_ω). *Suppose that* Σ *contains* $k \geqslant 3$ *distinct minimal sets. Then there is a continuous, continuously invertible*

matrix function $L:\Omega \to \mathrm{SL}(2,\mathbb{R})$ *such that the change of variables* $y = L(\omega \cdot t)x$ *takes Eqs.* (3.1_ω) *into the form*

$$y' = \begin{pmatrix} 0 & -b_0 \\ b_0 & 0 \end{pmatrix} y \quad (\omega \in \Omega)$$

where $b_0 \in \mathbb{R}$ *is a constant.*

We include a proof of this proposition since it illustrates several simple but useful techniques.

PROOF. We first show that, for each $\omega \in \Omega$, the fundamental matrix solution $\Phi_\omega(t)$ of (3.1_ω) is bounded. For this, let M_1, M_2, M_3 be three distinct minimal subsets of Σ. Fix $\omega \in \Omega$, and let $l_1, l_2, l_3 \in \mathbb{P}$ be points such that $(\omega, l_i) \in M_i$, $1 \leqslant i \leqslant 3$. Such points can be found because $(\Omega, \mathbb{R})$ is minimal, hence the projection $\pi(M_i)$ equals Ω $(1 \leqslant i \leqslant 3)$.

Let d be a metric on $\mathbb{P}$. Note that, if $i \neq j$, then (ω, l_i) and (ω, l_j) form a distal pair in the sense that $\inf_{t\in\mathbb{R}} d(\Phi_\omega(t)l_i, \Phi_\omega(t)l_j) > 0$. This is because the orbits $\{\hat{\tau}_t(\omega, l_i) \mid t \in \mathbb{R}\}$ lie in the mutually disjoint minimal sets M_i $(1 \leqslant i \leqslant 3)$. Let $\bar{x}_i \in l_i$ be a vector of norm 1, and set $x_i(t) = \Phi_\omega(t)\bar{x}_i$ $(1 \leqslant i \leqslant 3)$.

Suppose for contradiction that $x_1(t)$ is unbounded. Then there exists a sequence $(t_k) \subset \mathbb{R}$ such that $|t_k| \to \infty$ and $|x_1(t_k)| \geqslant k$. Let $\Psi(t)$ be the matrix solution of (3.1_ω) whose columns are $x_1(t)$ and $x_2(t)$. By Liouville's formula, $\det \Psi(t) = c = \mathit{constant} \neq 0$, and it follows from the distallity condition on (ω, l_1) and (ω, l_2) that $\lim_{k\to\infty} x_2(t_k) = 0$. In a similar way, one proves that $\lim_{k\to\infty} x_3(t_k) = 0$.

Now, however, we let $\Psi_*(t)$ be the matrix solution of (3.1_ω) whose columns are $x_2(t)$ and $x_3(t)$. Then $\det \Psi_*(t) = c_* = \text{constant} \neq 0$ for all $t \in \mathbb{R}$. But $\lim_{k\to\infty} \Psi_*(t_k) = 0$. This contradiction proves that $x_1(t)$ is bounded. In a similar way, one proves that $x_2(t)$ is bounded (and also $x_3(t)$ for that matter). Since a generic solution of (3.1_ω) is a linear combination of $x_1(t)$ and $x_2(t)$, we see that $\Phi_\omega(t)$ is bounded.

Let us apply Theorem 3.2 to find a continuous, continuously invertible function $L:\Omega \to \mathrm{SL}(2,\mathbb{R})$ such that $\omega \mapsto \frac{\mathrm{d}}{\mathrm{d}t} L(\omega \cdot t)|_{t=0}$ is well-defined and continuous, and such that the change of variables $x = L(\omega \cdot t)y$ brings Eq. (3.1_ω) to the form

$$y' = \begin{pmatrix} 0 & -b(\omega \cdot t) \\ b(\omega \cdot t) & 0 \end{pmatrix} y \tag{3.3_ω}$$

where $b:\Omega \to \mathbb{R}$ is continuous. Let ν be an ergodic measure on Ω, and let $b_0 = \int_\Omega b(\omega)\,\mathrm{d}\nu(\omega)$. For each $\omega \in \Omega$, consider the quantity

$$\int_0^t \big(b(\omega \cdot s) - b_0\big)\,\mathrm{d}s = \int_0^t b(\omega \cdot s)\,\mathrm{d}s - b_0 t.$$

According to Lemma 3.1 this quantity satisfies exactly one of the following conditions.

(α) There exists a continuous function $B:\Omega \to \mathbb{R}$ such that $B(\omega \cdot t) - B(\omega) = \int_0^t (b(\omega \cdot s) - b_0)\,\mathrm{d}s$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$.

(β) There is a residual set $\Omega_* \subset \Omega$ such that the oscillation conditions (ii) of Lemma 3.1 hold.

Suppose first that (α) holds. Make the change of variable

$$y = \begin{pmatrix} \cos B(\omega \cdot t) & -\sin B(\omega \cdot t) \\ \sin B(\omega \cdot t) & \cos B(\omega \cdot t) \end{pmatrix} z$$

in Eq. (3.3_ω), and note that, in the z-variable, Eqs. (3.1_ω) take the form

$$z' = \begin{pmatrix} 0 & -b_0 \\ b_0 & 0 \end{pmatrix} z.$$

So in this case Proposition 3.3 is proved.

Suppose the second possibility (β) holds. It turns out that, in this case, we cannot have $k \geqslant 3$. We indicate how this can be proved. Note that the circle group $\mathbb{T} = \mathbb{R}/\pi\mathbb{Z}$ acts on $\mathbb{P} = \mathbb{R}/\pi\mathbb{Z}$ by addition. Hence $\mathbb{T}$ acts on Σ in the following way: if $(\omega, l) \in \Sigma$ and $g \in \mathbb{T}$, then $g \cdot (\omega, l) = (\omega, g + l)$.

Let $(\Sigma, \{\hat{\tau}_t\})$ be the projective flow defined by Eqs. (3.3_ω). Then $\mathbb{T}$ commutes with $\{\hat{\tau}_t\}$ in the sense that, if $g \in \mathbb{T}$, then $g \cdot \hat{\tau}_t(\omega, l) = \hat{\tau}_t(g \cdot (\omega, l))$. Now, if $M \subset \Sigma$ is minimal, then $g \cdot M = \{g \cdot m \mid m \in M\}$ either coincides with M or is disjoint from M. Consider the subgroup G_0 of $\mathbb{T}$ defined as follows: $G_0 = \{g \in \mathbb{T} \mid g \cdot M = M\}$. Since $k > 1$, G_0 must be finite; let G_0 have r elements. Then M is an r-cover of Ω. It can be shown that $(M, \{\tau_t\})$ is an almost periodic minimal set [42,139].

Now however, let $\pi_2 : \Sigma \to \mathbb{P}$ be the projection on the second factor. Using almost periodicity of $(M, \mathbb{R})$, we see that the function

$$t \mapsto -b_0 t + \int_0^t b(\omega \cdot s)\, \mathrm{d}s \mod \pi$$

is almost periodic. Using an old theorem of Bohr (see Fink [54], Proposition 6.7), we see that $\int_0^t (b(\omega \cdot s) - b_0)\, \mathrm{d}s$ is an almost periodic function, hence is bounded. This last condition contradicts (β), so the proof of Proposition 3.3 is complete. In fact we have shown that, if (β) holds, then $k = 1$ for Eqs. (3.3_ω). That is, $(\Sigma, \mathbb{R})$ is minimal. $\square$

REMARKS 3.4. If $(\Omega, \mathbb{R})$ is assumed to be minimal but not necessarily almost periodic, then we can only prove a weaker version of Proposition 3.3. We indicate what can be shown. Suppose that $k \geqslant 3$. We can still apply Cameron's theorem and reduce Eqs. (3.1_ω) to the skew-symmetric form (3.3_ω). The group $\mathbb{T} = \mathbb{R}/\pi\mathbb{Z}$ commutes with the flow on Σ defined by Eqs. (3.3_ω), so if $M \subset \Sigma$ is minimal, and if $G_0 = \{g \in \mathbb{T} \mid g \cdot M = M\}$, then G_0 is a finite subgroup of $\mathbb{T}$. This shows, incidentally, that $k = \infty$ because two points $g_1, g_2 \in \mathbb{T}$ with $g_1 - g_2 \notin G_0$ give rise to distinct minimal sets $M_i = g_i \cdot M$ $(i = 1, 2)$. Thus one has the so-called 1–2–∞ theorem [139].

Suppose now that G_0 contains r elements. Then each minimal subset $M \subset \Sigma$ is an r-cover of Ω, and in fact a finite Abelian group extension of Ω. In these circumstances, it seems reasonable to look for a change of variable $y = L_1(\omega \cdot t) z$ which takes Eqs. (3.3_ω)

to constant-coefficient form. However, we have not been able to prove that such a change of variables always exists.

Let us now turn to the case $k = 2$, where k is the number of distinct minimal subsets of Σ. When the coefficient matrix $A(\cdot)$ in (3.1) is 2π-periodic, the condition $k = 2$ is equivalent to the statement that the corresponding family of Eqs. (3.1_ω) has an exponential dichotomy. In this situation, the projective flow is isomorphic to that determined by the system $x' = \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right)x$. Here the nonzero constant a is interpreted as a 2π-periodic function. We will see that other possibilities are present when (3.1) has an almost periodic coefficient $A(\cdot)$.

Unless otherwise stated, $(\Omega, \mathbb{R})$ will denote a minimal almost periodic flow. Let $A : \Omega \to \mathrm{sl}(2, \mathbb{R})$ be continuous, and let $(\Sigma, \mathbb{R})$ be the projective flow defined by the corresponding family of Eqs. (3.1_ω). Suppose that $(\Sigma, \mathbb{R})$ contains exactly two minimal sets.

Let us first note that, in this case, each Eq. (3.1_ω) admits an unbounded solution. For suppose that this is not the case, and that, for some $\omega \in \Omega$, the fundamental matrix solution $\Phi_\omega(t)$ of Eq. (3.1_ω) is bounded: $|\Phi_\omega(t)| \leqslant c < \infty$ for all $t \in \mathbb{R}$. If $\bar{\omega} \in \Omega$, let $(t_k) \subset \mathbb{R}$ be a sequence such that $\omega \cdot t_k \to \bar{\omega}$. Passing to a subsequence, we can assume that $\Phi_\omega(t + t_k)$ converges, uniformly on compact subsets of $\mathbb{R}$, to a matrix solution $\Psi(t)$ of Eq. $(1_{\bar{\omega}})$. Since $\det \Psi(t) = 1$ for all $t \in \mathbb{R}$, $\Psi(t)$ is an invertible matrix solution of $(1_{\bar{\omega}})$, and moreover $|\Psi(t)| \leqslant c$. It follows that all solutions of $(1_{\bar{\omega}})$ are bounded, as well.

Now however, we can apply Cameron's theorem to reduce Eqs. (3.1_ω) to the form (3.3_ω). Following the analysis of the family (3.3_ω) which was carried out in the proof of Proposition 3.3, we see that either $k = 1$ or $k = \infty$. So indeed each Eq. (3.1_ω) admits an unbounded solution.

It is of course possible that the family (3.1_ω) admits an exponential dichotomy. In this case, all solutions of all Eqs. (3.1_ω) are unbounded. However, we will encounter other, more interesting families of Eqs. (3.1_ω) for which $k = 2$.

First we consider some a priori properties of the minimal subsets of Σ when $(\Omega, \mathbb{R})$ is minimal and almost periodic, and $k = 2$. We begin with some terminology.

DEFINITIONS 3.5. Let Ω be a compact metric space with distance function d, and let $(\Omega, \mathbb{R})$ be a flow. Say that two points $\omega_1, \omega_2 \in \Omega$ form a *distal pair* if there exists $\delta > 0$ such that $d(\omega_1 \cdot t, \omega_2 \cdot t) \geqslant \delta$ for all $t \in \mathbb{R}$. Say that ω_1, ω_2 form a *proximal pair* if $\omega_1 \neq \omega_2$ and there exists a sequence $(t_k) \subset \mathbb{R}$ such that $\lim_{k\to\infty} d(\omega_1 \cdot t_k, \omega_2 \cdot t_k) = 0$. Say that $(\Omega, \mathbb{R})$ is *distal* if every pair $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \neq \omega_2$ is distal, and say that $(\Omega, \mathbb{R})$ is *proximal* is every pair $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \neq \omega_2$ is proximal. Let $(\Sigma, \mathbb{R})$ be another flow where Σ is a metric space, and let $\pi : \Sigma \to \Omega$ be a flow homomorphism. Say that Σ is a *distal extension* of Ω if whenever $\pi(\sigma_1) = \pi(\sigma_2)$ and $\sigma_1 \neq \sigma_2$, the pair (σ_1, σ_2) is distal with respect to the flow $(\Sigma, \mathbb{R})$. Similarly, say that Σ is a *proximal extension* of Ω if whenever $\pi(\sigma_1) = \pi(\sigma_2)$ and $\sigma_1 \neq \sigma_2$, the pair (σ_1, σ_2) is proximal with respect to the flow $(\Sigma, \mathbb{R})$.

In the context of systems of ordinary differential equations, the distallity concept is related to the well-known Favard property [52,158] and to properties of separatedness of the solutions. See [24,139].

PROPOSITION 3.6. *Let* $(\Omega, \mathbb{R})$ *be an almost periodic minimal flow, and let* $A : \Omega \to \mathrm{sl}(2, \mathbb{R})$ *be continuous. Suppose that the projective flow* $(\Sigma, \mathbb{R})$ *defined by Eqs.* (3.1_ω)

admits $k = 2$ minimal sets. Then each such minimal set $M \subset \Sigma$ is an almost automorphic extension of Ω.

For the definition of almost automorphic extension, see Section 2. It is of course understood that the flow homomorphism defining the extension is the projection $\pi : \Sigma \to \Omega$. Proposition 3.6 implies that there is a residual subset $\Omega_0 \subset \Omega$ such that, if $\omega \in \Omega_0$, then $\pi^{-1}(\omega) \cap M$ contains exactly one point. Of course, the possibility that $\Omega_0 = \Omega$ is not excluded.

PROOF. Let M_1 and M_2 be the minimal subsets of Σ. Let us first suppose that, for some $\omega \in \Omega$, the fiber $\pi^{-1}(\omega) \cap M_1$ contains two distinct points m_0, m_1. Further, let $m_2 \in M_2$ be a point such that $\pi(m_2) = \omega$. Let $m_i = (\omega, l_i)$, then choose $\bar{x}_i \in l_i$ with $|\bar{x}_i| = 1$, and set $x_i(t) = \Phi_\omega(t)\bar{x}_i$ $(i = 0, 1, 2)$. Note that (m_0, m_2) and (m_1, m_2) are distal pairs in Σ because M_1 and M_2 are compact and disjoint.

Suppose that $x_2(t)$ is unbounded, so that there is a sequence $(t_k) \subset \mathbb{R}$ such that $\lim_{k\to\infty} |x_2(t_k)| = \infty$. Let $\Psi_0(t)$ be the matrix solution of (3.1_ω) whose columns are $x_0(t)$ and $x_2(t)$. By Liouville's formula, $\det \Psi_0(t) = \det \Psi_0(0) \neq 0$ for all $t \in \mathbb{R}$. Since (m_0, m_2) is a distal pair, it must be the case that $\lim_{k\to\infty} x_0(t_k) = 0$. In an analogous way, we can conclude that $\lim_{k\to\infty} x_1(t_k) = 0$. Now, however, if $\Psi(t)$ is the matrix solution of (3.1_ω) whose columns are $x_0(t)$ and $x_1(t)$, then $0 \neq \det \Psi(0) = \lim_{k\to\infty} \det \Psi(t_k) = 0$, a contradiction. We conclude that $x_2(t)$ is bounded.

If $x_0(t)$ is also bounded, then the matrix solution $\Psi_0(t)$ is bounded. However, we have seen that this possibility is inconsistent with the hypothesis that $k = 2$. Therefore $x_0(t)$ is unbounded. Thus there is a sequence $(t_k) \subset \mathbb{R}$ such that $\lim_{k\to\infty} |x_0(t_k)| = \infty$. It follows that $x_2(t_k) \to 0$ as $k \to \infty$.

Suppose for contradiction that there is a point $m_3 \neq m_2$ in M_2 such that $\pi(m_3) = \omega$. Let $m_3 = (\omega, l_3)$, choose $\bar{x}_3 \in l_3$ such that $|\bar{x}_3| = 1$, and set $x_3(t) = \Phi_\omega(t)\bar{x}_3$. Reasoning as in the first two paragraphs of the proof, we see that $x_3(t_k) \to 0$ as $k \to \infty$. However, if we form the matrix solution of (3.1_ω) whose columns are $x_2(t)$ and $x_3(t)$, and if we apply Liouville's formula along the sequence (t_k), we obtain a contradiction. We conclude that m_2 is the unique point in the fiber $\pi^{-1}(\omega) \cap M_2$. This implies that M_2 is an almost automorphic extension of Ω.

We must still show that M_1 is an almost automorphic extension of Ω. If this is not the case, then for each $\omega \in \Omega$, the fiber $\pi^{-1}(\omega) \cap M_1$ contains at least two points. Arguing as above, we see that $\pi : M_2 \to \Omega$ is injective (actually a homeomorphism). Let $m_2 = (\omega, l_2) \in M_2$, and let $\bar{x}_2 \in l_2$ satisfy $|\bar{x}_2| = 1$. Carrying out the reasoning of the first two paragraphs of the proof, we see that $x_2(t) = \Phi_\omega(t)\bar{x}_2$ is bounded.

At this point, let us return to Eq. (3.3_ω) in Section 2. Letting $f : \Sigma \to \mathbb{R}$ be the function in that equation, and restricting f to M_2, we see that

$$\ln\left|x_2(t)\right| = \int_0^t f\big(\hat{\tau}(m_2)\big)\,\mathrm{d}s$$

for each $m_2 \in M_2$. Now, $(M_2, \mathbb{R})$ is a minimal flow and, according to the previous paragraph, $\int_0^t f(\hat{\tau}(m_2))\,\mathrm{d}s$ is bounded above for all $m_2 \in M_2$. It follows from the oscillation

Lemma 3.1 that $\int_0^t f(\hat{\tau}(m_2))\,\mathrm{d}s$ is uniformly bounded (above and below) for $m_2 \in M_2$, $t \in \mathbb{R}$. In particular, $x_2(t)$ is bounded away from zero for all $m_2 \in M_2$.

Again let $\omega \in \Omega$, let $m_0, m_1 \in M_1 \cap \pi^{-1}(\omega)$, and let $\Psi_0(t)$ be the matrix solution of (3.1_ω) whose columns are $x_0(t)$ and $x_2(t)$. Using the distallity of the pair (m_0, m_2), we see that $\Psi_0(t)$ is bounded. This is a contradiction, so we see that M_1 is an almost automorphic extension of Ω.

Until now, we have worked under the hypothesis that at least one fiber of M_1 contains two (or more) distinct points. To complete the proof of Proposition 3.6, we must examine the case that all fibers $\pi^{-1}(\omega) \cap M_1$ contain just one point. However, if this is true, and if M_2 is not an almost automorphic extension of Ω, then each fiber $\pi^{-1}(\omega) \cap M_2$ contains two or more points. We can thus repeat the reasoning just given; we obtain a contradiction; we then conclude that M_2 is an almost automorphic extension of Ω, after all. This conclude the proof of Proposition 3.6. □

We consider some examples in which $k = 2$. We will see that interesting phenomena are compatible with this hypothesis, which have no analogue in the periodic case. We begin with a class of simple equations which still have some interesting features. Let $(\Omega, \mathbb{R})$ be an almost periodic minimal flow, and let ν be the unique ergodic measure on Ω. Let $a : \Omega \to \mathbb{R}$ be a continuous function with mean value zero, that is to say $\int_\Omega a\,\mathrm{d}\nu = 0$. Let $C_0(\Omega)$ be the class of all such continuous functions on Ω. Next, let $a \in C_0(\Omega)$ be a function such that for some (hence all) $\omega \in \Omega$, the integral $\int_0^t a(\omega \cdot s)\,\mathrm{d}s$ is unbounded.

Consider the family of linear differential systems

$$x' = \begin{pmatrix} a(\omega \cdot t) & 0 \\ 0 & -a(\omega \cdot t) \end{pmatrix} x, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2. \tag{3.4$_\omega$}$$

Letting $l_i \in \mathbb{P}$ be the x_i-axis $(i = 1, 2)$, we see that there are two minimal sets in Σ, namely $M_i = \{(\omega, l_i) \mid \omega \in \Omega\}$, $i = 1, 2$. There are no other minimal subsets of Σ because, if there were, then all solutions of Eqs. (3.4_ω) would be bounded, and that is not true. It is clear that this example shows that the hypothesis $k = 2$ is compatible with the vanishing of the Lyapunov exponent of the family (3.4_ω).

Actually, this class of examples can be divided into two subclasses, as follows. We first consider two subsets $\mathcal{C}_1$ and $\mathcal{C}_2$ of $C_0(\Omega)$ defined as follows: $\mathcal{C}_1 = \{a \in C_0(\Omega) \mid$ there exists a ν-measurable but noncontinuous function $A : \Omega \to \mathbb{R}$ such that $A(\omega \cdot t) - A(\omega) = \int_0^t a(\omega \cdot s)\,\mathrm{d}s$ for all $\omega \in \Omega,\ t \in \mathbb{R}\}$ and $\mathcal{C}_2 = \{a \in C_0(\Omega) \mid \int_0^t a(\omega \cdot s)\,\mathrm{d}s$ in unbounded for some $\omega \in \Omega$, and there is no measurable function $A : \Omega \to \mathbb{R}$ such that $A(\omega \cdot t) - A(\omega) = \int_0^t a(\omega \cdot s)\,\mathrm{d}s$ a.e. in $\omega\}$. It can be shown that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are dense in $C_0(\Omega)$, though $\mathcal{C}_1$ is "bigger" in the sense that it is residual [74].

Next we note that the projective flow $(\Sigma, \mathbb{R})$ of the family (3.4_ω) can be explicitly described because, if $\theta(t)$ is the polar angle of a solution $x(t)$ of (3.4_ω), and if $u(t) = \cot\theta(t)$, then

$$u' = 2a(\omega \cdot t)u.$$

Suppose now that $a \in \mathcal{C}_1$. For each $\theta \in (0,\pi)$, the set $\{(\omega,\theta_\omega) \mid \cot\theta_\omega = \cot \mathrm{e}^{2A(\omega)}, \omega \in \Omega\}$ defines a measurable invariant section of Σ. This implies that, for each $\theta \in (0,\pi)$, there is an ergodic measure μ_θ on Σ defined as follows:

$$\int_\Sigma h\,\mathrm{d}\mu_\theta = \int_\Omega h(\omega,\theta_\omega)\,\mathrm{d}\nu(\omega), \quad h \in C(\Sigma).$$

That is, the flow $(\Sigma,\mathbb{R})$ laminates into measurable invariant sections which define ergodic measures on Σ.

On the other hand, if $a \in \mathcal{C}_2$, then it can be shown that there are only two ergodic measures μ_1 and μ_2, which are supported on M_1 and M_2 respectively. We leave the proof as an exercise for the interested reader.

Let us now consider a class of examples of almost periodic Eqs. (3.1_ω) for which $k=2$, and for which one of the minimal sets $M \subset \Sigma$ is almost automorphic but not almost periodic. We will see that examples exhibiting such minimal sets permit one to answer certain questions regarding linear nonhomogeneous almost periodic differential equations which do not satisfy the Favard separation property.

Let $(\Omega,\mathbb{R})$ be an almost periodic minimal flow, and let $a,b:\Omega\to\mathbb{R}$ be continuous functions. Consider the family of linear equations

$$x' = \begin{pmatrix} \frac{1}{2}a(\omega\cdot t) & b(\omega\cdot t) \\ 0 & -\frac{1}{2}a(\omega\cdot t)\end{pmatrix} x, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2. \tag{3.5_ω}$$

Introducing the polar angle θ in $\mathbb{R}^2$, and setting $u=\cot\theta$, we find that

$$u' = a(\omega\cdot t)u + b(\omega\cdot t). \tag{3.6_ω}$$

Let $(\Sigma,\mathbb{R})$ be the projective flow defined by Eqs. (3.5_ω). Note that, if $l_1 \in \mathbb{P}$ denotes the x_1-axis, then the set $M_0 = \{(\omega,l_1) \mid \omega\in\Omega\}$ is a minimal subset of Σ. The complement $\Sigma\setminus M_0$ is parameterized by points (ω,θ) where $0<\theta<\pi$, or equivalently by points (ω,u) where $-\infty<u<\infty$. Thus we can study the projective flow $(\Sigma,\mathbb{R})$ by studying solutions of the family (3.6_ω) of linear nonhomogeneous equations, and vice-versa.

Let ν be the unique invariant measure on Ω, and let $a_0 = \int_\Omega a\,\mathrm{d}\nu$. It is well-known that, if $a_0 \neq 0$, then each Eq. (3.6_ω) admits a unique almost periodic solution. For example, if $a_0>0$, then

$$u_\omega(t) = \int_t^\infty b(\omega\cdot s)\mathrm{e}^{\int_s^t a(\omega\cdot r)\,\mathrm{d}r}\,\mathrm{d}s$$

is the unique almost periodic solution of (3.6_ω). Of course, we are dealing here with a simple case of a general theory regarding the existence, uniqueness, and recurrence properties of bounded solutions of a linear nonhomogeneous differential system whose linear part admits an exponential dichotomy.

Our interest here is in the case when $a_0=0$. It is convenient to consider two subcases. First, suppose that $\int_0^t a(\omega\cdot s)\,\mathrm{d}s$ is bounded for some (hence all) $\omega\in\Omega$. One then checks

that, if some equation $(3.6_{\bar{\omega}})$ admits a bounded solution, then for every $\omega \in \Omega$, all solutions of (3.6_ω) are almost periodic. One can put this trivial observation in the context of the general Favard theory concerning bounded solutions of linear nonhomogeneous almost periodic differential systems in dimension n [52]. To see this, note that the family (3.6_ω) satisfies the Favard separation condition: if $\omega \in \Omega$ and if $u_1(t), u_2(t)$ are two bounded solutions of (3.6_ω), then $\inf_t |u_1(t) - u_2(t)| > 0$. The Favard theory then states that, if one Eq. $(6_{\bar{\omega}})$ admits a bounded solution, then every Eq. (3.6_ω) admits an almost periodic solution. In the one-dimensional case $n = 1$, this implies that all solutions of (3.6_ω) are almost periodic $(\omega \in \Omega)$.

We are going to consider the second case, which is when $\int_0^t a(\omega \cdot s)\,\mathrm{d}s$ is unbounded for some (hence all) $\omega \in \Omega$. In this situation, Eqs. (3.6_ω) do not all satisfy the Favard separation property (this follows from Lemma 3.1). For a long time it was not known if the existence of a bounded solution of some Eq. $(6_{\bar{\omega}})$ implies that Eqs. (3.6_ω) admit almost periodic solutions. The question was settled in the negative by examples of [158] and [73]. It turn out that, if some Eq. $(6_{\bar{\omega}})$ is assumed to have a bounded solution, then some Eq. (3.6_ω) (perhaps not the original equation) admits an almost automorphic solution; however this almost automorphic solution need not to be almost periodic. This follows from the discussion and the example in [73].

In a recent paper [126], Ortega and Tarallo carried out a systematic analysis of examples like those in [158] and [73]. They developed techniques which permit one to construct examples of "non-Favard type" for all almost periodic flows $(\Omega, \mathbb{R})$. In fact, they prove a result ([126], Theorem 2) which, for the one-dimensional family (3.6_ω), has the following consequence.

THEOREM 3.7. *Suppose that, for some $\bar{\omega} \in \Omega$, there holds*

$$\lim_{t\to-\infty} \int_0^t a(\bar{\omega} \cdot s)\,\mathrm{d}s = \lim_{t\to\infty} \int_0^t a(\bar{\omega} \cdot s)\,\mathrm{d}s = -\infty.$$

Then there exists a continuous function $b : \Omega \to \mathbb{R}$ such that each Eq. (3.6_ω) *admits a bounded solution, and for which no Eq.* (3.6_ω) *admits an almost periodic solution.*

We will not prove this theorem here, instead we refer to ([126], Section 6-7). There a particular concept of weak solution of an n-dimensional linear nonhomogeneous almost periodic system is developed. For related considerations see [127].

The $a(\cdot)$-term of the example in [158] and [73] satisfies the hypothesis of Theorem 3.7 for some $\bar{\omega} \in \Omega$. In fact, [158] makes use of a classical quasi-periodic example of Bohr [21], while [73] applies a limit periodic example of Conley and Miller [27].

We now extend Theorem 3.7 in that we discuss the recurrence properties of the bounded solutions of Eqs. (3.6_ω). We follow the discussion in [73]. We continue to assume that $k = 2$.

PROPOSITION 3.8. *Suppose that $\int_\Omega a\,\mathrm{d}\nu = 0$, and suppose that $\int_0^t a(\omega \cdot s)\,\mathrm{d}s$ is unbounded for some* (*hence all*) *$\omega \in \Omega$. Suppose further that some Eq.* (3.6_ω) *admits a bounded solution. Then there is a residual set $\Omega_* \subset \Omega$ such that, if $\omega \in \Omega_*$, then* (3.6_ω) *admits a unique bounded solution which is almost automorphic.*

Of course it is not excluded that the bounded solutions in this proposition are almost periodic.

PROOF. Let $\bar{\omega} \in \Omega$ be a point such that Eq. $(3.6_{\bar{\omega}})$ admits a bounded solution $\bar{u}(t)$. Let $\bar{\theta}(t) = \cot^{-1} u(t)$, and note that $\{(\tau_t(\bar{\omega}), \bar{\theta}(t)) \mid t \in \mathbb{R}\}$ is an orbit of the projective flow defined by Eqs. (3.5_ω). The orbit closure $\mathrm{cls}\{(\tau_t(\bar{\omega}), \bar{\theta}(t)) \mid t \in \mathbb{R}\}$ is a compact invariant subset of Σ which is disjoint from the minimal set $M_0 = \{(\omega, l_1) \mid \omega \in \Omega\}$. It therefore contains a minimal set M_1, such that $M_1 \cap M_0 = \emptyset$.

Now, it is clear that, each Eq. (3.5_ω) admits an unbounded solution. In fact there exists $\omega \in \Omega$ for which each solution $x(t) = \binom{x_1(t)}{x_2(t)}$ satisfying $x_2(t) \equiv 0$ is unbounded, and therefore every Eq. (3.5_ω) admits an unbounded solution. Taking account of Proposition 3.3, we see that M_0 and M_1 are the only minimal subsets of Σ. By Proposition 3.6, M_1 is an almost automorphic extension of Ω. This means that there is a residual subset $\Omega_1 \subset \Omega$ such that, if $\omega_1 \in \Omega_1$, then the fiber $\pi^{-1}(\omega_1) \cap M_1$ contains exactly one point $(\omega_1, \bar{\theta}_1)$. One checks that, if $\hat{\tau}_t(\omega_1, \bar{\theta}_1) = (\tau_t(\omega), \theta_1(t))$, then $\theta_1(t)$ is an almost automorphic function. Hence $u_1(t) = \cot \theta_1(t)$ is an almost automorphic solution of (3.6_{ω_1}).

It follows from the oscillation Lemma 3.1 that there is a residual set $\Omega_2 \subset \Omega$ such that, if $\omega \in \Omega_2$, Eq. (3.6_ω) admits a unique bounded solution. Set $\Omega_* = \Omega_1 \cap \Omega_2$; then Ω_* satisfies the condition of Proposition 3.8. This completes the proof. □

Proposition 3.8 is of course applicable to all one-dimensional examples of Ortega–Tarallo type, and hence to the examples in [158] and [73]. Let us note that almost automorphicity is a more specific property than that of Besicovich almost periodicity. It was noted in [158] that a bounded solution of (3.6_ω) satisfies this last property; see that reference for the definition of almost periodicity in the Besicovich sense. The nature of Besicovich almost periodic functions has recently been discussed by Andres and his collaborators [6].

Examples of Ortega–Tarallo type correspond to two-dimensional families (3.5_ω) whose projective flows $(\Sigma, \mathbb{R})$ admit exactly two minimal subsets, one of which, say M_1, is an almost automorphic, non-almost periodic extension of Ω. There is a question concerning the structure of this minimal set M_1, which, so far as we know, has not been answered. In fact, it is not known (to us) if there exists a family (3.5_ω) for which the minimal set M_1 supports more than one invariant measure. In other words, we ask if there is a family (3.5_ω) for which the projection $\Omega_* = \{\omega \in \Omega \mid \pi^{-1}(\omega) \cap M_1$ contains just one point$\}$ of the set of all automorphic points of M_1 has zero ν-measure: $\nu(\Omega_*) = 0$. It is known that the minimal set of the family (3.5_ω) in [73] supports only one invariant measure; see [76].

Another question – probably more difficult – also awaits an answer. Namely, one would like to find a family of type (3.1_ω) for which $k = 2$, and for which both minimal subsets of Σ are almost automorphic, non-almost periodic extensions of Ω. Such a family (3.1_ω) cannot have the form (3.5_ω), because for families of the latter type the minimal set $M_0 = \{(\omega, l_1) \mid \omega \in \Omega\} \subset \Sigma$ is flow isomorphic to Ω via the projection π.

We continue to assume that $(\Omega, \mathbb{R})$ is an almost periodic minimal flow, and turn to the case when the projective flow $(\Sigma, \mathbb{R})$ defined by Eq. (3.1_ω) contains just one minimal set, i.e., $k = 1$. It will turn out that the projective flow can exhibit an ample range of interesting structures, most of which have no analogues in the periodic case. We divide the discussion into three subsections, according to the dynamical behavior of the solutions of Eq. (3.1_ω).

For want of better terminology, we label these subsections as follows: (A) the strongly elliptic case; (B) the weakly elliptic case; (C) the weakly hyperbolic case. There is actually a fourth case, that in which Eqs. (3.1_ω) admit an exponential dichotomy. In this case, the two minimal sets are defined by the "traces" in $\Omega \times \mathbb{P}$ of the Sacker–Sell subbundles $W_\pm$ which were discussed in Section 2. See [78] for the details. We will not discuss this case further.

(A) Strongly elliptic case

The condition we impose here is that all solutions of all Eqs. (3.1_ω) are bounded. This condition is consistent with the requirement that $k = 1$. We examine the various flows $(\Sigma, \mathbb{R})$ which can arise. We will see that Σ is itself a minimal set. However it can exhibit various structures which were already understood by Furstenberg about fifty years ago [55].

First of all, we apply Cameron's theorem 3.2 to find a continuous, continuously invertible function $L : \Omega \to \mathrm{SL}(2, \mathbb{R})$ such that $\omega \mapsto \frac{\mathrm{d}}{\mathrm{d}t} L(\omega \cdot t)|_{t=0}$ is continuous and such that the change of variables $x = L(\omega \cdot t)y$ brings Eqs. (3.1_ω) to the form

$$y' = \begin{pmatrix} 0 & -b(\omega \cdot t) \\ b(\omega \cdot t) & 0 \end{pmatrix} y, \tag{3.7_ω}$$

Let ν be the unique ergodic measure on Ω, and let $b_0 = \int_\Omega b\,\mathrm{d}\nu$. We consider two possibilities: (1) $\int_0^t (b(\omega \cdot t) - b_0)\,\mathrm{d}s$ is bounded for some (hence all) $\omega \in \Omega$; (2) $\int_0^t (b(\omega \cdot t) - b_0)\,\mathrm{d}s$ is unbounded for some (hence all) $\omega \in \Omega$.

If the first possibility is realized, there is a continuous function $B : \Omega \to \mathbb{R}$ such that $B(\omega \cdot t) - B(\omega) = \int_0^t (b(\omega \cdot t) - b_0)\,\mathrm{d}s$, and, as in the proof of Proposition 3.3, the further change of variables $y = \begin{pmatrix} \cos B(\omega \cdot t) & -\sin B(\omega \cdot t) \\ \sin B(\omega \cdot t) & \cos B(\omega \cdot t) \end{pmatrix} z$ brings Eqs. (3.7_ω) to the form

$$z' = \begin{pmatrix} 0 & -b_0 \\ b_0 & 0 \end{pmatrix} z. \tag{3.8_ω}$$

The hypothesis that $k = 1$ imposes a condition on b_0 relative to the frequency module $\mathcal{M}_\Omega$ of $(\Omega, \mathbb{R})$, namely that there does not exist a rational number r such that $rb_0 \in \mathcal{M}_\Omega$. When this condition holds, the flow $(\Sigma, \mathbb{R})$ defined by Eqs. (3.8_ω) is minimal and almost periodic. If $(\Omega, \mathbb{R})$ is quasi-periodic, then $(\Sigma, \mathbb{R})$ is a Kronecker-type minimal flow.

If the second possibility is realized one has a less simple situation, which however can be understood thanks to [55]. First of all, it turns out that $(\Sigma, \mathbb{R})$ is minimal but not almost periodic. Further information can be obtained by distinguishing two subcases, as follows. For simplicity, set $b_0 = 0$.

(1) Suppose that there exists a ν-measurable but noncontinuous function $B : \Omega \to \mathbb{R}$ such that $B(\omega \cdot t) - B(\omega) = \int_0^t b(\omega \cdot s)\,\mathrm{d}s$ for all $t \in \mathbb{R}$, $\omega \in \Omega$. We saw earlier that the class $\mathcal{C}_1$ of functions $b \in C_0(\Omega)$ which have this property is dense in $C_0(\Omega)$. Let $\theta \in \mathbb{R}$ be an angular coordinate mod π on $\mathbb{P}$. For each fixed value $\theta \in [0, \pi)$, set $\theta_\omega = (\theta + B(\omega)) \bmod \pi$, then

note that the set $\{(\omega,\theta_\omega) \mid \omega\in\Omega\}$ is a measurable invariant section of Σ. There is an ergodic measure μ_θ on Σ defined as follows:

$$\int_\Sigma h\,\mathrm{d}\mu_\theta = \int_\Omega h(\omega,\theta_\omega)\,\mathrm{d}\nu(\omega) \quad \bigl(h\in C(\Sigma)\bigr).$$

That is, the flow $(\Sigma,\mathbb{R})$ laminates into measurable invariant sections which define ergodic measures μ_θ on Σ. A similar situation was considered when $k=2$.

(2) There exists no measurable function B with the above property. As we saw before, the class $\mathcal{C}_2\subset C_0(\Omega)$ of such functions b is residual in $C_0(\Omega)$. When $b\in\mathcal{C}_2$, the flow $(\Sigma,\mathbb{R})$ admits a unique ergodic measure μ (see [55], Lemma 2.1). It can be verified that μ is equivalent to the product of ν with the normalized Lebesgue measure m on $\mathbb{P}$.

Before considering the situations envisioned in subsections (B) and (C), we state a lemma which gives some a priori information concerning the structure of the projective flow in these cases.

LEMMA 3.9. *Let $(\Omega,\mathbb{R})$ be an almost periodic minimal flow, let $A:\Omega\to\mathrm{sl}(2,\mathbb{R})$ be a continuous function, and let $(\Sigma,\mathbb{R})$ be the projective flow defined by Eqs.* (3.1_ω). *Suppose that Σ contains exactly one minimal set M, and that some Eq.* (3.1_ω) *admits an unbounded solution.*

Then either M is a proximal extension of Ω, or M is a proximal extension of an almost periodic two-cover of Ω. In the latter situation, M is an almost automorphic extension of an almost periodic two-cover of Ω.

See Definitions 3.5 for the meaning of the term "proximal extension". A proof of Lemma 3.9 is given in ([83], Section 7). We only sketch it here.

To begin, suppose that $(M,\mathbb{R})$ is not a proximal extension of $(\Omega,\mathbb{R})$. Then there is a point $\bar\omega\in\Omega$ together with a distal pair $\bar m_1=(\bar\omega,\bar l_1),\bar m_2=(\bar\omega,\bar l_2)$ of points in M. Letting d be a metric on $\mathbb{P}$, and writing $\hat\tau_t(\bar\omega,\bar l_i)=(\tau_t(\bar\omega),l_i(t))$ $(i=1,2)$, we can find a number $\delta>0$ such that $d(l_1(t),l_2(t))\geqslant\delta$ for all $t\in\mathbb{R}$. Passing to the closure of one of the orbits $\{\hat\tau_t(\bar m_i)\mid t\in\mathbb{R}\}$ and using the minimality of M, one shows that, for each $\omega\in\Omega$, the fiber $\pi^{-1}(\omega)\cap M$ contains a distal pair (m_1,m_2). In particular, each fiber $\pi^{-1}(\omega)$ contains at least two points.

Using Lemma 3.1 together with by-now familiar reasoning using matrix solutions of Eqs. (3.1_ω), one can show that there is a residual subset $\Omega_*\subset\Omega$ such that, if $\omega\in\Omega_*$, then $\pi^{-1}(\omega)\cap M$ contains exactly two points. At this point we introduce the proximal relation $\mathcal{P}=\{(m_1,m_2)\in M\times M\mid (m_1,m_2)\text{ is a proximal pair}\}$. It is clear that $\mathcal{P}$ is invariant in the sense that, if $(m_1,m_2)\in\mathcal{P}$, then $(\hat\tau_t(m_1,m_2))\in\mathcal{P}$ for all $t\in\mathbb{R}$. It is also clear that, if $(m_1,m_2)\in\mathcal{P}$, then $\pi(m_1)=\pi(m_2)$. Now, it can be shown that $\mathcal{P}$ is a *closed* subset of $M\times M$. According to a general result of Ellis ([40], Lemma 5.17), this implies that $\mathcal{P}$ is an equivalence relation on M. Define Ω_2 to be the topological quotient $M/\mathcal{P}$. One checks that Ω_2 is a compact metrizable space, that it inherits a flow $\{\hat\tau_t^2\}$ from $\{\hat\tau_t\}$, and that $(\Omega_2,\{\hat\tau_t^2\})$ is an almost periodic 2-cover of Ω. One also checks that M is a proximal extension of Ω.

The statement of Lemma 3.9 follows from these considerations. Let us now turn to a discussion of the weakly elliptic case.

(B) Weakly elliptic case

Let $(\Omega, \mathbb{R})$ be an almost periodic minimal flow, and let $A \to \mathrm{sl}(2, \mathbb{R})$ be a continuous function. We impose the following conditions on Eqs. (3.1_ω):

(i) one Eq. (3.1_ω) admits an unbounded solution, hence all Eqs. (3.1_ω) have this property;

(ii) the Lyapunov exponent β of the family (3.1_ω) is zero.

Here the Lyapunov exponent is that with respect to the ergodic measure ν on Ω; see Section 2. The condition (ii) implies that $\frac{1}{t} \ln |\Phi_\omega(t)x_0|$ tends to zero uniformly in $\omega \in \Omega$ and $x_0 \in \mathbb{R}^2$ with $|x_0| = 1$ as $|t| \to \infty$; see Proposition 2.5.

We mention two types of minimal subset $M \subset \Sigma$ which are encountered when conditions (i) and (ii) hold.

First, there are weakly elliptic families (3.1_ω) for which the projective flow $(\Sigma, \mathbb{R})$ contains a (unique) minimal set M which is a genuine 2-cover of Ω. This is a remarkable phenomenon which has no analogue in the periodic case. The first example illustrating it seems to be in [88]; another is given in [123]. These examples both have the property that $(\Omega, \mathbb{R})$ is of limit periodic type. Apparently no example of quasi-periodic type has been constructed. It would be interesting to have a larger body of examples of families (3.1_ω) for which $(\Sigma, \mathbb{R})$ exhibits such a minimal set.

Second, there exist weakly elliptic families (3.1_ω) such that the projective flow $(\Sigma, \mathbb{R})$ is minimal and admits a unique invariant measure. According to Lemma 3.9 and its proof, the minimality of $(\Sigma, \mathbb{R})$ implies that it is a proximal extension of $(\Omega, \mathbb{R})$. Examples of this type were constructed in a more general context by Glasner and Weiss [61]; they used ideas of Fathi and Herman [51] and Anosov and Katok [7]. Subsequently Nerurkar and Sussmann [114,115] have developed very general methods which permit in particular to obtain examples of the present type for various particular classes of coefficients matrices $A(\cdot)$. They can treat the case – very special from their point of view – when $A(\cdot)$ has the form $\begin{pmatrix} 0 & 1 \\ q(\omega) & 0 \end{pmatrix}$, which corresponds to Eqs. (3.1_ω) of Hill–Schrödinger type; they can find functions $q(\cdot)$ for which the family (3.1_ω) gives rise to a minimal, uniquely ergodic projective flow Σ which is a proximal extension of Ω. It seems possible that such projective flows arise "frequently" even in suitable parameterized families of quasi-periodic equations of Schrödinger type with analytic potential q and Diophantine frequency vector; see, e.g., Eliasson [38,39], Puig [134,133].

We do not know of examples of weakly elliptic families (3.1_ω) whose projective flows have properties qualitatively different from those illustrated above. In this regard, it would be interesting to know if there is a weakly elliptic family for which the minimal set $M \subset \Sigma$ is an almost automorphic, non-almost periodic extension of an almost periodic 2-cover Ω_2 of Ω. It would also be satisfying to decide whether or not there are examples for which M is a proximal non-almost automorphic extension of Ω but $M \neq \Sigma$.

We turn to the

(C) Weakly hyperbolic case

Once again let $(\Omega, \mathbb{R})$ be a minimal almost periodic flow, and let $A : \Omega \to \mathrm{sl}(2, \mathbb{R})$ be a continuous function. We impose the following conditions:

(i) the family (3.1_ω) does not admit an exponential dichotomy;

(ii) the Lyapunov exponent β of the family (3.1_ω) is positive: $\beta > 0$.

We say that weakly hyperbolic families (3.1_ω) are of “M-V type”, in recognition of the papers of Millionščikov [108] and Vinograd [152]. Those authors constructed the first limit periodic resp. quasi-periodic examples of families (3.1_ω) of weakly hyperbolic type. They did not explicitly study the properties of the projective flows corresponding to their families. That was done later in [78,83] and other papers.

Generally speaking, the projective flow $(\Sigma, \mathbb{R})$ of a weakly hyperbolic family (3.1_ω) has remarkable properties. We will amplify this remark in the succeeding discussion. Let us begin by recalling from Section 2 that, in the weakly hyperbolic case, the projective flow $(\Sigma, \mathbb{R})$ admits exactly two ergodic measures μ_1 and μ_2. See Proposition 2.5 and the following discussion. These are supported on the measurable invariant sections of Σ which are defined by the “traces” in Σ of the Oseledets measurable invariant subbundles; see Section 2. Moreover, according to the discussion in the present Section 3, the projective flow $(\Sigma, \mathbb{R})$ contains exactly one minimal set $(M, \mathbb{R})$. It is an exercise to show that the two ergodic measures μ_1, μ_2 are actually supported on M. According to Proposition 3.9, $(M, \mathbb{R})$ is a proximal extension of $(\Omega, \mathbb{R})$.

It seems important to us to discuss the ideas underlying the construction of the examples in [108] and [152]. This is because those ideas, albeit in somewhat amplified form, have been applied by many authors since. So let us consider a fixed Bohr almost periodic function $A : \mathbb{R} \to \mathrm{sl}(2, \mathbb{R})$ together with the corresponding two-dimensional linear system

$$x' = A(t)x. \tag{3.9}$$

If x_1, x_2 are nonzero vectors in $\mathbb{R}^2$, let $\Theta(x_1, x_2)$ denote the counterclockwise angle between x_1 and x_2. Suppose that there exist a number $\delta > 0$, sequences $s_k \to -\infty$ and $t_k \to \infty$, and solutions $x_1^{(k)}(t), x_2^{(k)}(t)$ of (4.71) such that the following conditions hold:

$$\begin{aligned} &\bigl|x_1^{(k)}(0)\bigr| = 1 \quad \text{and} \quad \frac{1}{t_k} \ln\bigl|x_1^{(k)}(t_k)\bigr| \leqslant -\delta; \\ &\bigl|x_2^{(k)}(0)\bigr| = 1 \quad \text{and} \quad \frac{1}{s_k} \ln\bigl|x_2^{(k)}(s_k)\bigr| \geqslant \delta; \\ &\Theta\bigl(x_1^{(k)}(0), x_2^{(k)}(0)\bigr) < \frac{1}{k}. \end{aligned} \tag{$*$}$$

Let us note that Proposition 2.5 together with either of the first two conditions in $(*)$ imply that the family (3.1_ω) obtained from (3.9) via the Bebutov construction can be neither elliptic nor weakly elliptic. There remain just two possibilities: either the family (3.1_ω) has an exponential dichotomy, or it is weakly hyperbolic. Suppose for contradiction that it has an exponential dichotomy. Then it can be verified that Eqs. (4.71) admit solutions $x_1(t), x_2(t)$, with $|x_1(0)| = |x_2(0)| = 1$, such that the following conditions are valid:

$$\lim_{|t|\to\infty} \frac{1}{t} \ln\bigl|x_1(t)\bigr| \leqslant -\delta, \qquad \lim_{|t|\to\infty} \frac{1}{t} \ln\bigl|x_2(t)\bigr| \geqslant \delta. \tag{$**$}$$

We can write $x_i^{(k)}(0) = (\cos\theta_i^{(k)}, \sin\theta_i^{(k)})$ and $x_i(0) = (\cos\theta_i, \sin\theta_i)$ $(i = 1, 2)$. Passing to subsequences, we can assume that $\theta_i^{(k)} \to \bar{\theta}_i \in \mathbb{R}$ $(i = 1, 2)$. Let us show that $\bar{\theta}_1 =$

$\theta_1 \bmod \pi$. Suppose this is not true. Let $\Psi_k(t)$ be the fundamental matrix solution of (3.9) whose columns are $x_1^{(k)}(t)$ and $x_1(t)$. Then $|\sin(\theta_1^{(k)} - \theta_1)| = |\det \Psi_k(0)| = |\det \Psi_k(t)|$ is bounded away from zero for all sufficiently large k. However $\lim_{k\to\infty} \det \Psi_k(t_k) = 0$ by $(*)$ and $(**)$, so one has a contradiction.

In a similar way, one shows that $\bar{\theta}_2 = \theta_2 \bmod \pi$ (use the existence of the sequence (s_k)). Now, however, $\theta_2 = \theta_1 \bmod \pi$, because of the relation in the third line of $(*)$. We conclude that Eq. (3.9) gives rise to a family which is weakly hyperbolic.

Two points should be made concerning the above construction. First of all, Millionščikov's example [108] satisfies the condition $(*)$. Also, Vinograd's example [152] satisfies a variant of $(*)$ wherein one must translate the origin from $t = 0$ and make use of the stronger relations which substitute for $(**)$

$$\limsup_{|t-s|\to\infty} \frac{1}{t-s} \ln \frac{|x_1(t)|}{|x_1(s)|} \leqslant -\delta,$$

$$\liminf_{|t-s|\to\infty} \frac{1}{t-s} \ln \frac{|x_2(t)|}{|x_2(s)|} \geqslant \delta.$$

Second, conditions $(*)$ give a concrete meaning to the intuitive hypothesis that "the Oseledets bundles of the family (3.1_ω) are not bounded apart". This intuitive hypothesis has been rendered explicit and applied by later authors in ways similar to conditions $(*)$; see, e.g., [19,118].

Still another point to be made is that, for discrete two-dimensional system, a subharmonic trick due to Herman [68] allows one to construct discrete weakly hyperbolic cocycles with values in $SL(2, \mathbb{R})$. Herman's method and generalizations thereof have been fruitfully applied in the study of the discrete Schrödinger operator, as evidenced by a large literature which we do not attempt to review.

We now show that examples of M-V type illustrate various interesting phenomena related to the behavior of solutions of linear, two-dimensional differential systems.

We first consider a problem posed by Marcus and Moore [107] concerning disconjugate solutions of the almost periodic Hill–Schrödinger equation. Consider the second-order equation

$$-\varphi'' + q(t)\varphi = E\varphi \tag{3.10}$$

where $q(\cdot)$ is a Bohr almost periodic function, and E is a real parameter. It is well-known that there is a number E_0 such that, if $E \leqslant E_0$, then every nonzero solution of (3.10) admits at most one zero on $[0, \infty)$. That is to say, Eq. (3.10) is disconjugate on $[0, \infty)$. On the other hand, if $E > E_0$, then every nonzero solution of (3.10) has infinitely many zeroes which cluster at $t = \infty$. One says that (3.10) is oscillatory at $t = \infty$. See, e.g., [28,67]. The concept of disconjugacy has connections to the calculus of variations which we do not discuss here.

If Eq. (3.10) is disconjugate, then it turns out that it admits a "smallest", or *principal* solution $\varphi_0(t)$. In the quasi periodic case (and more generally in the case when $q(\cdot)$ is

Birkhoff recurrent), the principal solution is positive on all of $\mathbb{R}$, and is uniquely defined by the following property: if $\varphi_1(t)$ is a solution of (3.10) which is independent of $\varphi_0(t)$, then

$$\frac{\varphi_0(t)}{\varphi_1(t)} \to 0 \quad \text{as } t \to \infty.$$

This condition makes sense because $\varphi_1(t)$ is nonzero for all large t. An equivalent property is the following: φ_0 is, up to a constant multiple, the unique solution of (3.10) which satisfies

$$\int^{\infty} \frac{\mathrm{d}t}{\varphi_0(t)^2} = \infty.$$

Marcus and Moore posed the following problem. Let $\varphi_0(t)$ be the principal solution of Eq. (3.10); is it the case that the logarithmic derivative $\frac{\mathrm{d}}{\mathrm{d}t} \ln \varphi_0(t)$ is almost periodic? If q is a periodic function, then $\frac{\varphi_0'(t)}{\varphi_0(t)}$ is indeed periodic with the same period as q. Also, if q is almost periodic and $\frac{\varphi_0'(t)}{\varphi_0(t)}$ is almost periodic, then the frequency module of the latter is contained in that of the former [107]. Still another general fact is that, if $E < E_0$, then the Marcus–Moore question has a positive answer. So, in this regard, E_0 is the only problematic value of E.

It turns out that the Marcus–Moore question in general has a negative answer. We proceed to explain in outline how counterexamples can be constructed.

It is convenient to rephrase the Marcus–Moore problem by introducing the Bebutov hull $(\Omega, \mathbb{R})$ of the function q. Let $Q : \Omega \to \mathbb{R}$ be a continuous function and let $\omega_0 \in \Omega$ such that $Q(\omega_0 \cdot t) = q(t)$. We introduce the family of equations

$$-\varphi'' + Q(\omega \cdot t)\varphi = E\varphi \tag{3.10_ω}$$

and the corresponding family of differential systems

$$x' = \begin{pmatrix} 0 & 1 \\ -E + Q(\omega \cdot t) & 0 \end{pmatrix} x, \quad x = \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} \in \mathbb{R}^2. \tag{3.11_ω}$$

For each $E \in \mathbb{R}$, Eqs. (3.11_ω) determine a projective flow $(\Sigma, \{\tau_t\})$.

We state without proof some general facts; see, e.g., [86] for more discussion. To begin, if $E < E_0$, then Eqs. (3.11_ω) admit an exponential dichotomy. For each $\omega \in \Omega$, let $l_\omega^+ \in \mathbb{P}$ resp. $l_\omega^- \in \mathbb{P}$ be the fiber of the stable resp. unstable bundle. Thus in the notation of Definitions 2.2., l_ω^+ is the image of the projection $P(\omega)$, while l_ω^- is its kernel. If $\Phi_\omega(t)$ is the fundamental matrix solution of Eq. (3.11_ω) and if $x_0 \in l_\omega^\pm$, then $\Phi_\omega(t)x_0 \to 0$ exponentially as $t \to \pm\infty$.

Let θ be a π-periodic angular coordinate on $\mathbb{P}$, so that $\theta = -\pi/2$ and $\theta = \pi/2$ parameterize the φ'-axis in the $\binom{\varphi}{\varphi'}$-space, and other elements $l \in \mathbb{P}$ have parameter values in $(-\pi/2, \pi/2)$. Using the particular way in which Eqs. (3.11_ω) depend on E, one can prove

that, if $\omega \in \Omega$ and E is increased from $-\infty$ to E_0 then $l_\omega^+ \equiv l_\omega^+(E)$ moves counterclockwise on $\mathbb{P}$ starting near $\theta = -\pi/2$, while l_ω^- moves clockwise on $\mathbb{P}$ starting from $\theta = \pi/2$. Letting E tend to E_0 from the left, we see that, at $E = E_0$, there are two invariant sections $s^\pm$ of the projective bundle Σ, defined by $s^\pm(\omega) = \lim_{E \to E_0^-} l_\omega^\pm(E)$. These sections are semicontinuous in ω, because they can be expressed as pointwise limits of monotone sequences of continuous functions. See again [86] for a discussion of these facts. Furthermore, it can be shown [92] that, for each $\omega \in \Omega$, the line $s^+(\omega) \in \mathbb{P}$ determines the principal solution of Eqs. (3.10_ω) in the sense that, if $0 \neq \bar{x} = \binom{\bar{\varphi}}{\bar{\varphi}'} \in s^+(\omega)$, then the solution $\varphi_0(t)$ of (3.10_ω) with $\varphi_0(0) = \bar{\varphi}$, $\varphi_0'(0) = \bar{\varphi}'$ is principal; the converse statement also holds.

Let $S_\pm = \mathrm{cls}\{(\omega, s^\pm(\omega)) \mid \omega \in \Omega\} \subset \Sigma$, and let $M_\pm$ be a minimal subset of $S_\pm$. Let us observe that, by the semicontinuity of the sections $s_\pm$, there is a residual subset $\Omega_1 \subset \Omega$ such that, if $\omega \in \Omega_1$, then the fiber $\pi^{-1}(\omega) \cap S_\pm$ contains exactly one point, and hence $\pi^{-1}(\omega) \cap M_\pm$ also contains exactly one point, for each choice of sign $\pm$.

We claim that $M_+ = M_-$. For suppose that this is not true. Then M_+ and M_- are disjoint. Let $f : \Sigma \to \mathbb{R}$ be the function of Eqs. (2.3_ω), Section 2. Then if $\omega \in \Omega$, if $x(t)$ is a solution of (3.11_ω) with $|x_0(0)| = 1$, and if $l \in \mathbb{P}$ contains $x(0)$, then

$$\ln\bigl|x(t)\bigr| = \int_0^t f\bigl(\hat{\tau}_s(\omega, l)\bigr)\,\mathrm{d}s.$$

Let $m_1 \in M_+$ and let $\pi(m_1) = \omega \in \Omega$. Suppose that $\int_0^t f(\hat{\tau}_s(m_1))\,\mathrm{d}s$ is bounded (above and below) for all $t \in \mathbb{R}$. Using a familiar argument involving a matrix solution of Eq. (3.11_ω) (see the proofs of Proposition 2.5 and Proposition 3.3), one shows that, if $m_2 \in M_-$ and $\pi(m_2) = \omega$, then $\int_0^t f(\hat{\tau}_s(m_2))\,\mathrm{d}s$ is also bounded. This implies that all solutions of (3.11_ω) are bounded. But it is then an exercise to show that Eq. (3.10_ω) cannot have a principal solution. We conclude that $\int_0^t f(\hat{\tau}_s(m_1))\,\mathrm{d}s$ is unbounded for some (hence all) $m_1 \in M_+$. The same unboundedness holds for all $m_2 \in M_-$. Now however, using the oscillation Lemma 3.1, we can determine a residual subset $\tilde{\Omega}$ of Ω such that, if $\omega \in \tilde{\Omega}$, then Eq. ($3.10_\omega$) has no principal solution. So we must conclude that $M_+ = M_-$.

Let us write M for the common value of M_+ and M_-. Some additional reasoning shows that: (i) $(M, \mathbb{R})$ is an almost automorphic extension of $(\Omega, \mathbb{R})$ with projection equal to $\pi : M \to \Omega$; (ii) the Marcus–Moore question has an affirmative answer if and only if the projection π defines a flow isomorphism of $(M, \mathbb{R})$ onto $(\Omega, \mathbb{R})$. Furthermore, (iii) if π is a flow isomorphism, then the Lyapunov exponent of the family (3.11_ω) is zero. See [86].

Now however, one can construct families (3.10_ω) for which the corresponding families (3.11_ω) are of M-V type. One such family was determined in [86], using a rotation method of the sort applied by Millionščikov. Later other examples were found, in which $(\Omega, \mathbb{R})$ is quasi-periodic with Diophantine frequency vector and $Q : \Omega \to \mathbb{R}$ is real analytic (e.g., [14]). For families (3.11_ω) of M-V type, the minimal set $M \subset \Sigma$ must be an almost automorphic extension of Ω which is not almost periodic. The Marcus–Moore question has a negative answer for each Eq. (3.10_ω) if the family (3.11_ω) is of M-V type; this follows from the positivity of the Lyapunov exponent β for families of M-V type.

If Eqs. (3.11_ω) are of M-V type, then the principal solutions of the corresponding Eqs. (3.10_ω) exhibit quite remarkable behavior. We justify this remark. First, a variant of Lemma 3.1 allows one to find a residual subset $M_0 \subset M$ such that, if $m_0 \in M_0$, then

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t f\big(\hat{\tau}_s(m_0)\big)\,\mathrm{d}s = \beta, \qquad \liminf_{t\to\infty} \frac{1}{t}\int_0^t f\big(\hat{\tau}_s(m_0)\big)\,\mathrm{d}s = -\beta.$$

Let $\Omega_0 = \pi(M_0)$; one checks that Ω_0 is residual in Ω. On the other hand, there is a residual subset $\Omega_1 \subset \Omega$ such that, if $\omega \in \Omega_1$, then $\pi^{-1}(\omega) \cap M$ contains exactly one point (ω, l). If $0 \neq \bar{x} \in l$ and $\bar{x} = \binom{\bar{\varphi}}{\bar{\varphi}'}$, then the solution $\varphi_0(t)$ of (3.10_ω) satisfying $\varphi_0(0) = \bar{\varphi}$ and $\varphi_0'(0) = \bar{\varphi}'$ is a principal solution. Now $\Omega_* = \Omega_0 \cap \Omega_1$ is a residual subset of Ω. If $\omega \in \Omega_*$, then the principal solution $\varphi_0(t)$ exhibits both exponential growth and decay as $t \to \infty$, in the sense that $\limsup_{t\to\infty} \frac{1}{t}\ln\varphi_0(t) = \beta$ and $\liminf_{t\to\infty} \frac{1}{t}\ln\varphi_0(t) = -\beta$. This is surely a curious phenomenon.

As another consequence of the M-V property, we see that the principal solution of (3.10_ω) is not a continuous function of ω. In fact, in a family (3.11_ω) of M-V type, the section $s_+(\cdot)$ cannot be continuous because if they were, the projection $\pi : (M, \mathbb{R}) \to (\Omega, \mathbb{R})$ would be a flow isomorphism. Thus the lines $l_+(\omega) \in \mathbb{P}$ defining the initial conditions of the principal solutions of Eqs. (3.10_ω) have points of discontinuity; in fact $\omega \mapsto l_+(\omega)$ is only semicontinuous.

A second problem which can be solved by constructing a differential system of M-V type is the following. Return to the family (3.11_ω). The Lyapunov exponent β of this family is a function of the parameter E. It is natural to ask if $\beta = \beta(E)$ is continuous. We are in effect asking if the Lyapunov exponent of the almost periodic Schrödinger equation (3.10_ω) is always continuous as a function of E.

It is easy to see that β is continuous for values of E for which Eqs. (3.11_ω) admit an exponential dichotomy. One can further show that β is continuous at a point $E_* \in \mathbb{R}$ if $\beta(E_*) = 0$. Thus, if β is discontinuous at a point E_*, then the family (3.11_ω) with $E = E_*$ must be of M-V type, i.e., weakly hyperbolic.

There are reasons to think that β might be continuous at all values of E. For, the rotation number α of the family (3.11_ω) does depend continuously on E; this follows from Proposition 2.7 or from a direct argument of Krylov–Bogoliubov type. Moreover, the complex combination $w(E) = -\beta(E) + \mathrm{i}\alpha(E)$ admits a holomorphic extension from the real E-axis to the upper half of the complex E-plane [89]. Thus β and α are conjugate harmonic functions of E for $\Im E > 0$. One might now hope to apply the arguments available to the effect that regularity properties of $\alpha(\cdot)$ imply regularity properties of $\beta(\cdot)$ for real values of the argument E. See, e.g., [62].

It turns out, however, that one can construct an almost periodic function q for which $\beta = \beta(E)$ is discontinuous at the right end-point E_0 of the disconjugacy set $\{E \in \mathbb{R} \mid$ Eq. (3.10_ω) is disconjugate$\}$. Apparently the first example of this type was given in [80]. The family (3.11_ω) corresponding to $E = E_0$ is of M-V type. By repeating the considerations concerning the stable and unstable bundles which were made in the discussion of the Marcus–Moore problem, we see that the unique minimal subflow $(M, \mathbb{R})$ of the projective flow $(\Sigma, \mathbb{R})$ determined by Eqs. (3.11_ω) is an almost automorphic extension of $(\Omega, \mathbb{R})$.

It seems that few examples are known of almost periodic Schrödinger equations for which the Lyapunov exponent $\beta = \beta(E)$ has discontinuity points. Indeed, there are results concerning the discrete quasi-periodic Schrödinger equation which state, roughly speaking, that if the frequency vector is Diophantine and if the potential is highly regular, then the Lyapunov exponent β depends continuously on the energy parameter E. See, e.g., [23]. Assuming that such results are also valid for the continuous Schrödinger operator (3.10_ω), one is led to formulate questions concerning "minimal conditions" on $(\Omega, \mathbb{R})$ and on Q which imply that β is continuous in E. So far these questions appear to be unanswered.

We close the discussion by noting that there are very general results concerning Lyapunov exponents and so-called dominated splittings which, in the present case, yield discontinuity points $Q_* \in C(\Omega)$ of β when β is thought as a function of $Q \in C(\Omega)$. See [19] for an excellent discussion of this topic. However, it is not clear from these results how one might pick out families of functions $Q = Q(E_1, \dots, E_r)$, depending on finitely many parameters, for which $\beta = \beta(E_1, \dots, E_r)$ admits points of discontinuity.

A third theme which can be elaborated upon in a fruitful way using families (3.1_ω) of M-V type is that of strange nonchaotic attractors or SNAs. In the following lines, we will summarize a part of an excellent article by Jorba, Núñez, Obaya, and Tatjer [99]; the reader is referred to that paper for more information. The main point to be made is that certain families (3.1_ω) with the M-V property give rise to projective flows which contain SNAs.

Following [99], we discuss the concept of SNA. We work for a moment in the category of *discrete* flows: we recall that a discrete flow on a metric space Ω is defined by iterating a fixed homeomorphism $\tau : \Omega \to \Omega$ and its inverse τ^{-1}. Let $\Omega = \mathbb{T}^d$, the d-torus with $d \geqslant 1$. Let $\psi_1, \dots, \psi_d$ be angular coordinates mod 2π on $\mathbb{T}^d$, and let $\gamma_1, \dots, \gamma_d$ be real numbers such that the $d+1$-tuple $1, \gamma_1, \dots, \gamma_d$ is linearly independent over the rational field $\mathbb{Q}$. Consider the homeomorphism $\tau : \Omega \to \Omega$ defined by

$$\tau(\omega) = (\psi_1 + \gamma_1, \dots, \psi_d + \gamma_d) \quad \text{if } \omega = (\psi_1, \dots, \psi_d) \in \Omega.$$

Of course the angular quantities are taken mod 2π. It is well-known that, for each $\omega \in \Omega$, the orbit $\{\tau^n(\omega) \mid -\infty < n < \infty\}$ is dense in Ω, so one can think of the discrete flow (Ω, τ) as a discrete version of a Kronecker winding on $\Omega = \mathbb{T}^d$.

Now we introduce a continuous map $\hat{\tau} : \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}$ of the following type. Let $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume that the partial derivative $\frac{\partial\varphi}{\partial x}(\omega, x)$ is well-defined and continuous on $\Omega \times \mathbb{R}$, and that

$$\frac{\partial\varphi}{\partial x}(\omega, x) > 0 \quad \text{for all } (\omega, x) \in \Omega \times \mathbb{R}.$$

Then of course one has the monotonicity condition $\varphi(\omega, x_1) < \varphi(\omega, x_2)$ whenever $\omega \in \Omega$ and $x_1 < x_2$. We set

$$\hat{\tau}(\omega, x) = \big(\tau(\omega), \varphi(\omega, x)\big).$$

Clearly $\hat{\tau}$ is of skew-product type in the sense that the first coordinate on the right-hand side depends on ω but not on x.

Say that a Lebesgue measurable map $c : \Omega \to \mathbb{R}$ is a τ-invariant curve if it is bounded and $\varphi(\omega, c(\omega)) = c(\tau(\omega))$ for all $\omega \in \Omega$. The corresponding $\hat{\tau}$-invariant graph is $C = \{(\omega, c(\omega)) \mid \omega \in \Omega\} \subset \Omega \times \mathbb{R}$. Let ν be the unique τ-invariant measure on Ω (which coincides with the normalized Lebesgue measure on $\Omega = \mathbb{T}^d$). We can define a $\hat{\tau}$-invariant measure μ_c on $\Omega \times \mathbb{R}$ by the formula

$$\int_{\Omega\times\mathbb{R}} h(\omega, x)\,\mathrm{d}\mu_c(\omega, x) = \int_\Omega h\bigl(\omega, c(\omega)\bigr)\,\mathrm{d}\nu(\omega)$$

for each continuous function $h : \Omega \times \mathbb{R} \to \mathbb{R}$.

We introduce the *Lyapunov exponent* β_s of c, given by

$$\beta_s(c) = \int_\Omega \ln \frac{\partial\varphi}{\partial x}\bigl(\omega, c(\omega)\bigr)\,\mathrm{d}\nu(\omega) = \int_{\Omega\times\mathbb{R}} \left(\ln \frac{\partial\varphi}{\partial x}\right) \mathrm{d}\mu_c.$$

If the Lyapunov exponent $\beta_s(c)$ is negative, it measures the "vertical" contraction along the fiber $\mathbb{R}$ towards $c(\omega)$, for ν-a.a. $\omega \in \Omega$.

Next we follow T. Jäger [70] and make the following

DEFINITION 3.10. A strange nonchaotic attractor or SNA (in $\Omega \times \mathbb{R}$) is the graph of a noncontinuous, τ-invariant curve c with a negative Lyapunov exponent.

As is pointed out in [99], it is of interest to consider subsets $M \subset \Omega \times \mathbb{R}$ which are nonempty, compact, $\hat{\tau}$-invariant in the sense that $\hat{\tau}(M) \subset M$, and minimal with respect to these properties. Such a set M is called minimal. It turns out that, when restricted to M, $\hat{\tau}$ is invertible (we did not assume that $\hat{\tau}$ is invertible on $\Omega \times \mathbb{R}$), so that $(M, \hat{\tau})$ is a discrete flow. One can show that each orbit $\{\hat{\tau}^n(m) \mid -\infty < n < \infty\}$ is dense in M, and one can further show that $(M, \hat{\tau})$ is an almost automorphic extension of (Ω, τ). That is, if $\pi : \Omega \times \mathbb{R} \to \Omega$ is the projection, then there is a residual subset $\Omega_* \subset \Omega$ such that, if $\omega_* \in \Omega_*$, then $\pi^{-1}(\omega_*) \cap M$ contains exactly one point. Such minimal sets are analyzed in [99] and we refer to that paper for more information.

Whenever there exists a τ-invariant section c as above, the boundedness of the graph C ensures the existence of a minimal set M. Such an M need not contain an SNA. However, it is explained in [99] how a quasi-periodic family of Schrödinger equations with the M-V property at $E = E_0$ determines a minimal set which contains an SNA. We now discuss this matter.

Let us return momentarily to the framework of continuous (real) flows. Let $\Omega_c = \mathbb{T}^{d+1}$ be the $(d+1)$-torus $(d \geqslant 1)$, and let $(\psi_0, \psi_1, \dots, \psi_d)$ be angular coordinates mod 2π on Ω_c. Let $\gamma_1, \dots, \gamma_d$ be real numbers such that $1, \gamma_1, \dots, \gamma_d$ are linearly independent over the rational field $\mathbb{Q}$. If $\omega_c = (\psi_0, \psi_1, \dots, \psi_d)$ mod 2π, set $\tau_t(\omega_c) = (\psi_0 + t, \psi_1 + \gamma_1 t, \dots, \psi_d + \gamma_d t)$ mod 2π. Let $Q : \Omega_c \to \mathbb{R}$ be a continuous function such that Eqs. (10_{ω_c}) have the M-V property at $E = E_0$. Let $(\Sigma, \{\hat{\tau}_t\})$ be the projective flow defined by Eqs. (10_{ω_c}).

Next let θ be the polar angle in $\mathbb{R}^2$, which we regard as a π-periodic angular coordinate on $\mathbb{P}$. It follows from the description given earlier of the projective flow that $(\Sigma, \mathbb{R})$

contains a unique minimal subflow $(M_c, \mathbb{R})$, which is an almost automorphic extension of $(\Omega_c, \mathbb{R})$. Moreover, if $(\omega, l) \in M_c$ and θ parameterizes l, then

$$-\pi/2 < \theta < \pi/2$$

for an appropriate branch of θ.

Next, consider the differential equation (4_{ω_c}) for θ of Section 2. We have

$$\theta' = g\big(\hat{\tau}_t(\omega_c, \theta)\big). \tag{3.12}$$

Let $(\omega_c, \bar{\theta}) \in \Omega_c \times \mathbb{R}$. Let $\theta(t) = \bar{\theta} + \int_0^t g(\hat{\tau}_s(\omega_c, \bar{\theta}))\,\mathrm{d}s$ be the solution of (3.12) satisfying $\theta(0) = \bar{\theta}$. We abuse notation and write $\hat{\tau}_t(\omega_c, \bar{\theta}) = (\tau_t(\omega_0), \theta(t))$. Then $\{\hat{\tau}_t\}$ defines a real flow on $\Omega \times \mathbb{R}$. If values of θ are identified mod π, then $\{\hat{\tau}_t\}$ becomes a flow on Σ defined by Eqs. (3.10_{ω_c}). We identify M_c in the natural way with a subset of $\Omega_c \times (-\pi/2, \pi/2)$; we call this subset M_c, as well.

Let us identify the d-torus $\Omega = \mathbb{T}^d$ with $\{\omega_c = (0, \omega) \mid \omega = (\psi_1, \ldots, \psi_d) \bmod 2\pi\} \subset \Omega_c$. Thus Ω is the section $\psi_0 = 0$ of Ω_c. We define $\tau : \Omega \to \Omega$: $\tau(\omega) = \tau_1(0, \omega)$, where τ_1 is the time-1 map of $\{\tau_t\}$. Then (Ω, τ) is a minimal discrete flow. We also define $\hat{\tau} : \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}$ by $\hat{\tau}(\omega, \bar{\theta}) = \hat{\tau}_1(\omega, \bar{\theta})$. Let

$$M = M_c \cap \big\{(0, \omega, \bar{\theta}) \mid \omega \in \Omega, -\pi/2 < \bar{\theta} < \pi/2\big\}.$$

Then M is invariant with respect to $\hat{\tau}$, and it can be checked that $(M, \hat{\tau})$ is a minimal almost automorphic extension of (Ω, τ).

Now we return to the measurable invariant sections $s^{\pm} = s^{\pm}(\omega_c)$ of Σ which we encountered in our discussion of the Marcus–Moore problem. Let $\theta^{\pm}(\omega_c) \in (-\pi/2, \pi/2)$ be the corresponding values of the angular variable θ. Let $\beta > 0$ be the Lyapunov exponent of Eqs. (3.10_{ω_c}). Let us write $c^{\pm}(\omega) = \theta^{\pm}((0, \omega))$ for $\omega \in \Omega$. It can be shown that $\beta_s(c^{\pm}) = \mp\beta$. It follows that $C^+ = \{(\omega, c^+(\omega)) \mid \omega \in \Omega\}$ satisfies the condition of Definition 3.10, so C^+ is an SNA contained in M. See [99] for a detailed discussion and for further examples of SNAs obtained from quasiperiodic families (3.10_{ω}), and from quasiperiodic families of the more general form (3.1_{ω}) which have the M-V property.

We turn to a fourth issue which can be resolved by the construction of an almost periodic family (3.1_{ω}) which has the M-V property, namely a certain question concerning the behavior of the projective flow when Eqs. (3.1_{ω}) have unbounded mean motion. As usual, let $(\Omega, \mathbb{R})$ be an almost periodic minimal flow with unique invariant measure ν, let $A : \Omega \to \mathrm{sl}(2, \mathbb{R})$ be a continuous function, and let $(\Sigma, \{\hat{\tau}_t\})$ be the projective flow of the corresponding family (3.1_{ω}).

In Section 2 we defined the rotation number α of the family (3.1_{ω}) with respect to the measure ν. We recall the construction. If $\omega \in \Omega$ and $\bar{l} \in \mathbb{P}$, let $\hat{\tau}_t(\omega, \bar{l}) = (\tau_t(\omega), l(t))$. Introduce the polar angle θ in $\mathbb{R}^2$, and view θ as an angular coordinate mod π on $\mathbb{P}$. Letting $\theta(t)$ be a continuous parametrization of $l(t)$, we have Eq. (2.4_{ω}) of Section 2:

$$\theta' = g\big(\hat{\tau}_t(\omega, \bar{\theta})\big).$$

The rotation number α is by definition $\alpha = \lim_{t\to\infty} \frac{\theta(t)}{t}$. It turns out that the limit determining α is well-defined in the following sense: if ν is an ergodic measure on Ω, then there is a set $\Omega_1 \subset \Omega$ with $\nu(\Omega_1) = 1$ such that, if $(\omega, \bar{\theta}) \in \Omega_1 \times \mathbb{P}$, then the limit exists and does not depend on $(\omega, \bar{\theta})$. See [79]. In the present case, the flow $(\Omega, \mathbb{R})$ supports just one ergodic measure ν. Then, as noted in Proposition 2.7, the following stronger statement holds:

$$\alpha = \lim_{|t|\to\infty} \frac{1}{t} \int_0^t g\bigl(\hat{\tau}_s(\sigma)\bigr)\,\mathrm{d}s$$

where the limit is uniform in $\sigma \in \Sigma$.

It is natural to compare αt and $\int_0^t g(\hat{\tau}_s(\sigma))\,\mathrm{d}s$.

DEFINITION 3.11. Say that Eqs. (3.1$_\omega$) have *bounded mean motion* if, for some $\sigma \in \Sigma$, there holds

$$\sup_{t\in\mathbb{R}} \left| \int_0^t g\bigl(\hat{\tau}_s(\sigma)\bigr)\,\mathrm{d}s - \alpha t \right| < \infty. \tag{$*$}$$

We now follow the discussion of Yi ([154]; also [16,71]). Yi works with general nonlinear skew-product flows on a circle bundle over Ω, so our case is special from his point of view. First of all, if Eqs. (3.1$_\omega$) have bounded mean motion in the sense of Definition 3.11, then it turns out that condition $(*)$ holds for all $\sigma \in \Sigma$. See ([154], Proposition 3.1).

Second, if Eqs. (3.1$_\omega$) have bounded mean motion, then one can draw deep conclusions about the structure of a minimal subflow $(M, \mathbb{R})$ of the projective flow $(\Sigma, \mathbb{R})$. We refer to [154] for precise statements and proofs, and only summarize the situation here. Assume for simplicity that $(\Omega, \mathbb{R})$ is quasiperiodic. Let $\mathcal{M}_\Omega$ be the frequency module of $(\Omega, \mathbb{R})$; we can write $\mathcal{M}_\Omega = \{n_1\gamma_1 + \cdots + n_d\gamma_d \mid n_1, \ldots, n_d \in \mathbb{Z}\}$ where $\gamma_1, \ldots, \gamma_d \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$. Let $\mathcal{M}_1 = \{n_0\alpha + n_1\gamma_1 + \cdots + n_d\gamma_d \mid n_0, n_1, \ldots, n_d \in \mathbb{Z}\}$; then $\mathcal{M}_1$ is the frequency module of a quasiperiodic flow $(\Omega_1, \mathbb{R})$. The relation between $(\Omega_1, \mathbb{R})$ and $(\Omega, \mathbb{R})$ can be described in terms of the existence and nature of integral relations between $\alpha, \gamma_1, \ldots, \gamma_d$ [154].

It turns out that $(M, \mathbb{R})$ is an almost periodic minimal flow $(\Omega_*, \mathbb{R})$ which is "intermediate" between $(\Omega_1, \mathbb{R})$ and $(\Omega, \mathbb{R})$. If $n_0\alpha \notin \mathcal{M}_\Omega$ for all integers n_0, then $(\Omega_*, \mathbb{R}) = (\Omega_1, \mathbb{R})$. One thus has a general description of the minimal subsets of Σ when bounded mean motion is present. Yi's theory can be extended to the case of a general almost periodic minimal flow $(\Omega, \mathbb{R})$. One needs to define $\mathcal{M}_*$ to be the smallest subgroup of $\mathbb{R}$ which contains α and $\mathcal{M}_\Omega$.

The case when Eqs. (3.1$_\omega$) have unbounded mean motion is less well-understood. Suppose for example that there does not exist an integer $r \neq 0$ such that $r\alpha \in \mathcal{M}_\Omega$. We say that α is irrationally related to $\mathcal{M}_\Omega$. Suppose in addition that Eqs. (3.1$_\omega$) have unbounded mean motion. It has been asked in the literature if these conditions imply that $(\Sigma, \mathbb{R})$ is minimal.

The point we wish to make is that the answer to this question is no. In fact, as Yi notes, an example of [75] has the following properties: (i) there is no integer r such that $r\alpha \in \mathcal{M}_\Omega$;

(ii) Eqs. (3.1_ω) have the M-V property; (iii) the unique minimal set $M \subset \Sigma$ is an almost automorphic extension of Ω. In particular, $M \neq \Sigma$.

So far, we have seen that almost periodic families (3.1_ω) with the M-V property can be constructed which illustrate Marcus–Moore disconjugacy, a discontinuous Lyapunov exponent, and strange nonchaotic attractors. We have also seen that they may give rise to interesting minimal subsets $M \subset \Sigma$ when the bounded mean motion property is absent. All the minimal sets $M \subset \Sigma$ which have arisen have been almost automorphic extensions of Ω.

We note in passing that an almost automorphic minimal subset M of the projective flow $(\Sigma, \mathbb{R})$ determined by an almost periodic flow $(\Omega, \mathbb{R})$ may have interesting topological properties. For instance, the original Vinograd example [152] gives rise to an almost automorphic minimal set $M \subset \Sigma$ which is connected, locally connected at some points, but not locally connected at all points [84]. See [128] for other examples exhibiting this behavior which are invariant sets for 2-dimensional (nonlinear) differential systems. As another instance, we note that the minimal set M mentioned above in connection with mean motion property has "more" Čech cohomology than Ω in a precise sense, despite the fact that M is an almost automorphic extension of Ω. See [17].

We consider a final issue which can be resolved via the construction of an appropriate almost periodic family (3.1_ω) with the M-V property. The issue is that of the existence of minimal flows which have the property of Li–Yorke chaos, but which do not exhibit such traditional indicators of complicated orbital behavior as weak mixing or positive topological entropy. We refer to [18] for a general discussion of flows which do/do not exhibit Li–Yorke chaos. The concept itself derives from a classical paper by Li and Yorke [103].

Let us begin our discussion with the

DEFINITION 3.12. Let Ω be a metric space with metric d and let $(\Omega, \mathbb{R})$ be a flow. A pair $\{\omega_1, \omega_2\} \subset \Omega$ is a *Li–Yorke pair* if $\sup_{t>0}\{d(\omega_1 \cdot t, \omega_2 \cdot t)\} > 0$ and $\inf_{t>0}\{d(\omega_1 \cdot t, \omega_2 \cdot t)\} = 0$. A set $S \subset \Omega$ is called *scrambled* if each pair $\{\omega_1, \omega_2\} \subset S$ is a Li–Yorke pair. The flow $(\Omega, \mathbb{R})$ is *chaotic* in the *sense of Li and Yorke* if Ω contains an uncountable scrambled set.

Let us now focus on the problem of determining almost periodic families (3.1_ω) for which the projective flow $(\Sigma, \mathbb{R})$ admits a minimal subflow $(M, \mathbb{R})$ which exhibits Li–Yorke chaos. Let us note that such a minimal set cannot be weakly mixing. This is because a weakly mixing flow $(X, \mathbb{R})$ has by definition the property that there is a pair $(x_1, x_2) \in X \times X$ with $x_1 \neq x_2$ such that the set $\{(x_1 \cdot t, x_2 \cdot t) \mid t \in \mathbb{R}\}$ is dense in $X \times X$. If a pair $(m_1, m_2) \in M \times M$ had this property, then the pair $(\omega_1, \omega_2) \in \Omega \times \Omega$ of images $\omega_i = \pi(m_i)$ $(i = 1, 2)$ would have the property that $\{(\omega_1 \cdot t, \omega_2 \cdot t) \mid t \in \mathbb{R}\}$ is dense in $\Omega \times \Omega$. However, no such pair exists.

We now follow the remarkable thesis of Bjerklöv ([13]; see also [14]). Let Ω be the 2-torus $\mathbb{T}^2$ with angular coordinates (ψ_1, ψ_2) mod 1 Let γ be an irrational number, and set $\tau_t(\psi_1, \psi_2) = (\psi_1 + t, \psi_2 + t)$ mod 1. Let $Q : \Omega \to \mathbb{R}$ be a C^3 function which has a unique minimum on $\Omega = \mathbb{T}^2$. Suppose further that Q is quadratic in a neighborhood of the point ω_0 where it assumes its minimum value. Thus, assuming for convenience

that Q attains its minimum value at $(\psi_1, \psi_2) = (0, 0)$, it is required that $Q(\psi_1, \psi_2) = B(\psi_1, \psi_2) + \mathrm{O}((\psi_1^2 + \psi_2^2)^{3/2})$ where B is a positive definite bilinear form.

Let us consider the following family of linear differential systems of Schrödinger type:

$$x' = \begin{pmatrix} 0 & 1 \\ -E + \lambda Q\big(\tau_t(\omega)\big) & 0 \end{pmatrix} x. \tag{3.14_ω}$$

Here λ is a real parameter. Among other results, Bjerklöv shows that it is possible to find a (large) λ to which there correspond a (Diophantine) frequency γ and a (large) number E such that the corresponding family (3.14_ω) is of M-V type, and determines a projective flow $(\Sigma, \mathbb{R})$ which is minimal. Such a minimal projective flow differs essentially those of Glasner–Weiss type which we encountered in the weakly elliptic case, because (3.14_ω) has a positive Lyapunov exponent and hence $(\Sigma, \mathbb{R})$ supports exactly two ergodic measures.

Now, using an old triangularization technique (see [83], Section 5), one can show that there is a residual subset $\Omega_* \subset \Omega$ with the following property: if $\omega \in \Omega_*$, then $\pi^{-1}(\omega)$ is a scrambled subset of Σ. In fact, if $l_1, l_2 \in \pi^{-1}(\omega)$, let $\Theta(l_1, l_2)$ be the counterclockwise angle from l_1 to l_2. Let $\Phi_\omega(t)$ be the fundamental matrix solution of (3.14_ω), and set $l_i(t) = \Phi_\omega(t) l_i$ $(i = 1, 2)$. Then one can show that

$$\inf_{t \in \mathbb{R}} \Theta\big(l_1(t), l_2(t)\big) = 0 \quad \text{and} \quad \sup_{t \in \mathbb{R}} \Theta\big(l_1(t), l_2(t)\big) = \pi.$$

Thus these examples have the Li–Yorke property. They have zero topological entropy because of a general result of Bowen [22]. See [17] for details. We note that it turns out that the Glasner–Weiss examples also have the Li–Yorke property; see [17].

We close the section by posing a question concerning the projective flow determined by almost periodic families (3.1_ω) of linear differential systems. Namely, can one determine a family (3.1_ω) for which $(\Sigma, \mathbb{R})$ admits a minimal subset M with the property that, for a residual set of $\omega \in \Omega$, the fiber $\pi^{-1}(\omega) \cap M$ is perfect and nowhere dense in $\mathbb{P}$? That is, does there exist an almost periodic, $\mathrm{SL}(2, \mathbb{R})$-valued "Denjoy cocycle"? According to a private communication of T. Jäger, this cannot happen when Ω is a 2-torus. Can other phase spaces Ω be excluded as well?

4. Algebro-geometric Sturm–Liouville coefficients

This section is devoted to the discussion of an inverse spectral problem for the classical Sturm–Liouville operator. The present discussion has interest for various reasons, which we want briefly to explain. First, we exploit methods from the theory of nonautonomous differential systems in dimension 2, by using techniques from ergodic theory, dynamical systems and other fields as well. It is our opinion that such an approach to the study of the inverse spectral properties of the Sturm–Liouville operator allows us to retrieve more information than is present in other sources. Second, it is well known that certain Sturm–Liouville operators have the property that their isospectral classes are preserved by the solutions of a corresponding nonlinear evolution equation, and indeed by the solution of a whole commuting hierarchy of such equations. A prototypical and extensively studied

example of this phenomenon is the following. Let $u_t(x) = u(t,x)$ be the solution of the Korteweg–de Vries (K-dV) equation

$$\frac{\partial u}{\partial t} = 3u\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial^3 u}{\partial x^3},$$

with appropriate initial condition $u_0(x) = u(0,x)$; then the spectrum $\Sigma(L_t)$ of the Schrödinger operator

$$L_t = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u_t(x)$$

acting in $L^2(\mathbb{R})$ does not depend on t [56]. This fact permits one to explicitly solve the K-dV equation for an ample class of initial data u_0, as a vast literature testifies. The K-dV equation is one of an infinite family of commuting nonlinear evolution equations whose solutions preserve isospectral classes of the Schrödinger operator.

Another example is furnished by the Camassa–Holm equation

$$4\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} - 2u\frac{\partial^3 u}{\partial x^3} - 4\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + 24u\frac{\partial u}{\partial x} = 0.$$

In this case, the corresponding operator is of Sturm–Liouville type with a nonconstant density function. Actually more than one operator can be used. Following [11], we choose $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1$; introducing a positive density function y, the relevant spectral problem is

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \varphi = \lambda y(x)\varphi. \tag{4.1}$$

It turns out that, if $u_t(x) = u(t,x)$ is an appropriate solution of the Camassa–Holm equation, then there is a t-dependent density $y_t = y_t(x)$ such that the spectrum of (4.1) does not depend on t.

We mention that there are also nonlinear evolution equations which are related to isospectral classes of ordinary differential operators which are not of Sturm–Liouville type. As an example, we mention the (nonfocusing) nonlinear Schrödinger (NLS) equation:

$$-\mathrm{i}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 2|u|^2 u,$$

together with the AKNS operator

$$L = J^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}x} - \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\right], \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

acting on $L^2(\mathbb{R}, \mathbb{C}^2)$. Here, a and b are real-valued functions of x. For appropriate solutions $u_t(x) = u(t, x)$ of the NLS equation, there is a transformation $u_t \mapsto (a_t, b_t)$ such that the spectrum of

$$L_t = J^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}x} - \left(\begin{pmatrix} a_t & b_t \\ b_t & -a_t \end{pmatrix}\right)\right]$$

does not depend on t. See [1] and much additional literature for more information.

Roughly speaking, there are two classical methods which are used to study the isospectral classes of an operator of Sturm–Liouville type: the first is that of the “inverse scattering method” (see, for instance [11,132]), the second is that in which methods of the theory of algebraic curves are used [35,118,34,96,95,155].

These facts provide a wonderful demonstration of the interplay between the theory of nonautonomous differential systems, algebraic curves and partial differential equations.

In the last two decades, many papers concerning the inverse spectral theory of the Schrödinger operator have been published (see, among others, [11,35,118,132]). We take inspiration from these works, and extend some of those well-known results to a general one dimensional Sturm–Liouville operator L, namely

$$L(\varphi) = (p\varphi')' - q\varphi = -\lambda y\varphi, \quad ' = \frac{\mathrm{d}}{\mathrm{d}t} \tag{4.2}$$

acting on $L^2(\mathbb{R})$.

Here is a more detailed formulation of the problem we study. Equation (4.2) can be written in system form as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 0 & \frac{1}{p(t)} \\ q(t) - \lambda y(t) & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{4.3}$$

Here p, q, y are uniformly continuous functions, p, y are strictly positive, and $p \in C^1(\mathbb{R})$ has bounded uniformly continuous derivative. This system has, in the nonautonomous case, a maximal Lyapunov exponent $\beta = \beta(\lambda)$. We assume that:

(H1) The spectrum $\Sigma(L)$ of (4.2) is a finite union of intervals: $\Sigma(L) = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \cdots \cup [\lambda_{2g}, \infty)$, where $-\infty < \lambda_0 < \lambda_1 < \cdots < \lambda_{2g}$;

(H2) The Lyapunov exponent β vanishes on $\Sigma(L)$: $\beta(\lambda) = 0$ for all $\lambda \in \Sigma(L)$.

As we will see, these mild-looking assumptions are sufficient to permit the reconstruction of p, q, and y in terms of algebro-geometric data. In particular, we will see that $\sqrt{py}$ can be described as the restriction of a meromorphic function defined on a generalized Jacobian of the Riemann surface $\mathcal{R}$ of the algebraic relation

$$w^2 = -\prod_{i=0}^{2g}(\lambda - \lambda_i)$$

to a t-motion we will describe in detail and which is general nonlinear (unless $y/p = const$). Some words concerning the assumptions on the spectrum (H1) and (H2): in gen-

eral, the spectrum $\Sigma(L)$ of a Sturm–Liouville operator L can have a very variegated structure. It can contain isolated points, bands, Cantor parts, or even more complicated sets. One can say that, since L (at least when $p(t)$ is of class $C^1(\mathbb{R})$) defines a self-adjoint operator on the weighted space $L^2(\mathbb{R}, \sqrt{y})$, the spectrum $\Sigma(L)$ is contained in $\mathbb{R}$, is bounded below by a number λ_0 and the set $\mathbb{R} \setminus \Sigma(L)$ is at most a countable union of disjoint open intervals $(\lambda_{2i-1}, \lambda_{2i})$ $(i \in \mathbb{N})$: $\mathbb{R} \setminus \Sigma(L) = (-\infty, \lambda_0) \bigcup_{i=1}^{\infty} (\lambda_{2i-1}, \lambda_{2i})$. However, it is fairly clear that it can be very difficult to study the properties of the spectrum of a Sturm–Liouville operator, as well as to formulate a consistent setting for an inverse problem to be solved. For this reason it is necessary to give assumptions on the functions p, q, y in order to have spectral structures which are "not too wild": those assumptions are equivalent to requiring that the Green function $G(t, s, \lambda)$ associated to Eq. (4.2) behaves "well" at the endpoints of the spectral gaps. This happens, for instance, in the case when we consider potentials which are called "reflectionless", i.e. potentials for which the nontangential limit $\lim_{\varepsilon \to 0^+} \Re G(t, t, \lambda + \mathrm{i}\varepsilon) = 0$ for all $t \in \mathbb{R}$ and for Lebesgue a.a. $\lambda \in \Sigma(L)$. This class of potentials contains periodic potentials [106], ergodic potentials [34], algebro-geometric potentials [35,96,95], Sato–Segal–Wilson potentials [141,85], Eliasson potentials [38] and others as well: of course the potentials we consider here are reflectionless. It is possible to extend the results we will obtain under the Hypotheses (H1) and (H2) to some of the more general classes of potentials listed above, by using certain limit procedures (see [97]): in fact, all such potentials are reflectionless and have a spectrum $\Sigma(L)$ which is a countably infinite union of closed intervals such that the length of the spectral gaps satisfy certain specific assumptions. It is not yet known if all the potentials obtained by the limit procedure in [97] are almost periodic (see [34] for a detailed discussion in the case of the Schrödinger operator).

We follow the approach taken in [31] and [57], where analogous inverse problems were posed and solved for the AKNS operator and for the Verblunsky operator arising in the theory of orthogonal polynomials on the unit circle [148].

The key element we use to study our inverse spectral problem is a "dynamical" definition of the Weyl m-functions $m_{\pm}(\lambda)$. An ample discussion concerning the Weyl m-functions is present in the following. We will sketch the main results of the Gilbert–Person theory for the Schrödinger operator, which can be easily adapted to our more general Sturm–Liouville operator.

We now give a brief summary of the content of this section. It is divided into four subsections.

In Section 4.1, we review some basic facts concerning the Sturm–Liouville operator. Although these facts are very well known, we will systematically use them, hence it could be helpful for the reader to have a quick guide in order to retrieve the necessary information.

We begin Section 4.2 by describing the ergodic-dynamical setting. We define a family $\mathcal{F}$ of Sturm–Liouville systems (4.3), where the functions p, q, y vary in the hull of a given triple (p_0, q_0, y_0) of continuous functions which satisfies certain assumptions. We will apply the concept of exponential dichotomy, and define the Weyl m-functions in a dynamical way.

In Section 4.3 we consider the projective flow in the Sturm–Liouville setting. In particular, we let Ω be a compact metric space, and $\{\tau_t\}$ be a flow on Ω: the projective flow is built in analogy to that of the preceding sections, as $(\Sigma = \Omega \times \mathbb{P}(\mathbb{C}), \{\hat{\tau}_t\})$. Then we

introduce the Floquet exponent, the rotation number, the Lyapunov exponent and we extend Kotani's seminal work on the Schrödinger operator [101]. We use systematically the projective flow. We will draw many important conclusions about the behavior of the Weyl m-functions when λ crosses the real line. Moreover we prove that the spectra of the family of Sturm–Liouville equations $\mathcal{F}$ are invariant, i.e. that for a.e. equation in $\mathcal{F}$, the spectrum equals a fixed closed set with no isolated eigenvalues and coincides with the non-dichotomy set of the family $\mathcal{F}$.

In Section 4.4 we pose and solve the inverse problem. In particular, first we find relations occurring between the potentials p, q, y in terms of the finite poles of a meromorphic function which is directly linked to the Weyl m-functions, then we characterize all the ergodic potentials p, q, y satisfying (H1) and (H2).

4.1. *Basic results on the Sturm–Liouville operator*

This part mainly refers to Atkinson [10], Coddington and Levinson [32] and Weinberger [153]. Consider the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p(t)\frac{\mathrm{d}f}{\mathrm{d}t}\right) - q(t)f = -\lambda y(t) f(t), \quad t \in [0, b], \tag{4.4}$$

where p, q and y are real valued functions such that $p(t)$ is positive and of class C^1, $y(t)$ is positive and continuous, and $q(t)$ is continuous, together with the boundary conditions

(4.4a) $f(0)\cos\alpha + p(0)f'(0)\sin\alpha = 0$,

(4.4b) $f(b)\cos\beta + p(b)f'(b)\sin\beta = 0$,

where $\alpha, \beta \in [0, \pi)$.

DEFINITION 4.1. The values of $\lambda \in \mathbb{C}$ such that Eq. (4.4) together with the boundary conditions (4.4a) and (4.4b) has a nontrivial solution are the *eigenvalues*. The corresponding solutions are called *eigenfunctions*.

PROPOSITION 4.2. *Eigenfunctions corresponding to different eigenvalues are orthogonal with respect the weight function y.*

All the eigenvalues are real and positive. The set of eigenvalues is at most countable, and has a unique cluster point at ∞.

Equation (4.4) can be written in the following matrix form

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1/p \\ q - \lambda y & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{4.5}$$

which is equivalent to the system

$$\begin{cases} pu_1' = u_2, \\ u_2' = (q - \lambda y)u_1. \end{cases} \tag{4.6}$$

Let $u(t,\lambda)=\binom{u_1(t,\lambda)}{u_2(t,\lambda)}$ be the solution of (4.5) satisfying some fixed initial condition. As $|\lambda|\to\infty$, both u_1 and u_2 behave as follows:

$$u_1(t,\lambda), u_2(t,\lambda)=\mathcal{O}\left\{\mathrm{e}^{c\sqrt{|\lambda|}}\right\}, \tag{4.7}$$

where c is a constant.

From Eq. (4.7) we obtain:

(4.7a) the maps $\lambda\mapsto u_{1,2}(t,\lambda)$ are entire and of order at most $1/2$ for every fixed $t\in(0,b)$;

(4.7b) $u_{1,2}(t,\lambda)$ are entire functions if $t=b$, hence the map $\lambda\mapsto u_1(b,\lambda)\cos\beta+u_2(b,\lambda)\sin\beta$ is entire; this implies that this function can have only countably many zeros, and of course these zeroes are the eigenvalues;

(4.7c) $\sum_n\lambda_n^{-1/2-\varepsilon}<\infty$ for each $\varepsilon>0$.

For results concerning the exponential growth of the zeros of an entire function, i.e. to prove (4.7c), see, among others, [149, Section 8.22]. In addition the eigenvalues can cluster only at ∞.

Let us now consider Eq. (4.4) together with the boundary conditions

$$f(0)=f(b)=0. \tag{4.8}$$

Let λ_1 be the smallest eigenvalue of Eq. (4.4) with boundary conditions (4.8), and let χ_1 be the corresponding eigenfunction. Then

$$0\neq\lambda_1=\frac{\int_0^b(p(\chi_1')^2+q\chi_1^2)\,\mathrm{d}t}{\int_0^b y\chi_1^2\,\mathrm{d}t}.$$

It is easy to prove that χ_1 does not vanish in $(0,b)$. Since the eigenfunctions are orthogonal, all the other eigenfunctions *must* change sign in $(0,b)$: even more, if $\lambda<\rho$ are different eigenvalues, χ and ψ are the corresponding eigenfunctions and t_0,t_1 are two consecutive zeroes of χ, then there exists $\bar t\in(t_0,t_1)$ such that $\psi(\bar t)=0$. This implies that

PROPOSITION 4.3. *The kth eigenfunction χ_k of Eq.* (4.4) *together with the boundary conditions* (4.8) *has exactly $k-1$ zeroes in* $(0,b)$.

Starting from Eq. (4.4) (which is considered in the interval $[0,b]$, $b\in\mathbb{R}$), we now retrieve information concerning the solutions of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p(t)\frac{\mathrm{d}f}{\mathrm{d}t}\right)-q(t)f=-\lambda y(t)f(t),\quad t\in[0,\infty). \tag{4.9}$$

To this end we introduce the Weyl m-functions for the Sturm–Liouville operator.

Let φ_α and ψ_α two linearly independent solutions of (4.4) satisfying the following initial conditions at 0

(4.9a) $\varphi_\alpha(0,\lambda)=\sin\alpha$, $\psi_\alpha(0,\lambda)=\cos\alpha$

(4.9b) $p(0)\varphi_\alpha'(0,\lambda)=-\cos\alpha$, $p(0)\psi_\alpha'(0,\lambda)=\sin\alpha$,

where $0 \leqslant \alpha < \pi$. Note that φ satisfies the condition (4.4a). Any other solution χ of Eq. (4.4) (except for ψ) has the form

$$\chi(t) = \varphi_\alpha(t) + m(\alpha, \lambda)\psi_\alpha(t),$$

where $m(\alpha, \lambda) \in \mathbb{C}$. We omit for the moment the α-dependence of the functions involved. If we require that χ satisfies the boundary condition (4.4b), namely

$$\chi(b)\cos\beta + p(b)\chi'(b)\sin\beta = 0,$$

then m must satisfy

$$m = -\frac{\varphi(b, \lambda)\cot\beta + p(b)\varphi'(b, \lambda)}{\psi(b, \lambda)\cot\beta + p(b)\psi'(b, \lambda)}.$$

Clearly, now we have $m = m(\alpha, \beta, b, \lambda) \in \mathbb{C}$.

From the above observations, all the functions $\varphi, \varphi', \psi, \psi'$ are entire functions in λ, hence m is meromorphic in the variable λ. Let us set $z = \cot\beta$; then for fixed (λ, b) we obtain

$$m = -\frac{Az + B}{Cz + D} \tag{4.10}$$

where z varies on the real line, and A, B, C, D are determined by the expression above. The image of the real line is a circle $C_b(\lambda)$ in the m-plane. Thus χ satisfies the boundary conditions in b if and only if $m \in C_b(\lambda)$.

If we use the notation

$$[f, g](t) = p(t)\big(f(t)\bar{g}(t)' - f(t)'\bar{g}(t)\big), \qquad f, g \in C^1(0, b)$$

than the equation of the circle $C_b(\lambda)$ is

$$[\chi, \chi](b) = 0, \tag{4.11}$$

that is

$$C_b(\lambda) = \big\{m \in \mathbb{C} \mid [\chi, \chi](b) = 0\big\}.$$

The center c and the radius r of $C_b(\lambda)$ are

$$c = -\frac{[\varphi, \psi](b)}{[\psi, \psi](b)} \tag{4.12}$$

$$r = \frac{1}{|[\psi, \psi](b)|}, \tag{4.13}$$

The interior of the circle $C_b(\lambda)$ is given by the relation

$$\frac{[\chi,\chi](b)}{[\psi,\psi](b)} < 0.$$

Moreover

$$[\psi,\psi](b) = 2\mathrm{i}\Im\lambda \int_0^b y|\psi|^2\,\mathrm{d}t,$$

and

$$[\chi,\chi](b) - [\chi,\chi](0) = 2\mathrm{i}\Im\lambda \int_0^b y|\chi|^2\,\mathrm{d}t.$$

Since $[\chi,\chi](0) = -2\mathrm{i}\Im m$, the interior of the circle is given by

$$\int_0^b y|\chi|^2\,\mathrm{d}t < \frac{\Im m}{\Im\lambda} \quad (\Im\lambda \neq 0).$$

Points m are on the circle if and only if

$$\int_0^b y|\chi|^2\,\mathrm{d}t = \frac{\Im m}{\Im\lambda} \quad (\Im\lambda \neq 0). \tag{4.14}$$

The radius r is given, if $\Im\lambda \neq 0$, by

$$\frac{1}{r} = 2|\Im\lambda| \int_0^b y|\psi|^2\,\mathrm{d}t.$$

Let $0 < a < b < \infty$. If m is inside or on the circle $C_b(\lambda)$, we have

$$\int_0^a y|\chi|^2\,\mathrm{d}t < \int_0^b y|\chi|^2\,\mathrm{d}t \leqslant \frac{\Im m}{\Im\lambda},$$

so m is inside or on the circle $C_a(\lambda)$ as well. This means that if $a < b$, then $C_a(\lambda)$ contains $C_b(\lambda)$. For fixed λ with $\Im\lambda \neq 0$, the map $b \mapsto r(b) = (2|\Im\lambda| \int_0^b y|\psi|^2\,\mathrm{d}t)^{-1}$ is monotone decreasing, and hence the limit $\lim_{b\to+\infty} r(b) := r_+$ exists. We have two possibilities: either $r_+ = 0$ or $r_+ > 0$. In the first case the circles $C_b(\lambda)$ tend to a point $m_+(\lambda)$, in the second case the $C_b(\lambda)$ tend to a circle $C_+(\lambda)$. In the first case, since $r(b) \to 0$, then $\psi(t) \notin L^2([0,\infty), y)$, hence there exists only one solution (up to a constant multiple) which belongs to $L^2([0,\infty), y)$, and this solution is surely χ. In the other case, we can easily see that $\psi \in L^2([0,\infty), y)$. If $\hat{m}$ is any point of $C_+(\lambda)$ then $\hat{m}$ is inside every $C_b(\lambda)$, for all $b > 0$, so

$$\int_0^b y|\chi|^2\,\mathrm{d}t \leqslant \frac{\Im \hat{m}}{\Im \lambda},$$

and if $b \to \infty$, this proves that every solution $\chi = \varphi + \hat{m}\psi$ is in $L^2([0,\infty), y)$.

REMARK 4.4. The discussion we made above shows that solutions of (4.9) which are in $L^2([0,\infty), y)$ always exist: in the limit circle case, every solution which satisfies the boundary conditions mentioned above is square integrable, while in the limit point case there is (up to a constant multiple) only one solution which is square integrable, if $\Im\lambda \neq 0$, and this solution is $\chi(t)$.

We recall

PROPOSITION 4.5. *Let p, q, y be real valued bounded continuous functions such that $p(t)$ is positive and of class C^1, and $y(t)$ is positive. Then Eq.* (4.4) *together with the boundary conditions* (4.9a) *and* (4.9b) *is in the limit point case at ∞.*

As a consequence of Proposition 4.5, for every fixed λ with $\Im\lambda \neq 0$ there exists only one square integrable solution of (4.9) (up to a constant multiple) which is in $L^2([0,\infty), y)$. It turns out that the limit $\lim_{b\to\infty} C_b(\lambda)$ does not depend on the particular choice of β in the boundary conditions. In particular, if we put $\beta = 0$ we obtain

$$m_+(\alpha,\lambda) = -\lim_{b\to\infty} \frac{\varphi_\alpha(b,\lambda)}{\psi_\alpha(b,\lambda)}.$$

REMARK 4.6. From now on, unless explicitly stated otherwise, we only consider $\alpha = \frac{\pi}{2}$ and set $m_+(\lambda) = m_+(\pi/2,\lambda)$, to obtain the particularly important relation

$$m_+(\lambda) = \frac{p(0)\chi'(0)}{\chi(0)}, \tag{4.15}$$

where $\chi(t)$ is the only solution of (4.9) which is in $L^2([0,\infty), y)$.

One can retrieve $m_+(\alpha,\lambda)$ from $m_+(\lambda)$ and vice-versa, by using the following simple formulas:

$$m_+(\lambda) = -\frac{\cos\alpha - m_+(\alpha,\lambda)\sin\alpha}{\sin\alpha + m_+(\alpha,\lambda)\cos\alpha},$$

$$m_+(\alpha,\lambda) = \frac{\cos\alpha + m_+(\lambda)\sin\alpha}{\sin\alpha - m_+(\lambda)\cos\alpha}.$$

The function $m_+(\lambda)$ is the *Weyl m-function* at $+\infty$. As is easily seen, an analogous construction can be made for the problem in the negative half-line, giving a Weyl m-function at $-\infty$. We will denote this function with $m_-(\lambda)$. Let $\varphi_+(t,\lambda)$ and $\varphi_-(t,\lambda)$ be the (essentially unique) solutions of (4.9) which are square integrable on the positive and negative

half lines respectively (obviously with respect to the weighted measure $y\,\mathrm{d}t$). From (4.15) we have the following important formulas

$$m_+(\lambda) = \frac{p(0)\varphi_+'(0,\lambda)}{\varphi_+(0,\lambda)}, \qquad m_-(\lambda) = \frac{p(0)\varphi_-'(0,\lambda)}{\varphi_-(0,\lambda)}. \tag{4.16}$$

We have

PROPOSITION 4.7. *If $\Im(\lambda) \neq 0$, $m_\pm$ are analytic functions of λ. In addition,* $\mathrm{Sgn}(\Im m_\pm \Im\lambda) = \pm 1$. *If $m_\pm$ have zeroes or poles, they are all real and simple.*

Consider the operator

$$L(\varphi) = \frac{1}{y}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(p\frac{\mathrm{d}}{\mathrm{d}t}\varphi\right) - q\varphi\right]. \tag{4.17}$$

We will denote by L the self-adjoint closure of operator (4.17) on $L^2(\mathbb{R}, y)$ and with $L^\pm$ the closures of (4.17) on $L^2([0,\infty), y)$ and $L^2((-\infty,0], y)$ respectively.

Consider Eq. (4.4) in the interval $[0, b]$, together with the boundary conditions

$$u(0) = 0, \qquad p(b)u'(b) = 1.$$

Let $(\lambda_{b_n}) \subset \mathbb{R}$ be the sequence of eigenvalues of Eq. (4.4), and let (u_{b_n}) be the corresponding sequence of eigenfunctions. If ψ is the solution of (4.4) which satisfies the second half of the conditions (4.9a) and (4.9b), then ψ satisfies $\psi(0) = 0$, hence we must have $u_{b_n}(t) = r_{b_n}\psi(t, \lambda_{b_n})$, where $r_{b_n} \in \mathbb{C}$.

Let $\rho_b : \mathbb{R} \to \mathbb{R}$ be the monotone nondecreasing step function having jumps of $|r_{b_n}|^2$ at each eigenvalue λ_{b_n}. We take ρ_b to be right-continuous, and $\rho_b(0) = 0$. The function ρ defines a measure, that we still denote by ρ_b, called the spectral measure for Eq. (4.4) in the interval $(0, b)$. If σ is a measure of Lebesgue–Stieltjes type on $\mathbb{R}$, let $L^2(\mathbb{R}, \sigma)$ be the set of all the functions $h(\lambda)$ which are σ-measurable and such that

$$\int_{-\infty}^{\infty} \left|h(\lambda)\right|^2 \mathrm{d}\sigma(\lambda) < \infty.$$

We have the following important

THEOREM 4.8.

(i) *There is a monotone nondecreasing function $\rho(\lambda)$ defined on the real line, such that*

$$\rho(\lambda) - \rho(\mu) = \lim_{b\to\infty}\bigl(\rho_b(\lambda) - \rho_b(\mu)\bigr),$$

at every point of continuity λ, μ of ρ.

(ii) *If* $f \in L^2(0,\infty)$*, there exists a function* $g(\lambda) \in L^2(\mathbb{R},\rho)$ *such that*

$$\lim_{b\to\infty}\int_{-\infty}^{\infty}\left|g(\lambda)-\int_0^b f(t)y(t)\psi(t,\lambda)\,\mathrm{d}t\right|^2\mathrm{d}\rho(\lambda)=0$$

and

$$\int_0^{\infty} y(t)\left|f(t)\right|^2\mathrm{d}t=\int_{-\infty}^{\infty}\left|g(\lambda)\right|^2\mathrm{d}\rho(\lambda).$$

(iii) *The map*

$$t\mapsto\int_{-\infty}^{\infty}g(\lambda)\psi(t,\lambda)\,\mathrm{d}\rho(\lambda)$$

converges in $L^2(0,\infty)$ *to* fy.

(iv) *If* $m_+(\lambda)$ *is the Weyl function at* ∞*, then*

$$\frac{\rho(\mu+0^+)-\rho(\mu+0^-)}{2}-\frac{\rho(\lambda+0^+)-\rho(\lambda+0^-)}{2}+\int_{\lambda}^{\mu}\mathrm{d}\rho$$
$$=\lim_{\varepsilon\to0^+}\frac{1}{\pi}\int_{\mu}^{\lambda}\Im m_+(\nu+\mathrm{i}\varepsilon)\,\mathrm{d}\nu.$$

(v) *If* $\Im\lambda_0,\Im\lambda_1\neq0$*, then*

$$m_+(\lambda_0)-m_+(\lambda_1)=\int_{-\infty}^{\infty}\left(\frac{1}{\lambda-\lambda_0}-\frac{1}{\lambda-\lambda_1}\right)\mathrm{d}\rho(\lambda).$$

Analogous relations hold for the operator L^- *and the Weyl* m*-function* m_- *at* $-\infty$.

The above theorem can be extended to Sturm–Liouville operators on the whole line. As before, we start from the problem in an interval $I=[a,b]$ with $a<0<b$, and the boundary conditions

$$\cos\alpha u(a)+\sin\alpha p(a)u'(a)=0, \tag{4.18}$$

$$\cos\beta u(b)+\sin\beta p(b)u'(b)=0. \tag{4.19}$$

There is a matrix $P_I=(\rho_{jk})$ (called the *spectral matrix*) associated with Eq. (4.4) in the interval I. Its entries are nondecreasing step functions satisfying

$$\rho_{jk}(\lambda_n+)-\rho_{jk}(\lambda_n-)=r_j\bar{r}_k,$$

and otherwise constant. If we require that $P_I(\lambda+)=P_I(\lambda)$ then P_I is Hermitian, and, if $\lambda>\mu$, $P_I(\lambda)-P_I(\mu)$ is positive semidefinite. If we let $I\to\mathbb{R}$, then $P_I(\lambda)\to P(\lambda)$, where $P(\lambda)$ is a hermitian, positive semidefinite matrix.

REMARK 4.9. Since the matrix $P(\lambda)$ is Hermitian and nondecreasing, if follows that, for every interval $I=[\lambda,\mu]$,

$$\left|\rho_{jk}(\mu)-\rho_{jk}(\lambda)\right|^2 \leqslant \big(\rho_{jj}(\mu)-\rho_{jj}(\lambda)\big)\big(\rho_{kk}(\mu)-\rho_{kk}(\lambda)\big).$$

From this we conclude that the points of nonconstancy of the spectral matrix $P(\lambda)$ are the same of the points of nonconstancy of the trace $\mathrm{Tr}(P(\lambda))$.

We use the notation $\Sigma(L)$, $R(L)$ for the spectrum and the resolvent set of the full-line operator L and $\Sigma^{\pm}(L)$, $R^{\pm}(L)$ for the spectrum and the resolvent set of the positive and negative Sturm–Liouville operators $L^{\pm}$ respectively.

REMARK 4.10. The sets $\Sigma(L)$, $\Sigma^{\pm}(L)$ are closed subsets of $\mathbb{R}$. The set of the points of growth of the spectral matrix $P(\lambda)$ (i.e., of the trace $\mathrm{Tr}(P(\lambda))$) equals the set $\Sigma(L)$.

We finish this subsection by giving a brief summary of the theory of Gilbert and Pearson [60] adapted to the Sturm–Liouville operator. The original paper of Gilbert and Pearson deals with the Schrödinger operator. Using the weighted measure $\kappa=y\,\mathrm{d}t$ instead of the Lebesgue measure on $\mathbb{R}$, we can prove the results stated in the following lines.

We define a set $\mathcal{M}\subset\mathbb{R}$ to be a minimal support for a measure ι on $\mathbb{R}$ if:

(1) $\iota(\mathbb{R}\setminus\mathcal{M})=0$;

(2) if $\mathcal{M}_0\subset\mathcal{M}$ and $\iota(\mathcal{M}_0)=0$, then $\ell(\mathcal{M}_0)=0$, where ℓ is the Lebesgue measure on $\mathbb{R}$.

A minimal support of a measure ι determines where the measure is concentrated, and is defined up to a set of Lebesgue (and ι) 0 measure.

Let $E=\{\lambda\in\mathbb{R}\mid (\mathrm{d}\rho^+/\mathrm{d}\ell)(\lambda)\text{exists}\}$, where ρ^+ is the spectral measure for Eq. (4.9) and $\mathrm{d}\rho^+/\mathrm{d}\ell$ is the Radon–Nikodym derivative of ρ^+ with respect to the measure ℓ. Note that $\mathrm{d}\kappa/\mathrm{d}\ell=y>0$. Then $\ell(\mathbb{R}\setminus E)=0$ and the minimal support $\mathcal{M}$ of ρ^+ has the decomposition $\mathcal{M}=\mathcal{M}_{ac}\cup\mathcal{M}_s=\mathcal{M}_{ac}\cup\mathcal{M}_{sc}\cup\mathcal{M}_p$, where $\mathcal{M}_{ac}$ and $\mathcal{M}_s$ are the absolutely continuous part and the singular part of $\mathcal{M}$ respectively. In particular

(i) $\mathcal{M}=\{\lambda\in\mathbb{R}\mid 0<(\mathrm{d}\rho^+/\mathrm{d}\ell)(\lambda)\leqslant\infty\}$,

(ii) $\mathcal{M}_{ac}=\{\lambda\in\mathbb{R}\mid 0<(\mathrm{d}\rho^+/\mathrm{d}\ell)(\lambda)<\infty\}$,

(iii) $\mathcal{M}_s=\{\lambda\in\mathbb{R}\mid (\mathrm{d}\rho^+/\mathrm{d}\ell)(\lambda)=\infty\}$,

(iv) $\mathcal{M}_{sc}=\{\lambda\in\mathbb{R}\mid (\mathrm{d}\rho^+/\mathrm{d}\ell)(\lambda)=\infty, \lim_{\delta\to0^+}\rho^+(\lambda+\delta)-\rho^+(\lambda-\delta)=0\}$,

(v) $\mathcal{M}_p=\{\lambda\in\mathbb{R}\mid (\mathrm{d}\rho^+/\mathrm{d}\ell)(\lambda)=\infty, \lim_{\delta\to0^+}\rho^+(\lambda+\delta)-\rho^+(\lambda-\delta)>0\}$.

Let $\lambda\in\mathbb{C}$. A solution $u_s(t,\lambda)$ of Eq. (4.9) is called *subordinate* at infinity if for every other linearly independent solution $u(t,\lambda)$ of Eq. (4.9), one has

$$\lim_{N\to\infty}\frac{\|u_s(t,\lambda)\|_N}{\|u(t,\lambda)\|_N}=0,$$

where $\|\cdot\|_N$ denotes the $L^2((0,N),y)$-norm. Let $\alpha\in[0,\pi)$. Consider Eq. (4.9) together with the boundary condition (4.18) at $t=0$, i.e.,

$$u(0)\cos\alpha+p(0)u'(0)\sin\alpha=0.$$

The principal results of Gilbert and Pearson can be summarized as follows (see [60]):

THEOREM 4.11. *There exists a set $E_1 \subset E$ of full Lebesgue measure and full ρ^+ measure such that the nontangential limit*

$$\lim_{\varepsilon\to 0} m_+(\lambda + \mathrm{i}\varepsilon) = m_+(\lambda) \tag{4.20}$$

exists for every $\lambda \in E$, and

$$\frac{\mathrm{d}\rho^+}{\mathrm{d}\ell}(\lambda) = \frac{1}{\pi}\Im m_+(\lambda). \tag{4.21}$$

Moreover, we have

(i) $\mathcal{M} = \{\lambda \in \mathbb{R} \mid 0 < \Im m_+(\lambda) \leqslant \infty\}$,
(ii) $\mathcal{M}_{ac} = \{\lambda \in \mathbb{R} \mid 0 < \Im m_+(\lambda) < \infty\}$,
(iii) $\mathcal{M}_s = \{\lambda \in \mathbb{R} \mid \Im m_+(\lambda) = \infty\}$,
(iv) $\mathcal{M}_{sc} = \{\lambda \in \mathbb{R} \mid \Im m_+(\lambda) = \infty, \lim_{\delta\to 0^+} \rho^+(\lambda+\delta) - \rho^+(\lambda-\delta) = 0\}$,
(v) $\mathcal{M}_p = \{\lambda \in \mathbb{R} \mid \Im m_+(\lambda) = \infty, \lim_{\delta\to 0^+} \rho^+(\lambda+\delta) - \rho^+(\lambda-\delta) > 0\}$

and

(i) $\mathcal{M}^c = \{\lambda \in \mathbb{R} \mid$ *a subordinate solution of Eq.* (4.9) *exists and does not satisfy the boundary condition* (4.18)$\}$,
(ii) $\mathcal{M}_{ac} = \{\lambda \in \mathbb{R} \mid$ *no subordinate solution of Eq.* (4.9) *exists*$\}$,
(iii) $\mathcal{M}_s = \{\lambda \in \mathbb{R} \mid$ *a subordinate solution of Eq.* (4.9) *exists and satisfies the boundary condition* (4.18)$\}$,
(iv) $\mathcal{M}_{sc} = \{\lambda \in \mathbb{R} \mid$ *a subordinate solution of Eq.* (4.9) *exists and satisfies the boundary condition* (4.18), *but it is not in* $L^2([0,\infty), y)\}$,
(v) $\mathcal{M}_p = \{\lambda \in \mathbb{R} \mid$ *a subordinate solution of Eq.* (4.9) *exists, satisfies the boundary condition* (4.18) *and is in* $L^2([0,\infty), y)\}$.

Finally,

$$\Sigma^+(L) = \mathcal{M},$$

up to a set of 0 Lebesgue (and ρ^+) measure.

The theorem is still valid for Eq. (4.9) on $(-\infty, 0]$, with the spectral measure $\mathrm{d}\rho^-$ and the function m_-, with the appropriate changes of signs. From this theorem we derive the following important observations:

COROLLARY 4.12.

(i) *The spectral measures $\rho^\pm(\lambda)$, $P(\lambda)$ increase exactly in the points of the spectra $\Sigma^\pm(L)$, $\Sigma(L)$ respectively;*
(ii) *the nontangential limit* (4.20) *exists for Lebesgue a.e. $\lambda \in \mathbb{R}$, $\Im m_+(\lambda) > 0$ ($\Im m_-(\lambda) < 0$) for Lebesgue a.e $\lambda \in \Sigma^+(L)$ ($\Sigma^-(L)$), and $\Im m_\pm(\lambda) = 0$ for Lebesgue a.e. $\lambda \in R^\pm(L)$;*
(iii) *$|m_\pm(\lambda + \mathrm{i}\varepsilon)| \to \infty$ for $\varepsilon \to 0$ if and only if λ is an eigenvalue of $L^\pm$.*

PROOF. The only thing we have to prove is that the measure $P(\lambda)$ increases exactly in the points of the spectrum $\Sigma(L)$. From Remark 4.9, the points of nonconstancy of $P(\lambda)$ are exactly the points of nonconstancy of the trace $\mathrm{Tr}(P(\lambda))$, and these are exactly the points λ for which

$$\Im\left(\frac{1}{m_-(\lambda)-m_+(\lambda)}+\frac{m_-(\lambda)m_+(\lambda)}{m_-(\lambda)-m_+(\lambda)}\right) \tag{4.22}$$

is positive. We now rewrite (4.22) in the following way

$$\Im\left(\frac{(1+m_-m_+)(\overline{m_--m_+})}{\|m_--m_+\|^2}\right). \tag{4.23}$$

From Theorem 4.11, it easy to conclude that (4.23) is not 0 exactly in the points of the spectrum. □

4.2. *The Sturm–Liouville equation as a nonautonomous differential system*

Let Ω be a compact metric space, and let $(\Omega,\{\tau_t\})$ be a flow. We suppose that there is a $\{\tau_t\}$-ergodic measure ν on Ω such that $\operatorname{Supp}\nu=\Omega$. We recall that the topological support $\operatorname{Supp}\nu$ of the measure ν is the complement of the largest open subset $U\subset\Omega$ satisfying $\nu(U)=0$.

Suppose that $p,q,y:\Omega\to\mathbb{R}$ are continuous functions, and that p and y are strictly positive. For each $\omega\in\Omega$, we introduce the matrix function

$$t\mapsto\begin{pmatrix}0 & \frac{1}{p(\omega\cdot t)}\\ q(\omega\cdot t)-\lambda y(\omega\cdot t) & 0\end{pmatrix}$$

which by abuse of notation we call $\omega(t)$. We study the family of differential equations

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix}'=\begin{pmatrix}0 & \frac{1}{p(\omega\cdot t)}\\ q(\omega\cdot t)-\lambda y(\omega\cdot t) & 0\end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \tag{SL_ω}$$

where ω ranges over Ω and λ is a complex parameter. If the function $p':\Omega\to\mathbb{R}:\omega\mapsto\frac{\mathrm{d}}{\mathrm{d}t}p(\omega\cdot t)|_{t=0}$ is well defined and continuous, then (SL_ω) is equivalent to

$$-\big(p(\omega\cdot t)\varphi(t)'\big)'+q(\omega\cdot t)\varphi(t)=\lambda y(\omega\cdot t)\varphi(t). \tag{4.24_ω}$$

We have the following

PROPOSITION 4.13. *Suppose that* $\operatorname{Supp}\nu=\Omega$. *There is a set* $\Omega'\subset\Omega$ *with* $\nu(\Omega')=1$ *and such that, if* $\omega\in\Omega'$, *then the orbit* $\{\omega\cdot t\mid t\in\mathbb{R}\}$ *is dense in* Ω. *Moreover,* Ω' *can be chosen such that each* $\omega\in\Omega'$ *is both positively and negatively Poisson recurrent.*

Recall that $\omega \in \Omega$ is positively (resp. negatively) Poisson recurrent if there is a sequence $\{t_n\} \to \infty$ (resp. $\{t_n\} \to -\infty$) such that $\lim_{n\to\infty} \omega \cdot t_n = \omega$.

PROOF. The proof is an easy consequence of the Birkhoff ergodic theorem. For, let $V_1, \dots, V_n, \dots$ be a countable base for the topology of Ω. Let χ_i be the characteristic function corresponding to V_i. Using the Birkhoff ergodic theorem, one checks easily that the sets $\Omega_i = \{\omega \in \Omega \mid \lim_{t\to\infty} \frac{1}{t}\int_0^t \chi_i(\omega\cdot s)\,\mathrm{d}s = \nu(V_i) > 0\}$ have ν-measure 1. It follows that if $\omega \in \Omega_i$, then the orbit of ω intersects V_i $(i \in \mathbb{N})$. It is clear now that each $\omega \in \Omega_i$ has dense orbit (in fact, it has dense positive semi-orbit). The set $\Omega' = \bigcap_{i=1}^{\infty} \Omega_i$ is the desired set.

The set Ω' consists of those points with dense positive semi-orbit; such points are in particular positively Poisson recurrent.

Using the fact that $\lim_{t\to-\infty} \frac{1}{t}\int_0^t \chi_i(\omega\cdot s)\,\mathrm{d}s = \nu(V_i)$ for ν-a.a. $\omega \in \Omega$, we see that there is a subset of Ω' of ν-measure 1 consisting of points which are negatively Poisson recurrent as well; we abuse notation slightly and let Ω' denote this set. This completes the proof. $\square$

Let $\Phi_\omega(t)$ be the fundamental matrix solution of (SL_ω) with $\Phi_\omega(0) = I$. Define $\Sigma^{\mathrm{ed}}(L) = \{\lambda \in \mathbb{R} \mid$ the family (SL_ω) does NOT have exponential dichotomy$\}$, and $R^{\mathrm{ed}}(L) = \mathbb{C} \setminus \Sigma^{\mathrm{ed}}(L)$.

We next give a dichotomy-theoretic interpretation of the Weyl m-functions $m_\pm = m_\pm(\omega, \lambda)$ of Eqs. (4.24_ω) $(\omega \in \Omega)$.

To begin with, let $\omega \in \Omega$ and $\lambda \in \mathbb{C} \setminus \Sigma^{\mathrm{ed}}(L)$. Define $m_+(\omega,\lambda)$ to be the unique (extended) complex number such that $\operatorname{Im} P_\omega = \operatorname{Span}\binom{1}{m_+(\omega,\lambda)}$. Here $m_+(\omega,\lambda) = \infty$ if and only if $\operatorname{Im} P_\omega = \operatorname{Span}\binom{0}{1}$. One can show that $\Im m_+(\omega,\lambda)\Im\lambda > 0$ whenever $\Im\lambda \neq 0$. In a similar way, we define $\operatorname{Ker} P_\omega = \operatorname{Span}\binom{1}{m_-(\omega,\lambda)}$ where $\Im m_-(\omega,\lambda)\Im\lambda < 0$ if $\Im\lambda \neq 0$. For every $\omega \in \Omega$, if $\Im\lambda \neq 0$ the maps $\lambda \mapsto m_\pm(\omega,\lambda)$ coincide with the classical Weyl m-functions of problem (4.24_ω). In fact, let $\omega \in \Omega$ and $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$. Let $0 \neq u = (u_1, u_2) \in \mathbb{C}^2$ such that $\frac{u_2}{u_1} = m_+(\omega,\lambda)$ and let $u(t)$ be the solution of (SL_ω) with $u(0) = u$. Such a solution decays exponentially as $t \to \infty$, hence $u(t) \in L^2(0,\infty)$. Since (SL_ω) is in the limit-point case at ∞, the only L^2 solution at ∞ is $u(t)$. It is easy to check that $\frac{u_2}{u_1} = \frac{p(\omega)u_1'(0)}{u_1(0)}$. The same holds for $m_-(\omega,\lambda)$. We showed that the two definitions of the Weyl m-functions coincide. It follows that for each $\omega \in \Omega$, the maps $\lambda \mapsto m_\pm(\omega,\lambda)$ are holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and meromorphic on $\mathbb{C} \setminus \Sigma^{\mathrm{ed}}(L)$.

The next proposition provides a result concerning the behavior of the Weyl m-functions when ω varies along an orbit.

PROPOSITION 4.14. *Let $\omega \in \Omega$ have dense orbit. Let $\bar\omega \in \Omega$ and let $\{t_n\} \subset \mathbb{R}$ be a sequence with $\lim_{n\to\infty} \omega \cdot t_n = \bar\omega$. Then the Weyl m functions $m_\pm(\omega \cdot t_n, \lambda)$ are well defined and*

$$\lim_{n\to\infty} m_\pm(\omega \cdot t_n, \lambda) = m_\pm(\bar\omega, \lambda),$$

uniformly on compact subsets of the upper (*lower*) *half-plane.*

PROOF. Let $\chi(t,\lambda)$ be a solution of Eq. (4.4) (in the interval $[0,b]$) satisfying the boundary conditions (4.4b) at $t=b$, and let $C_b(\lambda)$ be the circle defined by this solution as in Section 4.1. Let $D_b(\lambda)$ be the closure of $C_b(\lambda)$. Since $D_{b'}(\lambda) \subsetneq D_b(\lambda)$ for every $b'>b$, one has that $m_+(\omega,\lambda)$ is contained in every $D_b(\lambda)$. In an analogous way, we define the (closed) discs $D_b^n(\lambda)$ corresponding to $\omega\cdot t_n$. Let $\varepsilon>0$, and choose $b\in\mathbb{R}$ such that $\operatorname{diam} D_b(\lambda)<\varepsilon$. Fix any $b'>b$: it is easy to check that $C_{b'}^n(\lambda)$ converges to $C_{b'}(\lambda)$ as $n\to\infty$. This immediately implies that $|m_+(\omega\cdot t_n,\lambda)-m_+(\omega,\lambda)|<\varepsilon$. □

Now we turn to the first basic result concerning the spectrum of the family (SL_ω), namely (see [79])

PROPOSITION 4.15. *Let $\omega\in\Omega$ have dense orbit. Then a complex number λ lies in $R(L_\omega)$ if and only if the family* (SL_ω) *admits an exponential dichotomy*: $R(L_\omega)=R^{\mathrm{ed}}(L)$.

PROOF. Let $\lambda\in R^{\mathrm{ed}}(L)$: then the family (SL_ω) admits an exponential dichotomy over Ω. An easy consequence of the definition of the exponential dichotomy is that for every $\omega\in\Omega$, there exist two solutions $\varphi_\pm(\cdot,\omega,\lambda)$ of (4.24_ω) which are square integrable on the positive and negative half-lines respectively. Indeed $\varphi_\pm(\cdot,\omega,\lambda)$ are exponentially decreasing at ∞ and $-\infty$ respectively, for every $\omega\in\Omega$. Since $\varphi_\pm(\cdot,\omega,\lambda)$ are linearly independent, the Wronskian $W(\varphi_+,\varphi_-)=p(\varphi_+\varphi_-'-\varphi_+'\varphi_-)\neq 0$. It follows that a Green's function

$$G(t,s,\omega,\lambda)=\begin{cases}\dfrac{\varphi_+(t)\varphi_-(s)}{W(\varphi_+,\varphi_-)}y(s) & t\geqslant s,\\[2mm] \dfrac{\varphi_-(t)\varphi_+(s)}{W(\varphi_+,\varphi_-)}y(s) & t\leqslant s\end{cases}$$

is well defined, hence $\lambda\in R(L_\omega)$.

Conversely, assume that $\lambda\in R(L_\omega)$. We must prove that the family (SL_ω) admits exponential dichotomy over Ω. From a result of Sacker–Sell [137] and Selgrade [142], it is sufficient to show that the only bounded solution of the family is the trivial one. Suppose for contradiction that there exists $\bar\omega\in\Omega$ such that the corresponding differential equation $(\mathrm{SL}_{\bar\omega})$ has a nonzero bounded solution (hence $\lambda\notin R(L_{\bar\omega})$). Let $\{t_n\}\to\infty$ be a real sequence such that $\lim_{n\to\infty}\omega\cdot t_n=\bar\omega$. The Weyl m-functions $m_\pm(\omega\cdot t_n,\lambda)$ are defined, and $m_\pm(\omega\cdot t_n,\lambda)\to m_\pm(\bar\omega,\lambda)$ uniformly on every compact subset of the upper complex half-plane as $n\to\infty$ (see Proposition 4.14). It's clear then that the measure $\operatorname{Tr}P_n(\omega,\lambda):=\operatorname{Tr}P(\omega\cdot t_n,\lambda)$ weakly converges to $\operatorname{Tr}P(\bar\omega,\lambda)$ (note that the spectral matrix P depends on $\omega\in\Omega$). This means that for every $f\in C_c(\mathbb{R})$ we have

$$\int_{\mathbb{R}} f(\lambda)\,\mathrm{d}\big(\operatorname{Tr}P_n(\omega,\lambda)\big)\to\int_{\mathbb{R}} f(\lambda)\,\mathrm{d}\big(\operatorname{Tr}P(\bar\omega,\lambda)\big).$$

Since the resolvent set $R(L_\omega)$ is open, there is an interval I such that $\lambda\in I\subset R(L_\omega)$. Now, the resolvent set is invariant under the flow $(\Omega,\{\tau_t\})$ (in fact, one can easily check that if a Green's function corresponding to $\omega\in\Omega$ is well defined, then a Green's function corresponding to $\omega\cdot t\in\Omega$ is well defined as well, for every $t\in\mathbb{R}$), hence $\operatorname{Tr}P_n$ is constant

on I. From Remark 4.10, it is straightforward to show that $\operatorname{Tr} P$ is constant on I, so $\lambda \in R(L_{\bar{\omega}})$, a contradiction. $\square$

It will be often useful to think at the functions $p, q, y, m_{\pm}$ as functions of t, as follows: for every fixed $\omega \in \Omega$, we consider the map $t \mapsto p(\omega \cdot t)$ obtained by "following the orbit through $\omega \in \Omega$". The same holds for all the functions involved; for instance, for fixed $\omega \in \Omega$, we consider the maps $m_{\pm}(\cdot, \omega, \lambda) : t \mapsto m_{\pm}(\omega \cdot t, \lambda)$. We will often suppress the dependence with respect to $\omega \in \Omega$, unless it is strictly necessary.

If $\omega \in \Omega$, then $m_{\pm}(t, \omega, \lambda)$ is obtained from $m_{\pm}(\omega, \lambda)$ by "translating" the (unique) L^2 solutions at $\pm\infty$ of Eq. (4.24_{ω}): if we denote these solutions with $\varphi_{\pm}(t, \omega, \lambda)$, then

$$m_{\pm}(t, \omega, \lambda) = \frac{p(\omega \cdot t)\varphi'_{\pm}(t, \omega, \lambda)}{\varphi_{\pm}(t, \omega, \lambda)}.$$

Both m_+ and m_- are analytic in the upper and lower λ-half-planes respectively. The following proposition is fundamental.

PROPOSITION 4.16. *Let $\omega \in \Omega$. For every fixed $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$ the Weyl m-functions $m_{\pm}(t, \lambda) \equiv m_{\pm}(t, \omega, \lambda)$ satisfy the Riccati equation*

$$m'_{\pm}(t, \lambda) + \frac{1}{p(t)} m^2_{\pm}(t, \lambda) = q(t) - \lambda y(t). \tag{4.25}$$

PROOF. Since

$$m_{\pm}(t, \lambda) = \frac{p(t)\varphi'_{\pm}(t)}{\varphi_{\pm}(t)},$$

differentiating with respect to t, we have

$$m'_{\pm} = \frac{(p\varphi'_{\pm})'\varphi_{\pm} - p(\varphi'_{\pm})^2}{\varphi^2_{\pm}}.$$

Hence

$$m'_{\pm} + \frac{1}{p} m^2_{\pm} = \frac{(p\varphi'_{\pm})'}{\varphi_{\pm}} = q - \lambda y.$$

$\square$

REMARK 4.17. Note that, if $\Im\lambda \neq 0$ and $\omega \in \Omega$, then

$$\varphi_{\pm}(t, \lambda) = \exp\left\{ \int_0^t \frac{m_{\pm}(s, \lambda)}{p(s)}\, \mathrm{d}s \right\} \tag{4.26}$$

is a solution of (4.24_{ω}) satisfying $\varphi_{\pm}(0, \lambda) = 1$ and $p(\omega)\varphi'_{\pm}(0, \lambda) = m_{\pm}(\omega, \lambda)$.

4.3. *The projective flow in the Sturm–Liouville setting*

We have already noted the importance of the projective flow in the study of nonautonomous differential systems. This subsection provides a further demonstration of this fact, since from the properties of the projective flow associated to a family of Sturm–Liouville differential systems, we will retrieve many important results concerning both the spectrum of the family (SL_ω) and the Weyl m-functions. We will use the notation and the terminology of Sections 2 and 3. Some proofs given in those sections will be repeated here, when they will lead to particular facts concerning the Sturm–Liouville operator.

Let Ω be a compact metric space, let $(\Omega, \{\tau_t\})$ be a real flow and let ν be a $\{\tau_t\}$-ergodic measure on Ω with $\operatorname{Supp}\nu = \Omega$. Consider the family of two-dimensional nonautonomous differential systems (SL_ω) defined as before. Let $(\Sigma, \mathbb{R})$ be the projective flow associated to $(\Omega, \mathbb{R})$, i.e. $\Sigma = \Omega \times \mathbb{P}(\mathbb{C})$ and $\hat{\tau}_t(\omega, l) = (\omega \cdot t, \Phi_\omega(t)l)$.

Fix $\omega \in \Omega$ and $\lambda \in \mathbb{R}$. The flow $(\Omega, \{\hat{\tau}_t\})$ induces a flow, which we still denote by $\{\hat{\tau}_t\}$, on $\Omega \times \mathbb{P}(\mathbb{R})$ as well. Following the developments of Section 2, we express Eqs. (SL_ω) in polar coordinates (r, θ), to obtain

$$\begin{cases} \dfrac{r'}{r} = \left(\dfrac{1}{p(\omega \cdot t)} + q(\omega \cdot t) - \lambda y(\omega \cdot t)\right) \sin\theta \cos\theta = f\big(\hat{\tau}_t(\omega, \theta)\big), \\ \theta' = \big(q(\omega \cdot t) - \lambda y(\omega \cdot t)\big)\cos^2\theta - \dfrac{1}{p(\omega \cdot t)} \sin^2\theta = g\big(\hat{\tau}_t(\omega, \theta)\big). \end{cases} \tag{4.27}$$

It is understood that we identify θ with the line through the origin it parametrizes, hence θ is taken mod π. If $\theta_0 \in \mathbb{P}(\mathbb{R})$ parametrizes the line l_0, then $\hat{\tau}_t(\omega, l_0) = \hat{\tau}_t(\omega, \theta_0) = (\omega \cdot t, \theta(t))$, where $\theta(t)$ satisfies the θ-equation in (4.27) with $\theta(0) = \theta_0$.

Recall that for every fixed $\omega \in \Omega$, the *rotation number* $\alpha_\omega(\lambda)$ is defined as

$$\alpha_\omega(\lambda) = -\lim_{t\to\infty} \frac{1}{t}\big(\theta(t, \lambda) - \theta_0(\lambda)\big) \quad (\lambda \in \mathbb{R}). \tag{4.28}$$

In the above formula, we explicitly wrote the fact that $\theta(\cdot)$ depends on the fixed $\lambda \in \mathbb{R}$. The questions we will answer in the following lines concern: (i) the existence of the limit (4.28); (ii) the independence of (4.28) with respect to the initial condition (ω, θ_0); (iii) continuity and monotonicity of the map $\lambda \mapsto \alpha(\lambda)$; (iv) the constancy of $\alpha(\lambda)$ in every interval $I \subset R^{\mathrm{ed}}(L)$. We will need the following

LEMMA 4.18. *Let $F : \Sigma \to \mathbb{R}$ be a continuous function such that $\int_\Sigma F(\sigma)\,\mathrm{d}\mu = 0$ for every invariant measure μ on Σ. Then $\frac{1}{t}\int_0^t F(\hat{\tau}_s(\sigma))\,\mathrm{d}s \to 0$ as $t \to \infty$, for all $\sigma \in \Sigma$. Moreover, the convergence is uniform with respect to $\sigma \in \Sigma$.*

PROOF. The proof uses techniques from Proposition 2.5; see, e.g. [89]. The space $C(\Sigma) = \{F : \Sigma \to \mathbb{R} \mid F \text{ is continuous}\}$ is separable, hence it contains a dense subset $C_1(\Sigma)$ which is generated by a countable family $\mathcal{F}$ of functions. We may assume that $F \in \mathcal{F}$. By con-

tradiction, suppose that there exist sequences $\{t_n\} \to \infty$ and $\{\sigma_n\} = \{(\omega_n, \theta_n)\} \subset \Sigma$ such that

$$\frac{1}{t_n} \int_0^{t_n} F\big(\hat{\tau}_s(\sigma_n)\big)\, \mathrm{d}s \to \Delta(F) \neq 0.$$

We can assume that $\{\sigma_n\} \to \sigma = (\omega, \theta) \in \Sigma$. Let $F_1 \in \mathcal{F}$, and choose subsequences $\{t_n^{(1)}\} \subset \{t_n\}$ and $\{\sigma_n^{(1)}\} \subset \{\sigma_n\}$ such that

$$\lim_{t_n^{(1)} \to \infty} \frac{1}{t_n^{(1)}} \int_0^{t_n^{(1)}} F_1\big(\hat{\tau}_s\big(\sigma_n^{(1)}\big)\big)\, \mathrm{d}s$$

exists. Call it $\Delta(F_1)$. Continuing this way, for every $F_k \in \mathcal{F}$, we can find subsequences $\{t_n^{(k)}\} \subset \{t_n^{(k-1)}\}$ and $\{\sigma_n^{(k)}\} \subset \{\sigma_n^{(k-1)}\}$ such that

$$\lim_{t_n^{(k)} \to \infty} \frac{1}{t_n^{(k)}} \int_0^{t_n^{(k)}} F_k\big(\hat{\tau}_s\big(\sigma_n^{(k)}\big)\big)\, \mathrm{d}s$$

exists. Call it $\Delta(F_k)$. Now we choose the diagonal sequences $\{t_n^{(n)}\}$ and $\{\sigma_n^{(n)}\}$. For every $G \in \mathcal{F}$, one has

$$\Delta(G) = \lim_{t_n^{(n)} \to \infty} \frac{1}{t_n^{(n)}} \int_0^{t_n^{(n)}} G\big(\hat{\tau}_s\big(\sigma_n^{(n)}\big)\big)\, \mathrm{d}s.$$

Note that $\Delta(G)$ is $\mathbb{Q}$-linear, $\Delta(G) \geqslant 1$ if $G \geqslant 0$, and $\Delta(1) = 1$. It follows that the above limit defines a linear functional $l_1 : C_1(\Sigma) \to \mathbb{R}$ with $|l_1| = 1$, hence it extends to a unique linear functional $l : C(\Sigma) \to \mathbb{R}$ (here $|\cdot|$ denotes the L^∞ norm). By the Riesz representation theorem, l defines a measure μ_l on Σ such that $l(G) = \int_\Sigma G\, \mathrm{d}\mu_l$ for every $G \in C(\Sigma)$. Further, l is invariant in the sense that, if $t \in \mathbb{R}$ and $G \in C(\Sigma)$, then $l(G(\hat{\tau}_{-t}(\cdot))) = l(G(\cdot))$; this is because $t_n^{(n)} \to \infty$. But now we have

$$0 \neq \Delta(F) = \lim_{t_n^{(n)} \to \infty} \frac{1}{t_n^{(n)}} \int_0^{t_n^{(n)}} F\big(\hat{\tau}_s\big(\sigma_n^{(n)}\big)\big)\, \mathrm{d}s = \int_\Sigma F d\mu_l = l(F) = 0,$$

a contradiction. □

We use Lemma 4.18 in the proof of the following important theorem, which answers the questions (i), (ii) and (iii) posed before (for an analogous result in the case of the Schrödinger operator, see [89]).

THEOREM 4.19.

(a) *There exists a set* $\Omega_\nu \subset \Omega$ *with* $\nu(\Omega_\nu) = 1$, *such that* $\alpha_\omega(\lambda)$ *exists for every* $\omega \in \Omega_\nu$ *and does not depend on the choice of* $(\omega, \theta) \in \Omega_\nu \times \mathbb{P}(\mathbb{R})$: $\alpha_\omega(\lambda) = \alpha(\lambda)$, *for every* $\omega \in \Omega_\nu$;

(b) *if ν is the only ergodic measure on Ω (for instance if $(\Omega, \mathbb{R})$ is minimal and almost periodic), then $\Omega_\nu = \Omega$, and the limit* (4.28) *is uniform in $t \in \mathbb{R}$ and $\sigma \in \Sigma$;*

(c) *for every $\omega \in \Omega$, the maps $\lambda \mapsto \alpha_\omega(\lambda)$ are continuous, monotone nondecreasing, and* $\lim_{\lambda\to\infty} \alpha_\omega(\lambda) = +\infty$.

PROOF. Recall that the angular variable θ is defined mod π. First of all, we claim that the limit (4.28) does not depend on the choice of $\theta \in \mathbb{P}(\mathbb{R})$. For, let $0 < \theta_1 - \theta_0 < \pi$, and let $\sigma_1 = (\omega, \theta_1), \sigma_0 = (\omega, \theta_0) \in \Sigma$. It follows that $0 < \theta(t, \sigma_1) - \theta(t, \sigma_0) < \pi$ for all $t \in \mathbb{R}$, otherwise there would exists $t_0 \in \mathbb{R}$ such that $\theta(t_0, \sigma_1) - \theta(t_0, \sigma_0) = \pi$, and hence, by uniqueness, $\theta(t, \sigma_1) - \theta(t, \sigma_0)$ for all $t \in \mathbb{R}$, a contradiction. The claim now is obvious, since $|\theta(t, \sigma_1) - \theta(t, \sigma_0)|$ is bounded, for every $t \in \mathbb{R}$ and $\sigma_0, \sigma_1 \in \Sigma$.

Now, let μ be a $\{\hat{\tau}_t\}$-ergodic measure on Σ. Using the Birkhoff ergodic theorem, one finds a set Σ_μ with $\mu(\Sigma_\mu) = 1$ and such that the limit (4.28) exists for all $\sigma \in \Sigma_\mu$. Moreover we have

$$\alpha_\omega(\lambda) = -\int_\Sigma g(\omega, \theta)\, \mathrm{d}\mu,$$

hence the limit (4.28) does not depend on the choice of $(\omega, \theta) \in \Sigma_\mu$. Note that if ν is any $\{\tau_t\}$-ergodic measure on Ω, then $\Omega_\nu := \pi(\Sigma_\mu)$ has ν-measure 1. From the observations above, the function $g(\cdot): \Sigma \to \mathbb{R}$ can be viewed as a function of the single variable $\omega \in \Omega$, since, when it exists, the limit (4.28) is independent of the choice of $\theta \in \mathbb{P}(\mathbb{R})$. It follows that for every $\omega \in \Omega_\nu$, the limit (4.28) exists and does not depend on the choice of $\omega \in \Omega_\nu$. We have proved (a).

To prove (b), we use Lemma 4.18. Let μ be any invariant measure on Σ. Then, by the Birkhoff ergodic theorem we conclude that there exists a set Σ_μ with $\mu(\Sigma_\mu) = 1$, and such that the limit (4.28) exists for all $\sigma = (\omega, \theta) \in \Sigma_\mu$. This limit defines a function $g^*: \Sigma_\mu \to \mathbb{R}$, which is μ-integrable and such that

$$\int_\Sigma g^*(\sigma)\, \mathrm{d}\mu = \int_\Sigma g(\sigma)\, \mathrm{d}\mu.$$

Arguing as above, the function g^* can be viewed as a function of the single variable $\omega \in \Omega$. Now we use the unique ergodicity of Ω. Since g^* is invariant with respect to the flow $(\Omega, \{\tau_t\})$, we conclude that $g^* = const$ on a set $\Omega_\nu \subset \Omega$ with $\nu(\Omega_\nu) = 1$. By the Birkhoff ergodic theorem, this constant is exactly the rotation number $\alpha(\lambda)$. At this point, (b) follows easily, applying Lemma 4.18 to the function $g(\cdot) + \alpha$.

To prove (c), let $\{\lambda_j\} \subset \mathbb{R}$ be a sequence which converges to $\lambda_0 \in \mathbb{R}$. Let $g_j(\sigma) = g(\sigma, \lambda_j)$, $g_0(\sigma) = g(\sigma, \lambda_0)$, and let $(\Sigma, \{\hat{\tau}_t^j\})$ be the corresponding projective flows (depending on λ_j). Let μ_j be any $\{\hat{\tau}_t^j\}$-ergodic measure on Σ. Then

$$\alpha(\lambda_j) = -\int_\Sigma g_j(\sigma)\, \mathrm{d}\mu_j.$$

Without loss of generality, we may assume that μ_j weakly converges to $\tilde{\mu}$, where $\tilde{\mu}$ is a $\{\hat{\tau}_t\}$-invariant measure on Σ. It follows that

$$\int_\Sigma g_0(\sigma)\,\mathrm{d}\mu_j \to \int_\Sigma g_0(\sigma)\,\mathrm{d}\tilde{\mu}.$$

Moreover

$$\left|g_j(\sigma) - g_0(\sigma)\right| \leqslant \max_{\omega\in\Omega}\left|y(\omega)\right| \cdot |\lambda_j - \lambda_0| \to 0$$

as $j \to \infty$. Using this inequality, we obtain

$$\alpha(\lambda_j) = -\int_\Sigma g_j(\sigma)\,\mathrm{d}\mu_j = -\int_\Sigma (g_j - g_0)(\sigma)\,\mathrm{d}\mu_j - \int_\Sigma g_0(\sigma)\,\mathrm{d}\mu_j \to \alpha(\lambda)$$

as $j \to \infty$. The monotonicity property and the asymptotic behavior of the map $\lambda \mapsto \alpha(\lambda)$ follow directly from the definition of the rotation number. □

REMARK 4.20. Any zero of the solution φ of Eq. (4.24$_\omega$) corresponds to a value $\bar{t} \in \mathbb{R}$ such that $\theta(\bar{t}, \lambda) = \frac{\pi}{2} \bmod \pi$. By (4.27), one has $\theta'(\bar{t}) = -\frac{1}{p(\omega\cdot\bar{t})} < 0$. It follows that θ decreases at such zero: this shows that there is exactly one more zero every time θ is increased by π. Let $N(t, \lambda)$ be the number of the zeros of the solution φ of (4.24$_\omega$) in $[0, t]$. Then we must have

$$\alpha(\lambda) = \pi \lim_{t\to\infty} \frac{1}{t} N(t, \lambda). \tag{4.29}$$

There remains to discuss the assertion that $\alpha(\lambda)$ is constant in every open interval $I \subset R^{\mathrm{ed}}(L)$. Actually, this and other considerations will enable us to prove a key fact concerning the spectrum of the family (SL$_\omega$). In fact, we will see that the points of growth of $\alpha(\lambda)$ *coincide* with the points of the spectrum $\Sigma(L_\omega)$, for Lebesgue a.a. $\lambda \in \mathbb{R}$. Since $\alpha(\lambda)$ does not depend on ω, for ν-a.a. $\omega \in \Omega$, it follows that $\Sigma(L_\omega)$ equals a fixed closed set $\Sigma(L)$ for ν-a.a. $\omega \in \Omega$. Moreover $\Sigma(L) = \Sigma^{\mathrm{ed}}(L)$.

We summarize these last remarks in the following theorem. In the remainder of Section 4.3 we will discuss its proof together with various facts concerning the spectral theory of the family (SL$_\omega$). We will also illustrate several methods which have proved useful in the study of 2-dimensional, nonautonomous linear systems.

THEOREM 4.21.

(i) *For ν-a.a. $\omega \in \Omega$, say $\omega \in \Omega_\nu$, the spectrum of Eq.* (4.24$_\omega$) *equals a closed set $\Sigma(L)$ which does not depend on $\omega \in \Omega_\nu$. Also, $R^{\mathrm{ed}}(L) = R(L)$; i.e., $R(L) := \mathbb{C} \setminus \Sigma(L) =$ $\{\lambda \in \mathbb{C} \mid$ the family (SL$_\omega$) has an exponential dichotomy over $\Omega\}$.*

(ii) *The set $\Sigma(L)$ contains no isolated points.*

(iii) *If $I \subset \mathbb{R}$ is an open interval, then $I \subset R(L)$ if and only if the rotation number α is constant on I.*

First, we note that Propositions 4.13 and 4.15 imply that there is a set $\Omega_\nu \subset \Omega$ with $\nu(\Omega_\nu) = 1$ such that, if $\omega \in \Omega_\nu$, then the spectrum of Eq. (4.24_ω) equals the complement in $\mathbb{R}$ of $R^{\mathrm{ed}}(L)$. Thus point (i) of Theorem 4.21 has already been proved.

Second, we turn to a proof of the above mentioned constancy property of the rotation number. We need some arguments of Schwarzmann [140]. We sketch some facts of his theory. Let $H(\Omega, S^1)$ be the set of all the homotopy classes of the maps $f:\Omega \to S^1 \subset \mathbb{C}$. Each class in $H(\Omega, S^1)$ contains a map f such that the map $f':\Omega \to \mathbb{C}:\omega \mapsto \frac{\mathrm{d}}{\mathrm{d}t} f(\omega \cdot t)_{|t=0}$ is well defined and continuous. Let $h: H(\Omega, S^1) \to \mathbb{R}$ be defined as follows: $h[f] = \Im \int_\Omega \frac{f'(\omega)}{f(\omega)} \,\mathrm{d}\nu(\omega)$, where, as usual, ν is a $\{\tau_t\}$-ergodic measure on Ω. Using the Birkhoff ergodic theorem, one can show that for ν-a.a. $\omega \in \Omega$, $h[f]$ equals $\lim_{t\to\infty} \frac{1}{t} \arg f(\omega \cdot t)$. Schwarzmann proved that the map h induces a homomorphism between the groups $H(\Omega, S^1)$ and $(\mathbb{R}, +)$. For, let C be the subgroup of $H(\Omega, S^1)$ consisting on the homotopy classes of the maps $g(\omega) = \mathrm{e}^{2\mathrm{i}s(\omega)}$, where $s:\Omega \to \mathbb{R}$ is a given continuous function. For every $g \in C$, one has $h[g] = 0$, hence the map h induces a homomorphism from $\tilde{H}(\Omega, \mathbb{Z}) = H(\Omega, S^1)/C$ into $(\mathbb{R}, +)$. This homomorphism is called the "Schwarzmann homomorphism". We can now prove the following result:

THEOREM 4.22. *Let $I \subset R(L) \cap \mathbb{R}$ be an open interval. Then $\alpha(\lambda)$ is constant in I.*

PROOF. We use the Schwarzmann homomorphism. Let us temporarily identify the projective circle $\mathbb{P}(\mathbb{R})$ with the circle $S^1 \in \mathbb{C}$ by mapping the line l parametrized by $\theta \in \mathbb{R}$ to $\mathrm{e}^{2\mathrm{i}\theta} \in S^1$ (recall that θ is a π-periodic coordinate on $\mathbb{P}(\mathbb{R})$). Let $i:\mathbb{P}(\mathbb{R}) \to S^1$ denote this map. Next note that, for each fixed $\lambda \in I$, the map $\tilde{g}_\lambda:\Omega \to \mathbb{P}(\mathbb{R}):\omega \mapsto m_+(\omega, \lambda)$ is a continuous function. The composition $g_\lambda = i \circ \tilde{g}_\lambda$ lies in $H(\Omega, S^1)$.

Let us now calculate the image of g_λ under the Schwarzmann homomorphism h. Let $\omega \in \Omega$, and let θ_0 be a parameter value of $m_+(\omega, \lambda)$. Let $\theta(t)$ be the solution of (4.27) with initial value θ_0. We then obtain for ν-a.a. $\omega \in \Omega$:

$$-h[g_\lambda] = -2 \lim_{t\to\infty} \frac{\theta(t)}{t} = 2\alpha(\lambda).$$

This shows that $2\alpha \in h(\tilde{H}(\Omega, \mathbb{Z}))$, and the continuity of $\alpha(\lambda)$ implies that $\alpha(\lambda)$ is constant on I. $\square$

The other useful quantity in the study of nonautonomous differential systems is the upper Lyapunov exponent $\beta(\lambda)$. It has been already defined and studied in detail in Section 2. All the properties of $\beta(\lambda)$ which were stated in that section retain validity in our case.

It is convenient to extend the concepts of the rotation number and the Lyapunov exponent to complex λ. To do this we define the Floquet exponent. We write z instead of λ, and use λ to denote real numbers.

Fix $z \in \mathbb{C}$ with $\Im z \neq 0$, and recall that for the family (SL_ω), the Weyl m-functions are defined. Such functions satisfy the Riccati equation (4.25).

Let $\omega \in \Omega$ be fixed. For $\Im z \neq 0$ we define the Floquet exponent $w(\omega, z)$ as follows

$$w(\omega, z) = \frac{1}{2} \lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{1}{p(\omega \cdot s)} \big(m_+(\omega \cdot s, z) - m_-(\omega \cdot s, z)\big) \,\mathrm{d}s. \tag{4.30}$$

Using the Birkhoff ergodic theorem, one sees that, for a fixed ergodic measure ν on Ω,

$$w(\omega, z) = \frac{1}{2} \int_\Omega \frac{[m_+(\omega, z) - m_-(\omega, z)]}{p(\omega)} \, \mathrm{d}\nu(\omega) = w(z), \tag{4.31}$$

for ν-a.a. $\omega \in \Omega$. In particular, it follows that $w(\omega, z)$ does not depend on the choice of $\omega \in \Omega$, for ν-a.a $\omega \in \Omega$, and we will write simply $w(z)$ instead of $w(\omega, z)$.

We have the following

PROPOSITION 4.23. *For $\Im z \neq 0$, we have*

$$w(z) = \int_\Omega \frac{1}{p(\omega)} m_+(\omega, z) \, \mathrm{d}\mu = -\int_\Omega \frac{1}{p(\omega)} m_-(\omega, z) \, \mathrm{d}\mu. \tag{4.32}$$

PROOF. For fixed $\omega \in \Omega$, consider the map (where the notation has an obvious meaning)

$$f(t, z) = \frac{m_+(t, z) + m_-(t, z)}{p(t)} = \frac{\varphi'_+(t, z)\varphi_-(t, z) + \varphi'_-(t, z)\varphi_+(t, z)}{\varphi_+(t, z)\varphi_-(t, z)}.$$

It is obvious that f is the derivative with respect to t of the function

$$\ln\bigl(\varphi_+(t, z)\varphi_-(t, z)\bigr).$$

Integrating between 0 and t we obtain

$$\begin{aligned}\int_0^t \frac{1}{p(s)} \bigl[m_+(s, z) + m_-(s, z)\bigr] \, \mathrm{d}s = {} & \ln\bigl(\varphi_+(t, z)\varphi_-(t, z)\bigr) \\ & - \ln\bigl(\varphi_+(0, z)\varphi_-(0, z)\bigr).\end{aligned}$$

If we divide both sides by t and pass to the limit as $t \to \infty$, we obtain (4.32). □

From (4.26), the solutions $\varphi_\pm(t, \lambda)$ have the asymptotic form

$$\varphi_\pm(t, \lambda) \sim \mathrm{e}^{\pm(\Re w(\lambda)t + i\Im w(\lambda)t)}. \tag{4.33}$$

It is quite natural to define the Lyapunov exponent and the rotation number for complex z in the following way:

DEFINITION 4.24. Let $z \in \mathbb{C} \setminus \mathbb{R}$. We define

$$\tilde{\beta}(z) = \Re w(z) \quad \text{(Lyapunov exponent)}$$

and

$$\tilde{\alpha}(z) = \Im w(z) \quad \text{(Rotation number)}.$$

Now we prove important results of convergence. Recall that for Lebesgue a.a. $\lambda \in \mathbb{R}$ there exists the nontangential limit of $m_\pm(z)$ as $z \to \lambda$, and that $|m_\pm(z)| \to \infty$ as $z \to \lambda$ if and only if λ is an eigenvalue for the operator $L^\pm$ respectively (see again Theorem 4.11 and Corollary 4.12).

Moreover, if $z \in \mathbb{C}$ is such that $\Im z \neq 0$, then the projective flow $(\Sigma, \mathbb{R})$ is defined as $(\Sigma = \Omega \times \mathbb{P}(\mathbb{C}), \{\hat{\tau}_t\})$, where $\{\hat{\tau}_t\}$ is the skew-product flow induced by $\{\tau_t\}$ and the solutions of (SL_ω). We prove the following

PROPOSITION 4.25. *For every* $\lambda \in \mathbb{R}$,

$$\lim_{\varepsilon \to 0} \tilde{\beta}(\lambda + \mathrm{i}\varepsilon) = \beta(\lambda). \tag{4.34}$$

PROOF. Recall that we have only two possibilities for $\lambda \in \mathbb{R}$: namely, either $\beta(\lambda) = 0$ or $\beta(\lambda) < 0$.

For every z with $\Im z \neq 0$ the Weyl m-function $m_+(z)$ defines an invariant (minimal) section in Σ, hence an ergodic measure μ_z on Σ expressed as follows

$$\int_\Sigma f(\sigma)\,\mathrm{d}\mu_z = \int_\Omega f\big(\omega, m_+(\omega, z)\big)\,\mathrm{d}\nu(\omega), \tag{4.35}$$

for every continuous function $f : \Sigma \to \mathbb{R}$. It is easy to check that μ_z is indeed $\{\hat{\tau}_t\}$-ergodic and that it projects to ν. Next, for every $z \in \mathbb{C}$, we define a continuous function $f_z : \Sigma \to \mathbb{R}$ as follows:

$$f_z(\sigma) = \Re \frac{\langle [y(\omega)\left(\begin{smallmatrix} 0 & 0 \\ -z & 0 \end{smallmatrix}\right) + C(\omega)]u, u\rangle}{\langle u, u\rangle}, \tag{4.36}$$

where $\sigma = (\omega, l) \in \Sigma$, u is any nonzero vector in l, $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{C}$, and

$$C(\omega) = \begin{pmatrix} 0 & 1/p(\omega) \\ q(\omega) & 0 \end{pmatrix}.$$

For every sequence $z_n \to \lambda$, f_{z_n} converges to f_λ uniformly on Σ. Moreover, if $u(t)$ is any nonzero solution of Eq. (SL_ω), then

$$\frac{1}{t} \ln \frac{|u(t)|}{|u(0)|} = \frac{1}{t} \int_0^t f_z\big(\hat{\tau}_s(\omega, l_u)\big)\,\mathrm{d}s, \tag{4.37}$$

where l_u is the line in $\mathbb{C}$ containing 0 and $u(0)$.

From the Birkhoff ergodic theorem, (4.37), the definition of μ_z, and (4.33), we have

$$\tilde{\beta}(z) = \int_\Sigma f_z(\sigma)\,\mathrm{d}\mu_z = \int_\Omega f_z\big(\omega, m_+(\omega, z)\big)\,\mathrm{d}\nu \quad (\Im z > 0). \tag{4.38}$$

Suppose first that $\beta(\lambda)=0$. Simple applications of the Choquet theorem [131], the Oseledets theorem [8,129] and the Birkhoff ergodic theorem show that for every invariant measure μ on Σ, we have

$$\int_\Sigma f_\lambda(\sigma)\,\mathrm{d}\mu=0. \tag{4.39}$$

In fact, suppose that there exists an invariant measure μ on Σ such that

$$\int_\Sigma f_\lambda(\sigma)\,\mathrm{d}\mu<0.$$

Then we apply the Choquet theorem to the set Γ of the $\{\hat{\tau}_t\}$-invariant measures on Σ which project to ν (Γ is weak-$*$ compact and convex), to show that there exists an ergodic measure μ_1 on Σ such that $\int_\Sigma f\,\mathrm{d}\mu_1=\eta<0$. Now, use the Birkhoff ergodic theorem to find a set $\Sigma_1\subset\Sigma$ with $\mu_1(\Sigma_1)=1$, and such that

$$\lim_{|t|\to\infty}\frac{1}{t}\int_0^t f\bigl(\hat{\tau}_s(\sigma)\bigr)\,\mathrm{d}s=\eta, \quad \text{for all } \sigma\in\Sigma_1.$$

Since $\beta(\lambda)=0$, we apply the Oseledets theorem to find a set Ω_* with $\nu(\Omega_*)=1$ and such that, if θ_0 is any initial value, then

$$\lim_{|t|\to\infty}\frac{1}{t}\int_0^t f\bigl(\hat{\tau}_s(\omega,\theta_0)\bigr)\,\mathrm{d}s=0, \quad \text{for all } a\in\Omega_*.$$

Let $\Omega_1=\pi(\Sigma_1)$, and choose $\omega\in\Omega_1\cap\Omega_*$. Take points $(\omega,l_1)\in\Sigma_1$, $(\omega,l_2)\in\Sigma_2=\pi^{-1}(\Omega_*)$. Clearly $l_1\neq l_2$. If we let x_1,x_2 be unit vectors in l_1 and l_2 respectively, then x_1 and x_2 are linearly independent. Let $x_i(t)=\Phi_\omega(t)x_i$ $(i=1,2)$. Let $\Psi(t)$ be the fundamental matrix solution of (SL_ω) whose columns are $x_1(t)$ and $x_2(t)$. By Liouville's formula, we argue that $\det\Psi(t)=\det\Psi(0)=\text{const.}$ for all $t\in\mathbb{R}$. But also $\lim_{t\to-\infty}\det\Psi(t)=0$, a contradiction.

We claim now that, if $\beta(\lambda)=0$, then $\tilde{\beta}(z)$ converges to $\beta(\lambda)$ as $z\to\lambda$, and this convergence need not be nontangential, i.e. is nonrestricted. Suppose for contradiction that there exists a sequence $\{z_n\}$ of complex numbers converging to λ and such that

$$\tilde{\beta}(z_n)\to\beta^*\neq\beta(\lambda).$$

Using weak compactness of measures, we can assume that $\mu_{z_n}\rightharpoonup\mu_\lambda$, where μ_λ is invariant with respect to the flow $(\Sigma,\{\hat{\tau}_t^{(\lambda)}\})$. So we have, using Lemma 4.18,

$$\tilde{\beta}(z_n)=\int_\Sigma f_{z_n}(\sigma)\,\mathrm{d}\mu_{z_n}\to\int_\Sigma f_{z_n}(\sigma)\,\mathrm{d}\mu_\lambda\to\int_\Sigma f_\lambda(\sigma)\,\mathrm{d}\mu_\lambda=0,$$

a contradiction.

Now we consider the case when $\beta(\lambda) < 0$. From the Oseledets theorem 2.1, there is a set $\Omega_\nu \subset \Omega$ with $\nu(\Omega_\nu) = 1$ which has the following property: if $\omega \in \Omega_\nu$, there is a vector $0 \neq u \in \mathbb{R}^2$ such that

$$\lim_{t\to\infty} \ln\left|\Phi_\omega(t)u\right| = \beta(\lambda) < 0. \tag{4.40}$$

In particular, it follows that the solution $\Phi_\omega(t)u \in L^2(0,\infty)$. Using Liouville's formula, one checks that u is unique up to a constant multiple.

Let $l_\omega \in \mathbb{P}(\mathbb{R})$ be the line containing u. Then l_ω contains the unit vector $\binom{\cos\alpha}{\sin\alpha}$ for exactly one number $\alpha \in [0,\pi)$. Consider the following spectral problem:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(p(\omega\cdot t)\dfrac{\mathrm{d}\varphi}{\mathrm{d}t}\right) - q(\omega\cdot t)\varphi = -zy(\omega\cdot t)\varphi, \quad t\in[0,\infty), \\ \varphi(0) = \cos\alpha. \\ p(0)\varphi'(0) = \sin\alpha. \end{cases} \tag{$*$}$$

Note that the number λ which we are considering is an eigenvalue of this spectral problem. As stated previously, a Weyl m-function $m_+(\alpha,\omega,z)$ is associated to the spectral problem $(*)$; see Proposition 4.5 and the discussion preceding it. We have

$$m_+(\omega,z) = m_+(\pi/2,\omega,z) \quad (\Im z > 0);$$

see Remark 4.6.

Let us temporarily write

$$m_\alpha(z) = m_+(\alpha,\omega,z), \qquad m(z) = m_+(\omega,z).$$

One can check that

$$m(z) = -\frac{\cos\alpha - m_\alpha(z)\sin\alpha}{\sin\alpha + m_\alpha(z)\cos\alpha} \quad (\Im z \neq 0)$$

(see Remark 4.6 again). Now, an analogue of the Gilbert–Pearson theory is valid for the spectral problem $(*)$; in particular, there holds a statement corresponding to point (iii) in Corollary 4.12. This means that

$$m_\alpha(z) \to \infty \quad \text{when } z \to \lambda \text{ nontangentially.}$$

This in turn means that $m(z) \to \tan\alpha$ as $z \to \lambda$ nontangentially. That is, if we identify $m(z)$ with the complex line it parametrizes, we have

$$m(z) \to l_\omega \quad \text{when } z \to \lambda \text{ nontangentially.}$$

We conclude that, if $z \to \lambda$ nontangentially, then $f_z(\omega, m_+(\omega,z)) \to f_\lambda(\omega, l_\omega)$ for all $\omega \in \Omega_\nu$. Using (4.38), we obtain

$$\tilde{\beta}(z) = \int_\Omega f_z\big(\omega, m_+(\omega, z)\big)\,\mathrm{d}\nu \to \int_\Omega f_\lambda(\omega, l_\omega)\,\mathrm{d}\nu = \beta(\lambda). \qquad \square$$

Now we turn to $\Im w(z)$, and prove an analogous proposition, i.e.

PROPOSITION 4.26. *For every $\lambda \in \mathbb{R}$, we have*

$$\lim_{\varepsilon\to 0} \tilde{\alpha}(\lambda + \mathrm{i}\varepsilon) = \alpha(\lambda). \tag{4.41}$$

PROOF. Before proving this proposition, we want to give an equivalent definition of the rotation number for each complex number z with $\Im z \neq 0$. Let $\omega \in \Omega$. For $\Im z > 0$, consider a solution $u(t)$ of (4.24_ω) such that

$$\Im W[u, \bar{u}] > 0, \tag{4.42}$$

where $W[u, v] = p(uv' - uv')$ is the Wronskian of u and v. Note that $W[u, \bar{u}]$ is not constant, since $\bar{u}(t)$ is not a solution of (4.24_ω).

Let

$$h(t, z) = -\frac{1}{t} \arg \frac{u(t, z)}{u(0, z)} = -\frac{1}{t} \int_0^t \Im \frac{u'(s, z)}{u(s, z)}\,\mathrm{d}s. \tag{4.43}$$

We define the rotation number as the limit

$$\lim_{t\to\infty} h(t, z). \tag{4.44}$$

Suppose for the moment that the limit (4.44) exists and does not depend on the particular choice of the solution. Then

$$\lim_{t\to\infty} h(t, z) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{\Im m_+(s, z)}{p(s)}\,\mathrm{d}s = -\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{\Im m_-(s, z)}{p(s)}\,\mathrm{d}s = \tilde{\alpha}(z), \tag{4.45}$$

and the given definitions turn out to be equivalent. To prove the existence of the limit (4.44), and its invariance with respect to the particular solution, we may proceed as follows. Since $\Im W[u, \bar{u}] = -2p|u|^2 \Im \frac{u'}{u}$, the assumption (4.42) is equivalent to

$$\Im \frac{u'}{u} < 0, \tag{4.46}$$

hence $h(t, z) > 0$. Now, let v any other solution of (4.24_ω) satisfying (4.42). We may assume that $v(0, z) = 1 = u(0, z)$ (the assumption (4.42) retains validity if we replace u by cu, where c is a non zero constant). We claim that

$$\left| \int_0^t \Im \left(\frac{u'}{u} - \frac{v'}{v} \right) \mathrm{d}s \right| < \pi. \tag{4.47}$$

If true, this proves that if the limit exists for a particular solution, then it exists for any other solution, and does not depend on the choice of the solution. Let us prove the claim. The assumption (4.42) implies that both $\Im u'(0,z)$ and $\Im v'(0,z)$ are negative. If our claim were false, then there would exist $t^* > 0$ such that $\arg(u(t^*,z) - v(t^*,z)) = \pm\pi$, and so there would exist a positive constant c such that

$$u(t^*,z) + cv(t^*,z) = 0.$$

But $u + cv$ is a solution of (4.24_ω) satisfying (4.42), which in turn implies that $u + cv$ has no zeros, a contradiction. The existence of the limit follows easily.

Now we turn to the proof of Proposition 4.26. The function $w(z)$ is analytic in the upper complex half plane and has positive imaginary part there: hence it is an Herglotz function. From the theory of Herglotz functions [36], there is a nondecreasing function $\rho(\lambda)$ such that

$$\Im w(z) = \Im z \int_{-\infty}^{\infty} \frac{\mathrm{d}\rho(\lambda)}{|\lambda - z|^2} + c,$$

where c is a constant (it is well known that the asymptotic behavior of m_+ as $\Im z \to \infty$ ensures that $c = 0$). In addition the following relation holds

$$\rho(\lambda_2) - \rho(\lambda_1) = \frac{1}{\pi} \lim_{\varepsilon\to 0} \int_{\lambda_1}^{\lambda_2} \Im w(\lambda + \mathrm{i}\varepsilon)\,\mathrm{d}\lambda$$

at all continuity points of ρ. If $\{w_n(z)\}$ is a sequence of analytic functions with $\Im w_n(z) > 0$ if $\Im z > 0$, and if $w_n(z) \to w(z)$ pointwise, then we have that the corresponding sequence of measures $\{\mathrm{d}\rho_n(\lambda)\}$ weakly converges to $\mathrm{d}\rho(\lambda)$ (with abuse of notation).

Let $\Im z > 0$, and put

$$h(t,z) = -\frac{1}{t} \int_0^t \Im \frac{\varphi_+'(s,z)}{\varphi_+(s,z)}\,\mathrm{d}s.$$

For every fixed $t \in \mathbb{R}$, the map $z \mapsto h(t,z)$ is analytic in the upper complex half plane, has positive imaginary part, and pointwise converges to $\Im w(z)$ as $t \to \infty$. It follows that the corresponding densities define measures which are weakly convergent to the measure $\mathrm{d}\rho(\lambda)$. If we denote these densities by $\rho(t,\lambda)$ we obtain easily

$$\rho(t,\lambda_2) - \rho(t,\lambda_1) = \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} h(t,\lambda)\,\mathrm{d}\lambda,$$

since $h(t,\lambda)$ is continuous for $\Im z \geqslant 0$. Using the fact that, as $t \to \infty$, $\rho(t,\lambda)$ converges to $\rho(\lambda)$ at all points λ of continuity of ρ, we can write

$$\rho(\lambda_2) - \rho(\lambda_1) = \frac{1}{\pi} \lim_{t\to\infty} \int_{\lambda_1}^{\lambda_2} h(t,\lambda)\,\mathrm{d}\lambda$$

at all continuity points λ_1, λ_2 of ρ.

Recall that if $\lambda \in \mathbb{R}$, then $h(t,\lambda) \to \alpha(\lambda)$ as $t \to \infty$ (this is an easy consequence of Corollary 4.12). Using Sturm's comparison theorem we can also prove that this convergence is bounded on $[\lambda_1, \lambda_2]$, hence we can write

$$\rho(\lambda_2) - \rho(\lambda_1) = \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \alpha(\lambda)\, \mathrm{d}\lambda.$$

From these facts we obtain

$$\Im w(z) = \tilde{\alpha}(z) = \frac{\Im z}{\pi} \int_{\mathbb{R}} \frac{\alpha(\lambda)\, \mathrm{d}\lambda}{|\lambda - z|^2}, \quad \Im z > 0. \tag{4.48}$$

Next we put $z = \eta + \mathrm{i}\varepsilon$ and compute the limit for $\varepsilon \to 0$: we have, using standard arguments,

$$\lim_{\varepsilon \to 0} \tilde{\alpha}(\eta + \mathrm{i}\varepsilon) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \frac{\alpha(\lambda)\, \mathrm{d}\lambda}{(\lambda - \eta)^2 + \varepsilon^2} = \alpha(\eta)$$

for all $\eta \in \mathbb{R}$. □

REMARK 4.27. Using (4.48), we can obtain the following representation

$$\frac{w(z) - w(z_0)}{z - z_0} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\alpha(\lambda)\, \mathrm{d}\lambda}{(\lambda - z_0)(\lambda - z)} \quad (\Im z, \Im z_0 > 0), \tag{4.49}$$

which allows us to determine $w(z)$ up to an additive constant if $\alpha(\lambda)$ is known.

Now we go deeper in the examination of the properties of the sets $\Sigma(L)$ and $R(L)$. In particular, we will retrieve some useful information about the existence of isolated eigenvalues for the full-line operator, as well for the half-line operators. We will prove that the assumptions on Ω and on the functions p, q, y ensure that the spectrum $\Sigma(L)$ of the family (SL_ω) has no isolated points. Moreover, at most one of the Weyl m-functions can have a pole in a given open interval $J \subset R(L)$; that is, if $J \subset R(L)$ is an open interval, then $m_+(\omega,\lambda)$ and $m_-(\omega,\lambda)$ have at most one pole $P(\omega) \in J$, and such a point $P(\omega)$ satisfies $\lim_{\varepsilon \to 0} |\Im m_+(\omega, P(\omega) + \mathrm{i}\varepsilon)| = \infty$ and $\lim_{\varepsilon \to 0} |m_-(\omega, P(\omega) + \mathrm{i}\varepsilon)| < \infty$, or vice-versa.

Let $\omega \in \Omega$, and define $\mathrm{RE}^{\pm}(L_\omega) = R^{\pm}(L_\omega) \cup \{\lambda \in \Sigma(L) \mid \lambda \text{is an isolated point}\}$. We begin with the following proposition, whose proof is straightforward.

PROPOSITION 4.28. *Let $\omega \in \Omega$. If $J \subset R(L)$ is any open interval, then both $m_+(\omega,\cdot)$ and $m_-(\omega,\cdot)$ extend meromorphically through J. In particular $m_\pm(\omega,\cdot)$ has a pole at every point $\lambda_0 \in J \cap \{\mathrm{RE}^{\pm}(L_\omega) \setminus R^{\pm}(L_\omega)\}$.*

REMARK 4.29. For every $\lambda \in J \subset R(L)$, $m_+(\omega,\lambda)$ and $m_-(\omega,\lambda)$ are elements of $\mathbb{P}(\mathbb{R})$. Moreover, the maps $\lambda \mapsto m_\pm(\omega,\lambda)$ rotate without rest in opposite directions in $\mathbb{P}(\mathbb{R})$ as λ increases in J. In particular, $m_+(\omega,\cdot)$ rotates counterclockwise, and $m_-(\omega,\cdot)$ rotates

clockwise. In fact, this is a simple restatement of the following elementary facts: (i) for $\Im\lambda = 0$, $\mathrm{d}m_-/\mathrm{d}\lambda < 0$ if $m_- \neq \infty$, and $\mathrm{d}(1/m_-)/\mathrm{d}\lambda > 0$ for $m_- \neq 0$; (ii) for $\Im\lambda = 0$, $\mathrm{d}m_+/\mathrm{d}\lambda > 0$ if $m_+ \neq \infty$, and $\mathrm{d}(1/m_+)/\mathrm{d}\lambda < 0$ for $m_+ \neq 0$.

We now consider point (ii) of Theorem 4.21.

PROPOSITION 4.30. *The spectrum $\Sigma(L)$ of the family* (SL_ω) *does not contain any isolated point.*

PROOF. We argue by contradiction. Suppose that $\lambda_0 \in \Sigma(L)$ is an isolated point. Let $D \subset \mathbb{C}$ be a disc centered in λ_0 such that $D \cap \Sigma(L) = \{\lambda_0\}$. Then $D \subset \mathrm{RE}^{\pm}(L_\omega)$ for every $\omega \in \Omega$, and both $m_+(\omega, \cdot)$ and $m_-(\omega, \cdot)$ extend meromorphically through $D \cap \mathbb{R} = J$, for ν a.a. $\omega \in \Omega$. Set $m(\omega) = m_+(\omega, \lambda_0)$. Consider the set $C = \{\omega \in \Omega \mid L_\omega \text{has no eigenvalue at} \lambda_0\}$. We claim that C is closed. To prove the claim, we can argue as follows: first, we note that the set C is $\{\tau_t\}$-invariant. Let $\bar{\omega} \in C$, and let $\{t_n\} \subset \mathbb{R}$ be a sequence such that $t_n \to \infty$ as $n \to \infty$. Let $\omega_0 = \lim_{n\to\infty} \bar{\omega} \cdot t_n$. Let $P_n(\bar{\omega})$ denote the spectral matrix corresponding to $\bar{\omega} \cdot t_n$, for every $n \in \mathbb{N}$. Since λ_0 is an isolated point, then the trace $\operatorname{Tr} P_n(\bar{\omega})$ is constant in an open interval I_0 centered at λ_0, for every $n \in \mathbb{N}$. Since $\operatorname{Tr} P_n(\bar{\omega})$ weakly converges to $\operatorname{Tr} P(\omega_0)$, it follows that $\operatorname{Tr} P(\omega_0)$ is constant on I_0 as well, and hence $\omega_0 \in C$ (because the points of growth of $\operatorname{Tr} P(\omega_0)$ coincide with the points in the spectrum of the operator L_{ω_0}). We have proved the claim.

The complement set $\Omega_1 = \Omega \setminus C$ is open and dense, since, if $\omega \in \Omega_1$ has dense orbit (such a point exists, since ν is ergodic), then $\omega \cdot t \in \Omega_1$, for all $t \in \mathbb{R}$. Consider the function $h_n : \Omega \to \mathbb{P}(\mathbb{R}) : \omega \mapsto m_+(\omega, \lambda_0 + 1/n)$. It follows that $m(\omega) = \lim_{n\to\infty} h_n(\omega)$ is a pointwise limit of continuous functions, hence it has a residual set Ω_2 of continuity points. It is easy to check that Ω_2 is $\{\tau_t\}$-invariant. Let Ω_3 be the set of all $\omega \in \Omega$ with dense orbit in Ω. Then Ω_3 is a residual set as well. Choose a point $\bar{\omega} \in \Omega_1 \cap \Omega_2 \cap \Omega_3$. Then λ_0 is an isolated eigenvalue of $L_{\bar{\omega}}$, the function $m(\omega)$ is continuous at $\bar{\omega}$, and $\{\omega \cdot t \mid t \in \mathbb{R}\}$ is dense in Ω. Let Σ_1 be the closure of the set

$$\big\{(\bar{\omega} \cdot t, m(\bar{\omega} \cdot t)) \mid t \in \mathbb{R}\big\} \subset \Omega \times \mathbb{P}(\mathbb{R}) \subset \Omega \times \mathbb{P}(\mathbb{C}) = \Sigma.$$

As we already noted, the flow $(\Sigma, \{\hat{\tau}_t\})$ induces a flow $\{\hat{\tau}_t\}$ in $\Omega \times \mathbb{P}(\mathbb{R})$ (abusing the notation slightly). It follows that Σ_1 is compact and $\{\hat{\tau}_t\}$-invariant . Moreover, it is not difficult to show that Σ_1 contains a residual set of points $\sigma = (\omega, l)$ such that $\{\hat{\tau}_t(\sigma) \mid t \in \mathbb{R}\}$ is dense in Σ_1; in fact, Σ_1 is the α-limit set of the point $(\bar{\omega}, m(\bar{\omega}))$. Let $\sigma_0 = (\omega, l_0)$ be such a point. Let $\Sigma_2 = \Sigma_1 \cap \{\Omega_1 \times \mathbb{P}(\mathbb{R})\}$. Then Σ_2 is an open, dense and $\{\hat{\tau}_t\}$-invariant subset of Σ_1. Let $\sigma = (\omega, l) \in \Sigma_2$ and let $f_{\lambda_0}(\sigma)$ as in Eq. (4.36). Let u_0 be a vector of norm 1 which is contained in l. Then

$$\ln\big|u(t)\big| = \int_0^t f_{\lambda_0}\big(\hat{\tau}_s(\sigma)\big)\,\mathrm{d}s \to -\infty$$

as $t \to \pm\infty$. By the Baire category theorem, there exists an integer N and a set $\Sigma_3 \subset \Sigma_2$ such that for every $\sigma \in \Sigma_3$,

$$\int_0^t f_{\lambda_0}\big(\hat{\tau}(\sigma)\big)\,\mathrm{d}s \leqslant N$$

for every large $t \in \mathbb{R}$. But then $\hat{\tau}_t(\sigma_0) \in \Sigma_3$ for every large t, a contradiction. $\square$

The following result is of independent interest.

PROPOSITION 4.31. *For ν a.a $\omega \in \Omega$, the following relations hold*:

$$\mathrm{RE}^+(L_\omega) = \mathrm{RE}^-(L_\omega) = R(L) = R^{\mathrm{ed}}(L).$$

PROOF. According to Proposition 4.13, there is a set $\Omega_\nu \subset \Omega$ with $\nu(\Omega_\nu) = 1$ such that, if $\omega \in \Omega_\nu$, then ω has dense orbit and is positive and negative Poisson recurrent in the sense that there are sequences $\{t_n\} \to \infty$ and $\{s_n\} \to -\infty$ such that $\lim_{n\to\infty} \omega \cdot t_n = \lim_{n\to\infty} \omega \cdot s_n = \omega$.

Let $T > 0$, and consider Eq. (4.24$_\omega$) on the interval $[0, T]$ together with the boundary conditions $\varphi(0) = \varphi(T) = 0$. There is a corresponding self-adjoint differential operator L_ω^T on the weighted space $L^2[0, T]$ with density $y(t)$. Let $J = (\lambda', \lambda'') \subset \mathbb{R}$ be an open interval containing exactly $k \geqslant 0$ eigenvalues of the half-line operator L_ω^+. We will use a beautiful argument of Hartman [66] to show that J contains no more than $k + 1$ eigenvalues of L_ω^T.

To prove this statement, let $\varphi(t, \lambda_1), \dots, \varphi(t, \lambda_k)$ be eigenfunctions of L_ω^+ corresponding to the eigenvalues $\lambda_1, \dots, \lambda_k \in J$. Suppose for contradiction that L_ω^T admits $N \geqslant k + 2$ eigenvalues in J with corresponding eigenfunctions $\varphi_1(t), \dots, \varphi_N(t)$; thus $\varphi_j(0) = \varphi_j(T) = 0$ $(1 \leqslant j \leqslant N)$. Write

$$z(t) = \sum_{j=1}^{N} c_j \varphi_j(t) \quad (0 \leqslant t \leqslant T), \tag{4.50}$$

then determine coefficients $c_1, \dots, c_N$, not all zero, such that

$$\begin{aligned} &z'(T) = 0,\\ &\int_0^T z(t)\varphi(t, \lambda_j) y(t)\,\mathrm{d}t = 0 \quad (1 \leqslant j \leqslant k). \end{aligned} \tag{4.51}$$

This is possible because $N \geqslant k + 2$, while (4.51) consists of only $k + 1$ linear relations.

Next extend $z(t)$ to all of $[0, \infty)$ by setting $z(t) = 0$ for $t \geqslant T$. Then z is in the domain of L_ω^+. Let $\bar{\lambda} = \frac{\lambda' + \lambda''}{2}$, and set $w = L_\omega^+(z) - \bar{\lambda} z$. Using (4.50) together with the spectral theorem for self-adjoint operators, we see that

$$\int_0^\infty w^2(t) y(t)\,\mathrm{d}t \geqslant \frac{(\lambda'' - \lambda')^2}{4} \int_0^\infty z^2(t) y(t)\,\mathrm{d}t.$$

On the other hand, condition (4.50) together with the fact that $L_\omega^T(z) = L_\omega^+(z)$ for $0 \leqslant t < T$ imply that

$$\int_0^\infty w^2(t)y(t)\,\mathrm{d}t = \int_0^T w^2(t)y(t)\,\mathrm{d}t < \frac{(\lambda'' - \lambda')^2}{4}\int_0^T z^2(t)y(t)\,\mathrm{d}t$$
$$= \frac{(\lambda'' - \lambda')^2}{4}\int_0^\infty z^2(t)y(t)\,\mathrm{d}t.$$

These relations are inconsistent, so $N \leqslant k+1$ as stated.

Continuing, we claim that J contains no more than $2k+2$ eigenvalues of $L^+_{\omega\cdot T}$. It is easiest to prove this by passing to the $u = \binom{u_1}{u_2}$-plane. The proof is by contradiction. Let $u(t,\lambda) = \binom{u_1(t,\lambda)}{u_2(t,\lambda)}$ satisfy (SL_ω) and $u_1(0,\lambda) = 0, u_2(0,\lambda) = 1$ $(\lambda \in J, 0 \leqslant t \leqslant T)$. Furthermore, write $m(t,\lambda) = m_+(\omega\cdot t,\lambda)$ $(\lambda \in J, 0 \leqslant t \leqslant T)$. Let us parametrize $u(t,\lambda)$ in a continuous way using the polar coordinate θ in the u-plane, then identify $u(t,\lambda)$ with its θ-value; we carry out the same operation on $m(t,\lambda)$ $(\lambda \in J, 0 \leqslant t \leqslant T)$. It is clear that the interval $(m(T,\lambda'), m(T,\lambda''))$ contains $2k+3$ consecutive numbers of the form $\frac{\pi}{2} + j\pi$, $j \in \mathbb{Z}$. A moment's thought shows that the interval $(u(T,\lambda'), u(T,\lambda''))$ contains $k+2$ consecutive numbers of the form $\frac{\pi}{2} + l\pi$, $l \in \mathbb{Z}$. But then L_ω^T has at least $k+2$ eigenvalues in J, contradicting the previous result. So in fact J contains non more than $2k+2$ eigenvalues of $L^+_{\omega\cdot T}$, for each $T > 0$.

Now, let $\omega_1 \in \Omega$, and let $\{t_n\} \subset \mathbb{R}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \omega\cdot t_n = \omega_1$. Let $\rho_n^+ := \rho^+_{\omega\cdot t_n}$ be the spectral measure corresponding to the half-line operator $L^+_{\omega\cdot t_n}$. Then ρ_n^+ weakly converges to ρ^+, where ρ^+ is the spectral measure associated to $L^+_{\omega_1}$. The eigenvalues of $L^+_{\omega_1}$ correspond to the discontinuity points of ρ^+. It follows that ρ^+ has no more than $2k+3$ discontinuity points, because the previous remarks apply to each t_n.

Now, consider Eq. $(4.24_{\omega\cdot(-T)})$ in the interval $[0,T]$, together with the boundary conditions $u(0) = u(T) = 0$ $(T > 0)$. It follows again that J contains no more than $2k+3$ eigenvalues of the operator $L^+_{\omega\cdot(-T)}$. If we consider Eq. (4.24_ω) with the boundary conditions

$$u(0) = u(-T) = 0$$

then the number of eigenvalues in J of the operator L_ω^{-T} is equal to the number of eigenvalues, always in J, of the operator $L^T_{\omega\cdot(-T)}$, hence it is $\leqslant 2k+3$.

Let $\{s_n\} \subset \mathbb{R}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \omega\cdot(-s_n) = \omega_1$. For every $n \in \mathbb{N}$, consider the spectral measure, which we still denote by ρ_n^+, corresponding to Eq. $(4.24_{\omega\cdot(-s_n)})$ in the interval $[-s_n, 0]$, together with the boundary conditions $u(0) = u(-s_n) = 0$. Recall that ρ_n^+ is piecewise constant, with a jump of discontinuity equal to

$$\left(\int_{-s_n}^0 y(t)\|\chi_i(t)\|^2\,\mathrm{d}t\right)^{-1}$$

at every eigenvalue λ_i of Eq. (4.24$_{\omega\cdot(-s_n)}$), where $\chi_i(t)$ is the normalized eigenfunction corresponding to the eigenvalue λ_i. It can be shown that ρ_n^+ converges weakly to $\rho_{\omega_1}^-$, where $\rho_{\omega_1}^-$ is the spectral measure for the negative half-line operator $L_{\omega_1}^-$. As before, we can prove that $\rho_{\omega_1}^-$ has at most $2k+3$ discontinuity points in J. We conclude that both $m_+(\omega,\cdot)$ and $m_-(\omega,\cdot)$ extend meromorphically through J. It follows that the spectral matrix M of the full line operator extends meromorphically through J, hence the full line operator L_ω has at most isolated eigenvalues on J. However, the spectrum of L_ω contains no isolated eigenvalues and in fact $R(L_\omega) = R(L)$, which in turn equals $R^{\mathrm{ed}}(L)$ by Proposition 4.15. All this implies that $R^+(L_\omega) \cup R^-(L_\omega) \subset R^{\mathrm{ed}}(L)$. Since $R^{\mathrm{ed}}(L) \subset R^+(L_\omega) \cup R^-(L_\omega)$, we have proved Proposition 4.31. $\square$

Let us note that the statement of Proposition 4.31 holds on the orbit closure of any point ω which is positive and negative Poisson recurrent.

The final result concerning the structure of the spectrum is stated in the following

THEOREM 4.32. *Let $\omega \in \Omega$ have dense orbit, and let J be an open interval with $J \subset \mathrm{RE}^+(L_\omega)$. Then the half-line operator L_ω^+ has at most one eigenvalue in J. If L_ω^+ has an eigenvalue in J, then L_ω^- has no eigenvalue in J, and vice-versa. In particular, if $m_+(\omega,\cdot)$ has a pole $\lambda_0 \in J$, then λ_0 is a regular point for $m_-(\omega,\cdot)$ and vice-versa.*

PROOF. By some results of Coppel [29], the bundles Ker P_ω and Im P_ω vary continuously with respect to λ. Since $J \subset \mathrm{RE}^+(L_\omega)$, we have that $m_+(\omega,\lambda) \neq m_-(\omega,\lambda)$ for all $\lambda \in J$. Moreover, the above observations ensure that only one of $m_+(\omega,\cdot)$ and $m_-(\omega,\cdot)$ can attain the value ∞, and such a value can be attained at most once. This concludes the proof, since, from Theorem 4.11, $m_+(\omega,\cdot)$ and $m_-(\omega,\cdot)$ attain the values ∞ only at those points $\lambda \in J$ which are isolated eigenvalues for L_ω^+ and L_ω^- respectively. $\square$

Our next goal is to complete the proof of point (iii) of Theorem 4.21. We need a formula for the z-derivative of $w(z)$ which involves the potential $y(\omega)$ and the Weyl m-functions as well. This formula will turn to be very useful also in the following, when we will discuss a theory of Kotani-type for the general Sturm–Liouville operator. In [59,31,89], and also in [81], one can find very interesting (trace) formulas for $\mathrm{d}w/\mathrm{d}z$ and its representation in terms of the trace of the spectral matrix $P(\lambda)$ arising from the Green (matrix) function. In our case, it is not very difficult to compute directly such a derivative. Let $\mathcal{L} = \{f : \Omega \to \mathbb{R} \mid f$ is continuous and strictly positive$\}$. Then $\mathcal{L}$ can be regarded as an open subset of the Banach space $C(\Omega)$ of continuous real-valued functions on Ω. By a direct observation of the Riccati equation (4.25), letting $\tilde{z} = zy$, the Floquet exponent $w(\cdot)$ can be considered as a functional from $\mathcal{L}$ into $\mathbb{C}$, as follows: $w(z,\cdot) : \mathcal{L} \to \mathbb{C} : y \mapsto w(zy)$, for each fixed $z \in \mathbb{C}$. We now adapt to our Sturm–Liouville setting the developments in [89].

PROPOSITION 4.33. *For every fixed $z \in \mathbb{C} \setminus \mathbb{R}$, the functional $w(z,\cdot)$ defined above admits a Frechet derivative $D_f w(z,\cdot)$ along every direction $f \in \mathcal{L}$. Moreover, for every $z \in \mathbb{C}$ with $\Im z \neq 0$, and for every ergodic measure ν on Ω, we have*

$$\frac{\mathrm{d}}{\mathrm{d}z} w(z) = \int_\Omega \frac{y(\omega)}{m_-(\omega,z) - m_+(\omega,z)}\, \mathrm{d}\nu(\omega). \tag{4.52}$$

PROOF. Let $z \in \mathbb{C} \setminus \mathbb{R}$. Consider the functionals $m_\pm : \mathcal{L} \to \mathbb{C} : y \mapsto m_\pm(z, y)$, and suppose for the moment that $m_\pm$ are Frechet differentiable at $y \in \mathcal{L}$. Let $f \in \mathcal{L}$ and denote by $D_f m_\pm(z, y)$ the Frechet derivatives of $m_\pm(z, \cdot)$ computed at $y \in \mathcal{L}$ in the direction $f \in \mathcal{L}$. Then we have

$$D_f w(z, y) = \int_\Omega \frac{D_f m_+(z, y(\omega)) - D_f m_-(z, y(\omega))}{2p(\omega)} \, \mathrm{d}\nu, \tag{4.53}$$

where ν is a fixed ergodic measure on Ω. This shows that w is Frechet differentiable at $y \in \mathcal{L}$.

We compute $D_f m_\pm(z, y)$. From the Riccati equation (4.25) it follows that $D_f m_\pm(z, y)$ must satisfy

$$\big(D_f m_\pm(z, y)\big)' + \frac{2}{p} m_\pm D_f m_\pm(z, y) = -zf. \tag{4.54}$$

The solutions $\hat{D}_f m_\pm(z, y)$ of these first order differential equations are

$$\begin{aligned} \hat{D}_f m_-(z, y) &= \int_{-\infty}^t -zf(s) \exp\left\{2 \int_t^s \frac{m_-}{p} \, \mathrm{d}r\right\} \mathrm{d}s, \\ \hat{D}_f m_+(z, y) &= \int_t^\infty zf(s) \exp\left\{2 \int_t^s \frac{m_+}{p} \, \mathrm{d}r\right\} \mathrm{d}s \end{aligned} \tag{4.55}$$

hence

$$\hat{D}_f m_-(z, y) = \int_{-\infty}^t -zf(s) \frac{\varphi_-^2(s)}{\varphi_-^2(t)} \, \mathrm{d}s \tag{4.56}$$

and

$$\hat{D}_f m_+(z, y) = \int_t^\infty zf(s) \frac{\varphi_+^2(s)}{\varphi_+^2(t)} \, \mathrm{d}s. \tag{4.57}$$

Now we show that $D_f m_\pm(z, y)$ actually exist and equal the solutions $\hat{D}_f m_\pm(z, y)$ defined in (4.56) and (4.57). We prove this only for m_+. Let us simplify the notation and omit the dependence with respect to $z \in \mathbb{C} \setminus \mathbb{R}$. Let $\Delta m_+ = m_+(y + \varepsilon f) - m_+(y)$. Then the Riccati equation (4.25) ensures that

$$(\Delta m_+)' + \frac{1}{p}\big(m_+(y + \varepsilon f) + m_+(y)\big)\Delta m_+ = -z\varepsilon f. \tag{4.58}$$

Since $\Im z \neq 0$, the family (SL_ω) admits an exponential dichotomy, and an estimate of Coppel ([29], page 34) ensures that $m_+(y + \varepsilon f) = m_+(y) + \mathcal{O}(\varepsilon f)$. It follows that Eq. (4.58) admits a unique bounded solution which (obviously) must be Δm_+. Moreover

$$\Delta m_+(t) = \int_t^\infty z\varepsilon f(s) \exp\left\{\int_t^s \frac{2m_+(y) + \mathcal{O}(\varepsilon f)}{p} \, \mathrm{d}r\right\} \mathrm{d}s.$$

It is now easy to see that

$$\lim_{\varepsilon\to 0}\frac{|\Delta m_+ - \varepsilon\hat{D}_f m_+(z,y)|}{\varepsilon}=0,$$

where the limit is appropriately uniform in $f\in\mathcal{L}$. This shows that $D_f m_+(z,y)$ exists and equals $\hat{D}_f m_+(z,y)$ for every $f\in\mathcal{L}$. The same holds for m_-.

It remains to prove the formula (4.52). Let us consider the Frechet derivatives $D_y m_\pm(z,y)$ along the direction y. By a direct computation, we get

$$\begin{aligned}D_y w(z,y) &= \lim_{\varepsilon\to 0}\frac{w(z(y+\varepsilon y))-w(zy)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{w(zy(1+\varepsilon))-w(zy)}{\varepsilon}\\ &= z\frac{\mathrm{d}}{\mathrm{d}z}w(z,y).\end{aligned}$$

We calculate $D_y w(z,y)$: we have

$$\begin{aligned}&\frac{D_y m_+(z,y)-D_y m_-(z,y)}{p}\\ &\quad=\frac{z}{p}\left[\left(\int_t^\infty\frac{\varphi_+^2(s)}{\varphi_+^2(t)}y(s)\,\mathrm{d}s+\int_{-\infty}^t\frac{\varphi_-^2(s)}{\varphi_-^2(t)}y(s)\,\mathrm{d}s\right)\right].\end{aligned}\tag{4.59}$$

Next we observe that

$$\left(\frac{\varphi_+}{\varphi_-}\right)'=\frac{\varphi_+'\varphi_- - \varphi_+\varphi_-'}{\varphi_-^2}=-\frac{W}{p\varphi_-^2}\quad\text{and}\quad\left(\frac{\varphi_-}{\varphi_+}\right)'=\frac{\varphi_-'\varphi_+ - \varphi_-\varphi_+'}{\varphi_+^2}=\frac{W}{p\varphi_+^2}$$

where W is the Wronskian of φ_+ and φ_-. We obtain

$$\begin{aligned}&\frac{D_y m_+(z,y)-D_y m_-(z,y)}{2p}\\ &\quad=\frac{z}{2W}\left[\left(\frac{\varphi_-}{\varphi_+}\right)'\int_t^\infty\varphi_+^2(s)y(s)\,\mathrm{d}s-\left(\frac{\varphi_+}{\varphi_-}\right)'\int_{-\infty}^t\varphi_-^2(s)y(s)\,\mathrm{d}s\right].\end{aligned}\tag{4.60}$$

It follows that

$$\begin{aligned}D_y w(z,y)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{z}{2W}\Bigg[\left(\frac{\varphi_-}{\varphi_+}\right)'\int_t^\infty\varphi_+^2(s)y(s)\,\mathrm{d}s\\ -\left(\frac{\varphi_+}{\varphi_-}\right)'\int_{-\infty}^t\varphi_-^2(s)y(s)\,\mathrm{d}s\Bigg]\mathrm{d}t.\end{aligned}\tag{4.61}$$

To make the computation easier, we add in the integral the derivative with respect to t of the function

$$h(t) = \frac{z}{2W}\left[\left(-\frac{\varphi_-}{\varphi_+}\right)\int_t^\infty \varphi_+^2(s)y(s)\,\mathrm{d}s + \left(\frac{\varphi_+}{\varphi_-}\right)\int_{-\infty}^t \varphi_-^2(s)y(s)\,\mathrm{d}s\right]. \tag{4.62}$$

Note that $h'(t)$ gives no contribution to the integral (4.61). We obtain

$$\begin{aligned}
&\lim_{T\to\infty}\frac{z}{T}\int_0^T \frac{1}{2W}\left[\left(\frac{\varphi_-}{\varphi_+}\right)'\int_t^\infty \varphi_+^2(s)y(s)\,\mathrm{d}s - \left(\frac{\varphi_+}{\varphi_-}\right)'\int_{-\infty}^t \varphi_-^2(s)\,\mathrm{d}s\right]\mathrm{d}t\\
&\quad= \lim_{T\to\infty}\frac{z}{T}\int_0^T \frac{1}{2W}\left[\left(\frac{\varphi_-}{\varphi_+}\right)'\int_t^\infty \varphi_+^2(s)y(s)\,\mathrm{d}s\right.\\
&\qquad\left. - \left(\frac{\varphi_+}{\varphi_-}\right)'\int_{-\infty}^t \varphi_-^2(s)y(s)\,\mathrm{d}s + h'(s)\right]\mathrm{d}t\\
&\quad= \lim_{T\to\infty}\frac{z}{T}\int_0^T \frac{y(t)(\varphi_+(t)+\varphi_-(t))}{2(m_-(\omega\cdot t,z)-m_+(\omega\cdot t,z))}\,\mathrm{d}t.
\end{aligned} \tag{4.63}$$

Using the Birkhoff ergodic theorem we have

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \int_\Omega \frac{y(\omega)(\varphi_+(0)+\varphi_-(0))}{2(m_-(\omega,z)-m_+(\omega,z))}\,\mathrm{d}\nu(\omega) = \int_\Omega \frac{y(\omega)}{m_-(\omega,z)-m_+(\omega,z)}\,\mathrm{d}\nu(\omega)$$

for every $\{\tau_t\}$-ergodic measure ν on Ω. The proof is complete. □

We can now return to point (iii) of Theorem 4.21. Arguing as in [89, Section 6], and using our formula (4.52) for $\frac{\mathrm{d}w}{\mathrm{d}z}$ together with (4.49), we can show that, if $\alpha(\lambda)$ is constant in an interval $I \subset \mathbb{R}$, then $I \subset R(L)$. We omit the details as they are similar to those in the cited paper [89].

In the rest of Section 4.3 we discuss a theory of Kotani-type for the Sturm–Liouville operator. We begin with the following lemmas.

LEMMA 4.34. *If $\Im z \neq 0$, then*

$$\Re w(z) = \mp\frac{\Im z}{2}\int_\Omega \frac{y(\omega)}{\Im m_\pm(\omega,z)}\,\mathrm{d}\nu. \tag{4.64}$$

In particular,

$$\Re w(z) \leqslant 0.$$

PROOF. We consider the imaginary parts in the Riccati equation for m_+, obtaining

$$\Im m_+' + \frac{2}{p}\Re m_+ \Im m_+ = -y\Im z.$$

Now, dividing by $\Im m_+$ on both sides of the equation, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln(\Im m_+) + \frac{2}{p}\Re m_+ = -\frac{y\Im z}{\Im m_+},$$

and hence the proof follows integrating both sides on Ω with respect to the ergodic measure ν, and using the Birkhoff ergodic theorem. □

Let us denote by $H^{\pm}$ the upper and lower complex half-planes respectively.

LEMMA 4.35. *Let $I \subset \mathbb{R}$ be an open interval and let $f : H^+ \to H^+$ be analytic.*

Suppose that $\lim_{\varepsilon\to 0^+}\Re(f(\lambda + \mathrm{i}\varepsilon)) = 0$ for Lebesgue a.a. $\lambda \in I$. Then f extends analytically through I, and its extension $\bar{f}$ has positive imaginary part on I.

Lemma 4.35 is a standard result from the theory of complex functions, hence we will not give the proof. Instead we refer to a standard text such as [64] for more details.

Recall that $\lim_{\varepsilon\to 0}\Re w(\lambda + \mathrm{i}\varepsilon) = \beta(\lambda)$ and $\lim_{\varepsilon\to 0}\Im w(\lambda+\mathrm{i}\varepsilon) = \alpha(\lambda)$, for every $\lambda \in \mathbb{R}$. Motivated by Kotani's seminal work on the Schrödinger operator [101], and using the above formulas (4.64) and (4.52), we obtain, after omitting to write the dependence with respect to z of the functions involved:

$$\begin{aligned}
&-4\left(\frac{\Re w}{\Im z} + \Im\frac{\mathrm{d}w}{\mathrm{d}z}\right)\\
&\quad = \int_\Omega \frac{y(\omega)(\Im m_-(\omega) - \Im m_+(\omega))}{\Im m_-(\omega)\Im m_+(\omega)}\left(1 + 4\frac{\Im m_-(\omega)\Im m_+(\omega)}{|m_-(\omega) - m_+(\omega)|^2}\right)\mathrm{d}\nu\\
&\quad = \int_\Omega y(\omega)\left(\frac{1}{\Im m_+(\omega)} - \frac{1}{\Im m_-(\omega)}\right)\\
&\qquad \times\left(\frac{[\Re(m_-(\omega) - m_+(\omega))]^2 + [\Im(m_-(\omega) + m_+(\omega))]^2}{|m_-(\omega) - m_+(\omega)|^2}\right)\mathrm{d}\nu.
\end{aligned}$$

Let $I \subset \mathbb{R}$ be an open interval such that $\beta(\lambda) = 0$ for Lebesgue a.a. $\lambda \in I$. Note that the function $-w(\cdot)$ is analytic in the upper half-plane H^+ and has positive real part there. In addition $-w(\cdot)$ is bounded in every compact subset of H^+. We apply the Schwarz reflection principle [136], to prove that $w(\cdot)$ admits an analytic extension from H^+ through I to the set $H = I \cup \{\lambda \in \mathbb{C} \mid \Im\lambda \neq 0\}$. Since $\beta(\lambda) = 0$ for Lebesgue a.a. $\lambda \in I$, we have $\Re w(\lambda) = 0$. It follows that

$$\Im\frac{\mathrm{d}w}{\mathrm{d}\lambda}(\lambda) = -\lim_{\varepsilon\to 0^+}\frac{\Re w(\lambda + \mathrm{i}\varepsilon)}{\varepsilon}.$$

Since the limits $\lim_{\varepsilon\to 0^+} m_\pm(\omega, \lambda + \mathrm{i}\varepsilon) := m_\pm(\omega,\lambda)$ exist for Lebesgue a.a. $\lambda \in I$ and for ν a.a. $\omega \in \Omega$, we can use Fatou's Lemma and the computation above to obtain that $\Re m_+(\omega,\lambda) = \Re m_-(\omega,\lambda)$ and $\Im m_+(\omega,\lambda) = -\Im m_-(\omega,\lambda)$, for Lebesgue a.a. $\lambda \in I$ and

for ν a.a. $\omega \in \Omega$. We claim that the maps $(m_+ - m_-)(\omega, \cdot)$ and $\frac{m_- m_+}{m_- - m_+}(\omega, \cdot)$ extend analytically through I to H. To prove the claim, we first describe a general fact concerning functions which are analytic in H^+. Let $f(z)$ be an analytic function on H^+ with $\Im f(z) > 0$ for all $z \in H^+$. Put

$$F(z) = \ln f(z) = \ln\bigl|f(z)\bigr| + \mathrm{i} \arg f(z) = u(z) + \mathrm{i}v(z),$$

where $0 \leqslant \arg f(z) < \pi$. It follows that $u(z)$ is harmonic in H^+, and that $v(z)$ is harmonic and bounded in H^+. For $F(z)$ a generalized version of the Schwarz reflection principle holds [64]: namely, if $\Im F(\lambda) = 0$ for Lebesgue a.a $\lambda \in I$, then $F(z)$ extends analytically through I from $H \cap H^+$ to $H \cap H^-$. In our case, it follows that for ν-a.a $\omega \in \Omega$, the function $m(\omega, \cdot) := m_+(\omega, \cdot) - m_-(\omega, \cdot)$ has the properties that $\Im m(\omega, z) > 0$, for every $z \in H^+$, $\Re m(\omega, \lambda) = 0$ and $\Im m_+(\omega, \lambda) = -\Im m_-(\omega, \lambda)$, for Lebesgue a.a $\lambda \in I$. Letting $G(z) = \ln m(\omega, z)$, we have $G(z) = \ln|m(\omega, z)| + i\pi/2$. The function $F(z) = G(z) - \mathrm{i}\pi/2$ satisfies $\Im F(\lambda) = 0$ for Lebesgue a.a. $\lambda \in I$. From the above observations, it follows that $m_+(\omega, z) - m_-(\omega, z)$ extends analytically through I to H^-. The same holds for the function $\frac{m_- m_+}{m_- - m_+}$. Now, it is easy to prove that both m_+ and m_- extend analytically through I from $H \cap H^+$ to $H \cap H^-$. We proved the first part of the following important

THEOREM 4.36. *Let $I \subset \mathbb{R}$ be an open interval, and suppose that $\beta(\lambda) = 0$ for Lebesgue a.a. $\lambda \in I$. Then, for ν a.a. $\omega \in \Omega$, the Weyl m-functions $m_\pm(\omega, \cdot)$ admit analytic extensions through I from H^+ to H^-. If we denote these extensions with $h_\pm(\omega, \cdot)$, then*

$$h_+(\omega, \lambda) = \begin{cases} m_+(\omega, \lambda), & \Im\lambda > 0, \\ m_-(\omega, \lambda), & \Im\lambda < 0 \end{cases} \quad \textit{and} \quad h_-(\omega, \lambda) + \begin{cases} m_-(\omega, \lambda), & \Im\lambda > 0, \\ m_+(\omega, \lambda), & \Im\lambda < 0. \end{cases}$$

PROOF. We give the proof only for $h_+(\omega, \cdot)$. Let $\Omega_1 \subset \Omega$ such that both m_+ and m_- extend analytically through I. Let $\Omega_2 \subset \Omega$ be the set consisting on those points $\omega \in \Omega$ with dense orbit. Then $\nu(\Omega_1 \cap \Omega_2) = 1$. Choose $\bar{\omega} \in \Omega_1 \cap \Omega_2$. Let $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$, and define $h_+(\lambda) := h_+(\bar{\omega}, \lambda)$. We write down the Riccati equation (4.25)

$$m' + \frac{1}{p} m^2 = q - \lambda y.$$

Suppose that $\Im\lambda < 0$. Then, for every initial condition $m_0 \in \mathbb{R}$ and for every $t > 0$, we have $\Im m(t, \lambda) > 0$ and $\Im n(t, \lambda) := \Im \frac{1}{m(t,\lambda)} < 0$, where $m(t, \lambda)$ is the solution of Eq. (4.25) satisfying $m(0, \lambda) = m_0 \in \mathbb{R}$. We define a flow $\{\tilde{\tau}_t\}$ on $\Omega \times \mathbb{P}(\mathbb{C}) = \Sigma$ by setting

$$\tilde{\tau}_t(\omega, m) = \bigl(\omega \cdot t, m(t)\bigr),$$

where $m(t)$ is the solution of Eq. (4.25) satisfying $m(0) = m \in \mathbb{C}$. Note that the flow $\{\tilde{\tau}_t\}$ can be defined in an equivalent manner by considering the action of the flow $\{\hat{\tau}_t\}$ on complex lines in $\mathbb{P}(\mathbb{C})$, i.e.,

$$\tilde{\tau}_t(\omega, m) = \hat{\tau}_t\left(\omega, \begin{pmatrix} 1 \\ m \end{pmatrix}\right).$$

From the above observations, it follows easily that for every fixed $\omega \in \Omega$, $\lambda \in H^-$ and $t > 0$, the map $\tilde{\tau}_t^\omega : \mathbb{P}(\mathbb{C}) \to \mathbb{P}(\mathbb{C}) : m \mapsto \tilde{\tau}_t(\omega, m) = (\omega \cdot t, m(t))$ maps the closure of the set $\{\infty\} \times \{z \in \mathbb{C} \mid \Im z \geqslant 0\}$ into H^+. Alternatively, identifying H^+ with a disc B contained in $\mathbb{P}(\mathbb{C})$, $\tilde{\tau}_t^\omega$ maps the closure of B into B, for every $\omega \in \Omega$, $\lambda \in H^-$ and $t > 0$.

These things being said, let $S^\pm(\lambda) = \{(\omega, m_\pm(\omega, \lambda)) \mid a \in \Omega\} \subset \Sigma$. Then $S^\pm(\lambda)$ are $\{\tilde{\tau}_t\}$-invariant sections of Σ. Recall that $S^\pm(\lambda)$ are minimal subsets of the projective flow $(\Sigma, \{\hat{\tau}_t\})$.

We claim that $S^\pm(\lambda)$ are the only $\{\tilde{\tau}_t\}$-invariant sections of Σ. To prove this, we use methods already exploited in Section 3. Suppose for contradiction that S is another invariant section of Σ. Let $\omega \in \Omega$ have dense orbit. Then there exists $l \in \mathbb{P}(\mathbb{C})$ such that $(\omega, l) \in S$. It follows that $(\omega, m_+(\omega, \lambda))$, $(\omega, m_-(\omega, \lambda))$ and (ω, l) form distal pairs in Σ. Choose unit vectors $u^+ \in m_+(\omega, \lambda)$, $u^- \in m_-(\omega, \lambda)$, $u \in (\omega, l)$. Then u^+, u^-, u are pairwise linearly independent. Let $\Phi_\omega(t)$ be the fundamental matrix solution of Eq. (SL_ω) with $\Phi_\omega(0) = I$. Let $u^\pm(t) = \Phi_\omega(t)u^\pm$, $u(t) = \Phi_\omega(t)u$. Since $\Im\lambda \neq 0$, the family (SL_ω) admits an exponential dichotomy over Ω, hence for every fixed $\omega \in \Omega$,

$$\lim_{t \to -\infty} \left|u^+(t)\right| = \infty, \qquad \lim_{t \to \infty} \left|u^-(t)\right| = \infty.$$

Suppose that $u(t)$ is unbounded at ∞. Let $\Psi(t)$ be the fundamental matrix solution of Eq. (SL_ω) whose columns are $u^-(t)$ and $u(t)$. By Liouville's formula, $\det \Psi(t) = \det \Psi(0) = const \neq 0$, and since $u^-(t)$ is unbounded at ∞, we must have $|u(t)| \to 0$, a contradiction. Hence $u(t)$ is bounded at ∞. We carry out the same reasoning at $-\infty$ to prove that $u(t)$ decays as $|t| \to \infty$. However, in this case, let $\Psi_1(t)$ be the fundamental matrix solution of Eq. (SL_ω) whose columns are $u^+(t)$ and $u(t)$. Then $\liminf_{t \to \infty} \det \Psi_1(t) = 0$, a contradiction. We proved that $S^\pm(\lambda)$ are the unique $\{\tilde{\tau}_t\}$-invariant minimal sections of Σ.

Now, let D_0 be a complex domain with $D_0 \supset (I \cup \{\lambda \mid \Im\lambda \neq 0\})$ and such that $h_+(\lambda)$ is analytic and has positive imaginary part in D_0. For every $t > 0$, we define a function $g_t : D_0 \to \mathbb{C}$, by setting

$$\big(\tilde{\tau}_t(\bar{\omega}), g_t(\lambda)\big) = \tilde{\tau}_t\big(\bar{\omega}, h_+(\lambda)\big),$$

i.e., $g_t(\lambda) = h_+(t, \lambda)$, where $h_+(t, \lambda)$ is the solution of Eq. (4.25) satisfying $h_+(0, \lambda) = h_+(\lambda)$. It follows that, for every $t > 0$, g_t maps D_0 into H^+. In fact, if $\Im\lambda > 0$, this follows from the invariance of the section $S^+(\lambda)$; if $\lambda \in I$, it follows from $\Im h_+ > 0$, and if $\Im\lambda < 0$, it follows from $\Im h_+ > 0$ and the first part of the proof. Moreover, for every $t > 0$, the map g_t in analytic in D_0, since $g_t(\lambda) = h_+(t, \lambda) = \Phi_{\bar{\omega}}(t)\binom{1}{h_+(\lambda)}$ and $\Phi_{\bar{\omega}}(t)$ is linear. It follows that the family $\{g_t(\lambda), t > 0\}$ is a normal family. Let $\omega \in \Omega$, and choose a sequence $\{t_n\} \to \infty$ such that $\bar{\omega} \cdot t_n \to \omega$ and such that $g_{t_n}(\lambda)$ converges to a holomorphic function $g_\omega(\lambda)$ on compact subsets of D_0 (if necessary, we pass to a subsequence). From the invariance and the continuity of the section $S^+(\lambda)$ with respect to λ, we conclude that $g_\omega(\lambda) = m_+(\omega, \lambda)$ for every λ with $\Im\lambda > 0$. Moreover, one easily sees that $g_{t_n}(\lambda)$ converges to $g_\omega(\lambda)$ on compact subsets of D_0, for every sequence $\{t_n\} \subset \mathbb{R}$ such that $\bar{\omega} \cdot t_n \to \omega$ as $n \to \infty$.

We can repeat this construction for every $\omega \in \Omega$. Now, using standard arguments about normal families [37], we prove that the section $S(\lambda) = \{(\omega, g_\omega(\lambda)) \mid \omega \in \Omega\}$ is continuous

and $\{\tilde{\tau}_t\}$-invariant for every $\lambda \in \mathbb{C}$ with $\Im\lambda > 0$. Of course $S(\lambda) = S^-(\lambda)$ if $\Im\lambda < 0$, and the proof is complete (for m_- we use similar arguments). □

We summarize the results we obtained about the behavior of the Weyl m-functions when crossing the real line, in the following

THEOREM 4.37. *For ν a.a. $\omega \in \Omega$, the Weyl m-functions extend meromorphically through every interval $J \subset R(L)$ from H^+ to H^-. Moreover, $m_\pm(\omega, \lambda) \in \mathbb{P}(\mathbb{R})$, for all $\lambda \in J$.*

If $I \subset \mathbb{R}$ is an open interval such that $\beta(\lambda) = 0$ for Lebesgue a.a. $\lambda \in I$, then $m_\pm$ extend analytically through I from H^+ to H^-. The extensions $h_\pm(\omega, \cdot)$ satisfy $h_\pm(\omega, z) = m_\mp(\omega, z)$ for all $z \in H^-$, and $\Re h_+(\omega, \lambda) = \Re h_-(\omega, \lambda)$ for Lebesgue a.a. $\lambda \in I$.

We finish this subsection by observing that actually $\beta(\lambda)$ can attain the value 0 only at points $\lambda \in \Sigma(L)$. In fact, if $\lambda \in R(L)$, it is a simple consequence of the Oseledets theorem that $\beta(\lambda) < 0$ (the Oseledets bundles are disjoint, and are contained in the bundles of exponential dichotomy, hence there are exactly two Lyapunov exponents, $\beta_1(\lambda) = -\beta_2(\lambda) \neq 0$).

4.4. *The inversion problem*

In this part, we pose and solve our inverse algebro-geometric problem for the Sturm–Liouville operator. We impose the following basic

HYPOTHESIS 4.38.

(H1) The spectrum of the family (SL_ω) is a finite union of closed intervals: $\Sigma(L) = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \cdots \cup [\lambda_{2g}, \infty)$ $(\lambda_0 > 0)$.

(H2) The Lyapunov exponent vanishes almost everywhere in the spectrum: $\beta(\lambda) = 0$ for Lebesgue a.a. $\lambda \in \Sigma(L)$.

We claim that the hypothesis that $\lambda_0 > 0$ is not restrictive, i.e., that if $\lambda_0 \leqslant 0$, then we can "translate" Eqs. (SL_ω) in such a way that the translated spectrum has $\tilde{\lambda}_0 > 0$, without any loss of regularity of the functions involved. For, suppose that $\lambda_0 \leqslant 0$, and choose $\eta > 0$. It is easy to check that the "translated" family of equations

$$(p\varphi')' - \big[q - (\lambda_0 - \eta)y\big]\varphi = -\lambda y \varphi \tag{5.2^η_ω}$$

has a spectrum Σ_η which equals the set $\Sigma(L) - \lambda_0 + \eta$. Moreover, the family (5.2^η_ω) satisfies Hypotheses 4.38. See ([95], Section 4) for more details concerning this fact.

Let $\mathcal{R}$ be the Riemann surface of the algebraic relation

$$w^2 = -(\lambda - \lambda_0)(\lambda - \lambda_1)\cdots(\lambda - \lambda_{2g}).$$

The surface $\mathcal{R}$ can be viewed as the union of two copies of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ cut open along $\Sigma(L)$ and glued together in the standard way. Alternatively, $\mathcal{R}$ can be viewed as an analytic g torus, with holes obtained by cutting the Riemann sphere along

(closed) paths c_i issuing from λ_{2i-1} ($i = 1, \ldots, g$) such that $\lambda_{2i} \in c_i$ for all $i = 1, \ldots, g$. Let $\pi : \mathcal{R} \to \mathbb{C} \cup \{\infty\}$ be the projection of $\mathcal{R}$ to the Riemann sphere. Then π is 2-1 except at the ramification (or branch) points $\{\lambda_0, \lambda_1, \ldots, \lambda_{2g}, \infty\}$, where it is 1-1. We abuse notation slightly, and write $\lambda_1, \lambda_2, \ldots, \lambda_{2g}, \infty \in \mathcal{R}$. Let $\pi^{-1}(0) = \{0^+, 0^-\}$. Define a meromorphic function k on $\mathcal{R}$ by letting $k(0^+)$ be the positive square root of $\prod_{i=0}^{2g} \lambda_i$ and letting $k(P)$ be the appropriate square root of $-\prod_{i=0}^{2g}(\pi(P) - \lambda_i)$ for each $P \in \mathcal{R}$.

Let $\omega \in \Omega$, and define a function M_ω on $\mathcal{R} \setminus \{\lambda_0, \lambda_1, \ldots, \lambda_{2g}\}$ as follows: put $M_\omega(0^+) = m_+(\omega, 0)$, $M_\omega(0^-) = m_-(\omega, 0)$. We define $M_\omega(P)$ by continuation along a curve in $\mathcal{R}$ joining 0^+ to P, for each $P \in \mathcal{R}$. Theorem 4.37 ensures that the function M_ω on $\mathcal{R} \setminus \{\lambda_0, \lambda_1, \ldots, \lambda_{2g}\}$ defined in this way is meromorphic in a neighborhood of every point $P \in \mathcal{R}$ which is not a ramification point of $\mathcal{R}$.

Let us prove the following important

THEOREM 4.39. *Suppose that Hypotheses* (H1) *and* (H2) *hold, and let* $\omega \in \Omega$. *Then* $m_+(\omega, \cdot)$ *and* $m_-(\omega, \cdot)$ *define a single meromorphic function* $M_\omega : \mathcal{R} \to \mathbb{C} \cup \{\infty\}$. *Moreover, the map* $M : \Omega \times \mathcal{R} \to \mathbb{C} \cup \{\infty\} : (\omega, P) \mapsto M_\omega(P)$ *is jointly continuous.*

PROOF. To extend M_ω to a meromorphic function defined on all $\mathcal{R}$, we examine the behavior of the Weyl m-functions near a ramification point $\{\lambda_0, \lambda_1, \ldots, \lambda_{2g}, \infty\}$. So, let $\lambda_* \in \{\lambda_0, \lambda_1, \ldots, \lambda_{2g}\}$. Let $\omega \in \Omega$, and let $D_* \subset \mathbb{C}$ be a punctured disc centered at λ_* such that D_* has diameter equal to the union of an interval $I_* \subset \mathbb{R} \setminus \Sigma(L)$ and an interval contained in $\Sigma(L)$. If $m_+(\omega, \cdot)$ is continued around a circle in D_* centered in λ_*, it goes into $m_-(\omega, \cdot)$, and vice-versa. If we introduce the coordinate $z = \sqrt{\lambda - \lambda_*}$, then the inverse image of D_* in the z-plane is a punctured disc $D_\#$ centered at $z = 0$, and $m_\pm(\omega, \cdot)$ are branches of a meromorphic function M_ω in $D_\#$. Shrinking the disc $D_\#$ if necessary, we see that M_ω omits infinitely many (real) values in $D_\#$. By the Picard theorem [64], M_ω is meromorphic on the full disc $D_\# \cup \{0\}$. For the ramification point ∞, let D_∞ be a disc in the Riemann sphere centered at ∞ such that $D_\infty \cap \mathbb{R} \subset (-\infty, \lambda_0) \cup (\lambda_{2g}, \infty)$. Arguing as above, we observe that M_ω omits infinitely many real values in the punctured disc $\pi^{-1}(D_\infty)$, and applying the Picard theorem again, we conclude that M_ω extends meromorphically at ∞. Moreover, the bundles $\operatorname{Im} P_\omega$ and $\operatorname{Ker} P_\omega$ of the exponential dichotomy vary continuously with respect to $\omega \in \Omega$, ensuring that the map $M : \Omega \times \mathcal{R} \to \mathbb{C} \cup \{\infty\}$ is jointly continuous. $\square$

Let $(\Omega, \{\tau_t\})$ be a topological flow, and let ν be a $\{\tau_t\}$-ergodic measure on Ω. We suppose that the topological support of ν equals Ω. Let p, q, and $y : \Omega \to \mathbb{R}$ be continuous functions such that p and y assume only positive values. We assume that the derivative $\omega \mapsto \frac{\mathrm{d}}{\mathrm{d}t} p(\omega \cdot t)|_{t=0}$ exists and is continuous. Again a word concerning the way we let the functions involved vary with $t \in \mathbb{R}$: if we fix $\omega \in \Omega$, we can consider p, q, y as functions of t: for instance, $p(t)$ is identified with $p(\omega \cdot t)$, and similarly for q, y. Suppose that Hypotheses 4.38 hold for these data.

Our goal is to determine all such functions p, q, y. Some of the following material follows closely the exposition in a recent paper in the journal Discrete and Continuous Dynamical Systems A **18**; see reference [96] below. We thank the editor Prof. Shouchuan Hu of that journal for permission to present this material here. Let $\omega \in \Omega$, $\lambda \in \mathbb{C} \setminus \Sigma(L)$,

and let $m_\pm(t) = m_\pm(\omega \cdot t, \lambda)$ be the corresponding m-functions. They satisfy the Riccati equation

$$m' + \frac{1}{p} m^2 = q - \lambda y.$$

It is understood that this equation holds for each $t \in \mathbb{R}$.

We change the meaning of z again, and let $z^2 = -\lambda$ be a parameter near $\lambda = \infty$ on the Riemann surface $\mathcal{R}$. We view points $P \in \mathcal{R}$ near ∞ as being parameterized by z. Write $M'_\omega(z)$ for the derivative of the function $t \mapsto M_{\omega \cdot t}(z)$ in $t = 0$: thus $M'_\omega(z) = \frac{\mathrm{d}}{\mathrm{d}t} M_{\omega\cdot t}(z)|_{t=0}$ for each $\omega \in \Omega$. Then

$$M'_\omega(z) + \frac{1}{p} M^2_\omega(z) = q + z^2 y. \tag{4.65}$$

We expand the function $z \mapsto M_\omega(z)$ in a Laurent series near $z = \infty$, obtaining

$$M_\omega(z) = \alpha_1 z + \alpha_0 + \sum_{n=1}^{\infty} \alpha_{-n} z^{-n}.$$

Using the Riccati equation (4.65) one finds

$$\alpha_1^2 = yp, \quad \alpha_0 = -\frac{(py)'}{4y}, \quad \alpha_0' + \frac{\alpha_0^2}{p} + \frac{2\alpha_1\alpha_{-1}}{p} = q. \tag{4.66}$$

In particular, the above derivatives exist; in fact, by using the Cauchy integral formula along a small circle centered at $z = \infty$, one sees that α_{-1} exists and is continuous in $\omega \in \Omega$. It follows that the derivative α_0' exists and is continuous in $\omega \in \Omega$, so the same holds for y, and moreover py has second derivative which is continuous in $\omega \in \Omega$.

Let $\sigma : \mathcal{R} \to \mathcal{R}$ be the hyperelliptic involution $(\lambda, w) \mapsto (\lambda, -w)$ (i.e. the map which interchanges the sheets). It is sometimes convenient to think of M_ω as "the meromorphic extension of $m_+(\omega, \cdot)$ to $\mathcal{R}$". From this point of view, one has $m_-(\omega, \cdot) = M_\omega \circ \sigma(\cdot)$. We then have

$$\begin{aligned} M_\omega(z) &= m_+(z) = \sqrt{py}\, z - \frac{(py)'}{4y} + \sum_{n=1}^{\infty} \alpha_{-n} z^{-n}, \\ M_\omega(z) \circ \sigma(z) &= m_-(z) = -\sqrt{py}\, z - \frac{(py)'}{4y} + \sum_{n=1}^{\infty} (-1)^n \alpha_{-n} z^{-n}. \end{aligned} \tag{4.67}$$

We adopt the convention that $\sqrt{py} < 0$ if g is even, while $\sqrt{py} > 0$ if g is odd. The derivatives in (4.66) and (4.67) are calculated at $t = 0$. Thus, for example, $(py)'$ means $\frac{\mathrm{d}}{\mathrm{d}t}(py)(\omega \cdot t)|_{t=0}$. Let us now write P_i $(1 \leqslant i \leqslant g)$ for the finite poles of M_ω. Note that these poles P_i depend on $\omega \in \Omega$.

We know that $\pi(P_i) \in [\lambda_{2i-1}, \lambda_{2i}]$ for each $i \leqslant i \leqslant g$. Our aim in the next lines is to obtain an explicit formula for M_ω in terms of the poles and of p, q, y.

Let P denote a generic point of the Riemann surface $\mathcal{R}$. Abbreviating $m_\pm(\omega, P)$ to $m_\pm(P)$, we first note that $m_+ - m_- = M_\omega - M_\omega \circ \sigma$ is meromorphic on $\mathcal{R}$ and has zeroes $\{\lambda_0, \dots, \lambda_{2g}\} \subset \mathcal{R}$. It has poles at the inverse images $\pi^{-1}\pi(P_i)$ $(1 \leqslant i \leqslant g)$ and at ∞. Taking account of (4.67), we have

$$(m_+ - m_-)(P) = \frac{2\sqrt{py}\,k(P)}{H(\lambda)}, \tag{4.68}$$

where

$$H(\lambda) = \prod_{i=1}^{g} \bigl[\lambda - \pi(P_i)\bigr],$$

and $k(P)$ is defined by the conditions

$$k(P) = \sqrt{-\bigl(\pi(P) - \lambda_0\bigr)\cdots\bigl(\pi(P) - \lambda_{2g}\bigr)}, \quad k\bigl(0^+\bigr) > 0. \tag{4.69}$$

Next, for each fixed $\omega \in \Omega$, the function $m_+ + m_-$ is meromorphic on the Riemann sphere; i.e., it is a rational function. It has poles at $\pi(P_1), \dots, \pi(P_g)$ and moreover tends to $2\alpha_0$ as $\lambda \to \infty$. This implies that

$$(m_+ + m_-)(P) = \frac{2Q(\lambda)}{H(\lambda)}$$

where $\lambda = \pi(P)$ and $Q(\lambda) = q_g\lambda^g + \cdots + q_1\lambda + q_0$ is a polynomial of degree $\leqslant g$ in λ. It is of degree g whenever $\alpha_0 \neq 0$. Putting together all this information we obtain

$$M_\omega(P) = m_+(\omega, P) = \frac{Q(\lambda) + \sqrt{py}\,k(P)}{H(\lambda)},$$
$$M_\omega \circ \sigma(P) = m_-(\omega, P) = \frac{Q(\lambda) - \sqrt{py}\,k(P)}{H(\lambda)}.$$

Here M_ω is viewed as a meromorphic function of the generic point $P \in \mathcal{R}$. The polynomials Q and H are functions of $\lambda = \pi(P)$, and so can also be viewed as functions on $\mathcal{R}$, for each fixed $\omega \in \Omega$.

Let us now suppose for a moment that none of the poles are ramification points of $\mathcal{R}$. Then $m_-(P_i)$ is finite for each $1 \leqslant i \leqslant g$ and so we must have

$$Q(P_i) = \sqrt{py}\,k(P_i) \quad 1 \leqslant i \leqslant g. \tag{4.70}$$

The above formulas retain validity even if one or more poles P_i are ramification points of $\mathcal{R}$: in fact, if this were not true, then such a point P_i would not be a simple pole for M_ω

(to see this, suppose that $Q(P_i) \neq \sqrt{py}\,k(P_i)$, then compute the derivative of $M_\omega(\cdot)$ with respect to λ at $\lambda = P_i$, and use the local coordinate $z^2 = \lambda - P_i$.)

Equation (4.70) provides g linear relations for the $g+1$ coefficients of Q. To get another relation, note that $\lambda = 0$ is to the left of $\Sigma(L)$. We point out again the fact that we may assume without any loss of generality that $\lambda_0 > 0$. Hence $m_+(\omega, 0) = m_+^0$ and $m_-(\omega, 0) = m_-^0$ have real values. We therefore have

$$q_0 = Q(0) = \frac{(-1)^g [m_+^0 + m_-^0] \prod_{i=1}^g \pi(P_i)}{2}.$$

Now we can determine the coefficients of Q from the following van der Monde system:

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & \pi(P_1) & \dots & \pi(P_1)^g \\ \vdots & \vdots & & \vdots \\ 1 & \pi(P_g) & \dots & \pi(P_g)^g \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_g \end{pmatrix} = \begin{pmatrix} Q(0) \\ \sqrt{py}\,k(P_1) \\ \vdots \\ \sqrt{py}\,k(P_g) \end{pmatrix}. \tag{4.71}$$

From now on, we will often abuse the notation, and write P_i instead of $\pi(P_i)$ ($i = 1, \dots, g$), when no confusion arises (i.e., we will omit π). In particular, we can determine the highest-order coefficient $q_g = \alpha_0 = -\frac{(py)'}{4y}$. In fact, letting Δ be the determinant of the matrix on the right-hand side of (4.71), we obtain

$$\begin{aligned} q_g \Delta = \det &\begin{pmatrix} P_1 & P_1^2 & \dots & P_1^{g-1} & \sqrt{py}\,k(P_1) \\ \vdots & \vdots & & \vdots & \vdots \\ P_g & P_g^2 & \dots & P_g^{g-1} & \sqrt{py}\,k(P_g) \end{pmatrix} \\ &+ (-1)^g q_0 \det \begin{pmatrix} P_1 & P_1^2 & \dots & P_1^{g-1} \\ \vdots & \vdots & & \vdots \\ P_g & P_g^2 & \dots & P_g^{g-1} \end{pmatrix}. \end{aligned}$$

Working out the determinants, we get

$$q_g = \frac{(-1)^g q_0}{\prod_{i=1}^g P_i} + \sum_{i=1}^g \frac{\sqrt{py}\,k(P_i)}{P_i \prod_{r \neq i} (P_i - P_r)}. \tag{4.72}$$

Now we obtain a formula for α_{-1}. The Laurent expansions (4.67) for $m_\pm$ together with (4.68) give

$$2\sqrt{py}\,z + \frac{2\alpha_{-1}}{z} + 2\sum_{k=1}^\infty \frac{\alpha_{-2k-1}}{z^{2k+1}} = \frac{2\sqrt{py}\,k(z)}{\prod_{i=1}^g (-z^2 - P_i)}.$$

Multiplying both sides of this relation by $z\prod_{i=1}^{g}(z^2+P_i)$ and comparing the z^{2g}-coefficients , we obtain

$$\alpha_{-1} = \sqrt{py}\left(\frac{1}{2}\sum_{i=0}^{2g}\lambda_i - \sum_{i=1}^{g}P_i\right).$$

Let us now fix $\omega \in \Omega$. Now we will "follow the trajectory $\{\omega \cdot t\}$ through ω". As already remarked, this allows us to think at the functions involved as varying with $t \in \mathbb{R}$. Thus write $m_\pm(t,z) = m_\pm(\omega \cdot t, z),\ p = p(t), q = q(t), y = y(t), P_i = P_i(t)\ (-\infty < t < \infty)$, $Q = Q(\lambda, t)$, $H = H(\lambda, t)$. Using the formula for q in (4.66) we obtain

$$q(t) = y(t)\left(\sum_{i=0}^{2g}\lambda_i - 2\sum_{i=1}^{g}P_i(t)\right) + q_g'(t) + \frac{q_g(t)^2}{p(t)}. \tag{4.73}$$

Here q_g is given by (4.72).

Now, consider

$$m_+' = \left(\frac{\sqrt{p(t)y(t)}\,k(P) + Q(\lambda,t)}{H(\lambda,t)}\right)', \quad ' = \frac{\mathrm{d}}{\mathrm{d}t}.$$

From the Riccati type equation for m_+ we obtain

$$m_+' = \left(\frac{\sqrt{p(t)y(t)}\,k(P) + Q(\lambda,t)}{H(\lambda,t)}\right)' = -\frac{1}{p}m_+^2 + q - \lambda y,$$

hence,

$$m_+' = \frac{(\sqrt{py}\,k + Q)'H - H'(\sqrt{py}\,k + Q)}{H^2} = -\frac{1}{p}\frac{(\sqrt{py}\,k + q)^2}{H^2} + q - \lambda y,$$

and multiplying by H^2 we obtain

$$(\sqrt{py}\,k + Q)'H - (\sqrt{py}\,k + Q)H' + \frac{1}{p}(\sqrt{py}\,k + Q)^2 = (q - \lambda y)H^2. \tag{4.74}$$

Let us write $k_r = k(P_r)$. If we compute (4.74) at a pole P_r we obtain, since $H(P_r) = H_r = 0$ and $\sqrt{py}\,k(P_r) = \sqrt{py}\,k_r = Q(\pi(P_r))$,

$$-(\sqrt{py}\,k_r)H_r' + \frac{1}{p}(2\sqrt{py}\,k_r)^2 = 0, \tag{4.75}$$

hence we obtain the following important relation

$$\frac{H_r'(t)}{k_r(t)} = \frac{2}{p(t)}\sqrt{p(t)y(t)}. \tag{4.76}$$

What we want to do now is to express the above fraction (4.76) as a product of the poles $P_1(t), \ldots, P_g(t)$ of the function $M_{\omega\cdot t}(P)$ To this aim, we compute m_- at $\lambda = 0$, and we have

$$m_-^0(t) = \frac{Q(0,t) - \sqrt{py}\, k(0^+)}{(-1)^g \prod_{i=1}^g P_i(t)},$$

hence

$$\sqrt{py} = \frac{(-1)^{g+1} m_-^0(t) \prod_{i=1}^g P_i(t) + Q(0,t)}{k(0^+)},$$

and (4.76) becomes

$$\frac{H_r'}{k_r} = \frac{2(-1)^{g+1} m_-^0(t) \prod_{i=1}^g P_i(t) + 2Q(0,t)}{p(t)k(0^+)}. \tag{4.77}$$

Hence we have

$$\frac{H_r'(t)}{k_r(t)} = \frac{(-1)^{g+1}[m_-^0(t) - m_+^0(t)] \prod_{i=1}^g P_i(t)}{p(t)k(0^+)}. \tag{4.78}$$

Finally

$$\frac{H_r'(t)}{P_r(t)k_r(t)} = \frac{(-1)^{g+1}[m_-^0(t) - m_+^0(t)] \prod_{s\neq r} P_s(t)}{p(t)k(0^+)}. \tag{4.79}$$

We proved the following

THEOREM 4.40. *Suppose that Hypotheses* 4.38 *hold. Then the functions* p, q *and* y *satisfy the following relations involving the* t*-motion of the poles of the meromorphic function* $M_\omega(P)$ *on* $\mathcal{R}$,

$$\frac{2\sqrt{p(t)y(t)}}{p(t)} = \frac{H_r'(t)}{k_r(t)} = \frac{(-1)^{g+1}[m_-^0(t) - m_+^0(t)]\prod_{i=1}^g P_i(t)}{p(t)k(0^+)},$$

and

$$q(t) = y(t)\left(\sum_{i=0}^{2g} \lambda_i - 2\sum_{i=1}^{g} P_i(t)\right) + q_g'(t) + \frac{q_g^2(t)}{p(t)},$$

where

$$q_g(t) = -\frac{(p(t)y(t))'}{4y(t)} = \frac{(-1)^g Q(0,t)}{\prod_{i=1}^g P_i(t)} + \sum_{j=1}^{g} \frac{\sqrt{p(t)y(t)}k(P_j(t))}{P_j(t)\prod_{r\neq j}(P_j(t) - P_r(t))}$$
$$= -\frac{p(t)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\ln\left(\left[m_-^0(t) - m_+^0(t)\right]\prod_{i=1}^{g} P_i(t)\right)\right\}.$$

Now we study the nature of the t-dependence of p, q, y. Clearly this issue is related to that of the pole motion $t \mapsto \{P_1(t), \ldots, P_g(t)\}$. We will see that a good deal of information about the pole motion can be obtained by introducing certain “nonstandard” Abel–Jacobi coordinates. In what follows, we will use the language of classical algebraic geometry. We will not pause to give much information concerning general facts of the theory of algebraic curves. We will only give a general outline of the classical theory, and spend a few more lines in the explanation of the main concepts occurring in the theory of the “singular Riemann surfaces”. For more details, the reader is referred to such standard texts as [146,112]. For the theory of generalized Jacobian varieties, see [53].

Let $\mathcal{R}$ be the Riemann surface of the algebraic relation

$$w^2 - (\lambda - \lambda_0)\cdots(\lambda - \lambda_{2g}).$$

It was already pointed out that $\mathcal{R}$ is homeomorphic to an analytic g-torus. Let $\{\alpha_i, \beta_i\}$ $(i = 1, \ldots, g)$ be a homology basis of $\mathcal{R}$. There is one simple standard way to take such a basis: all α_i, β_i $(i = 1, \ldots, g)$ consist of a simple closed curves satisfying the intersection relations

$$\begin{aligned} &\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \emptyset, \quad 1 \leqslant i, j \leqslant g, \\ &\alpha_i \cap \beta_j = \begin{cases} \{A_i\} & i = j, \\ \emptyset, & i \neq j \end{cases} \end{aligned} \tag{4.80}$$

where $\{A_i\}$ denotes a point in $\mathcal{R}$ which varies with i, for every $i = 1, \ldots, g$. Let $w_1, \ldots, w_g$ be a standard basis of normalized differential forms of the first kind on $\mathcal{R}$: to be clearer, $w_1, \ldots, w_g$ are holomorphic differentials such that

$$\int_{\alpha_i} w_j = \delta_{ij} \quad \text{and} \quad \int_{\beta_i} w_j = z_{ij},$$

where $z_{ij} \in \mathbb{C}$. The $g \times g$ matrix $Z = (z_{ij})$ is called the *period matrix* of the surface $\mathcal{R}$. This matrix is symmetric and has positive imaginary part [146]. It turns out that the $\mathbb{Z}$-lattice in $\mathbb{C}^g$ spanned by all the vectors of the form

$$\int_c (w_1, \ldots, w_g), \quad c = \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$$

is completely determined once the matrix $P = \binom{I}{Z}$ is given, where I is the $g \times g$ identity matrix. We denote this $\mathbb{Z}$-lattice with Λ^{2g}. Clearly the rank of Λ^{2g} is $2g$. We define the Jacobian variety as

$$J(\mathcal{R}) = \mathbb{C}^g / \Lambda^{2g}.$$

There is another way to describe the Jacobian variety. Any formal sum $\mathcal{D} = \sum_{i=1}^{p} k_i Q_i$, where each $k_i \in \mathbb{Z}$, and Q_i are points in $\mathcal{R}$, is called a *divisor*. The sum of two divisors $\mathcal{D} = \sum_{i=1}^{p} k_i Q_i$ and $\mathcal{D}_1 = \sum_{i=1}^{s} n_i R_i$ is the divisor

$$\mathcal{D} + \mathcal{D}_1 = \sum_{i=1}^{p} k_i Q_i + \sum_{i=1}^{s} n_i R_i.$$

The degree of a divisor $\mathcal{D} = \sum_{i=1}^{p} k_i Q_i$ is the integer $\sum_{i=1}^{p} k_i$. With the usual rules of the summation, the set of all the divisors of the Riemann surface $\mathcal{R}$ is an Abelian additive group. The inverse of a divisor $\mathcal{D} = \sum_{i=1}^{p} k_i Q_i$ is the divisor $-\mathcal{D} = \sum_{i=1}^{p} (-k_i) Q_i$.

Let $f : \mathcal{R} \to \mathbb{C}$ be any meromorphic function. Then f has the same number of zeroes and poles. If $Q_1, \dots, Q_p$ are the zeroes of f and $R_1, \dots, R_p$ are the poles of f (all counted with their multiplicities), then we can associate to f a divisor (f) of degree 0 defined as

$$(f) = \sum_{i=1}^{p} (k_i Q_i - n_i R_i) \quad \left(\sum_{i=1}^{p} (k_i - n_i) = 0 \right)$$

where $k_i, n_i \in \mathbb{Z}^+$ are the multiplicities of the points Q_i, R_i as zeroes or poles of f $(i = i, \dots, p)$. If f and g are two meromorphic functions on $\mathcal{R}$, then $(fg) = (f) + (g)$, $(fg^{-1}) = (f) - (g)$, and $(f) = (g)$ implies $f = cg$, where $c \in \mathbb{C}$ is a constant.

Every divisor of degree g can be thought as an element of $\mathrm{Symm}^g(\mathcal{R})$, the set of unordered g-tuples of points in $\mathcal{R}$. Given any divisor $\mathcal{D}$ of degree g (note that g is the genus of the surface $\mathcal{R}$) we form its class $[\mathcal{D}]$ consisting of all the divisors $\mathcal{D}_1$ of the same degree g such that $\mathcal{D} - \mathcal{D}_1$ is the divisor of a meromorphic function $f : \mathcal{R} \to \mathbb{C}$. Now, let P_* be any fixed point on $\mathcal{R}$ different from the ramification points. It is a classical fact that two divisors $\mathcal{D} = (Q_1, \dots, Q_g)$ and $\mathcal{D}_1 = (R_1, \dots, R_g)$ belong to the same class if and only if

$$\sum_{i=1}^{g} \int_{Q_i}^{R_i} (w_1, \dots, w_g) = Zt,$$

where t is a column vector in $\mathbb{Z}^g$.

To any divisor $\mathcal{D} = (Q_1, \dots, Q_g) \in \mathrm{Symm}^g(\mathcal{R})$ we associate the complex g-vector $(s_1, \dots, s_g) \in \mathbb{C}^g$, given by

$$(s_1, \dots, s_g) = \sum_{i=1}^{g} \int_{P_*}^{Q_i} (w_1, \dots, w_g).$$

We define a map $I : \mathrm{Symm}^g(\mathcal{R}) \to \mathbb{C}^g/\Lambda^{2g}$ as follows

$$I(Q_1, \dots, Q_g) = \sum_{i=1}^{g} \int_{P_*}^{Q_i} (w_1, \dots, w_g) \mod \Lambda^{2g} \in J(\mathcal{R}).$$

This map is called the *Abel map*. It is surjective and, when restricted to an appropriate open dense subset of $\mathrm{Symm}^g(\mathcal{R})$, is bijective. It is clear that $I(Q_1, \dots, Q_g) = I(R_1, \dots, R_g)$ if and only if $(Q_1, \dots, Q_g)$ and $(R_1, \dots, R_g)$ belong to the same class. It is easy to check that $J(\mathcal{R})$ can be identified with the set of the classes $[\mathcal{D}]$ of the divisors $\mathcal{D}$ on $\mathcal{R}$ of degree g. Equivalently, it can be proved that $J(\mathcal{R})$ can be identified as the quotient space of the set of all the divisors on $\mathcal{R}$ of degree 0, modulo the set of the divisors of meromorphic functions on $\mathcal{R}$ (see [146] for more details concerning this last remark).

Now we define the Riemann theta function. Let Z be any g-period matrix (i.e., any symmetric complex $g \times g$ matrix with positive imaginary part). Define $\Theta : \mathbb{C}^g \to \mathbb{C}$ as follows:

$$\Theta(s_1, \dots, s_g) = \Theta(s) = \sum_{t \in \mathbb{Z}^g} \exp\bigl\{\pi \langle t, Zt\rangle + 2\pi \mathrm{i} \langle t, s\rangle\bigr\},$$

where $(s_1, \dots, s_g) \in \mathbb{C}^g$. The function Θ has many important properties; we list some of them:

(i) $\Theta(-s) = \Theta(s)$;

(ii) $\Theta(s + t) = \Theta(s)$, for every integer column $t \in \mathbb{Z}^g$;

(iii) $\Theta(s + Zt) = \exp\{-\pi \langle t, Zt\rangle - 2\pi \mathrm{i} \langle t, s\rangle\}\Theta(s)$, for every column vector $t \in \mathbb{Z}^g$.

It was a Riemann's fundamental idea to compute the zeroes of the function Θ. To this aim, we define the so-called *vector of Riemann constants* $\Delta = (\Delta_1, \dots, \Delta_g) \in \mathbb{C}^g$, where

$$\Delta_k = -\frac{1}{2} z_{kk} + \sum_{j \neq k}^{g} \int_{\alpha_j} \left(\int_{P_*}^{P} w_k \right) w_j \quad (k = 1, \dots, g). \tag{4.81}$$

In order to simplify the notation, if $P \in \mathcal{R}$, we will write

$$w(P) = \int_{P_*}^{P} (w_1, \dots, w_g) \in \mathbb{C}^g.$$

THEOREM 4.41. *Let $s \in \mathbb{C}^g$ be fixed. If the map $F : P \mapsto \Theta(w(P) - s + \Delta)$ does not vanish identically, then it has exactly g zeroes $Q_1, \dots, Q_g \in \mathcal{R}$ such that*

$$s = \sum_{i=1}^{g} \int_{P_*}^{Q_i} (w_1, \dots, w_g) \mod \Lambda^{2g}.$$

Theorem 4.41 provides an answer to the Riemann inversion problem, which consists in associating to each $s \in \mathbb{C}^g$ a divisor $Q_1, \dots, Q_g$ such that $I(Q_1, \dots, Q_g) = s \mod \Lambda^{2g}$.

In fact, when restricted to $J(\mathcal{R})$, the equation $\Theta(s, Z) = 0$ defines the 1-codimensional analytic subvariety Υ_g of the zero-locus of Θ, called the Θ-*divisor*. It can be shown that Υ_g admits a parametrization

$$\Upsilon_g = \left\{ s \in \mathbb{C}^g \;\middle|\; s = \sum_{j=1}^{g-1} \int_{P_*}^{X_j} (w_1, \dots, w_g) - \Delta \bmod \Lambda^{2g} \right\}, \tag{4.82}$$

where X_j $(j = 1, \dots, g-1)$ are arbitrarily chosen points on $\mathcal{R}$ which are different from the ramification points. If, for some $s \in \mathbb{C}^g$, the map $F : P \mapsto \Theta(w(P) - s + \Delta) \equiv 0$, then $s \in \Upsilon_g$ and there exist points $X_1, \dots, X_{g-1} \in \mathcal{R}$ satisfying (4.82). The g-tuple of points $P_*, X_1, \dots, X_g$ defines a divisor such that $I(P_*, X_1, \dots, X_{g-1}) = s \bmod \Lambda^{2g}$. See [146, Chapter 4, Vol. 2].

Now we go deeper in the examination of the zeroes of Θ. Let $A, B \in \mathcal{R}$, and consider the differential $w_{B,A}$ of the third kind on $\mathcal{R}$, having simple poles at A and B with residues 1 and -1 respectively. Let $s = I(Q_1, \dots, Q_g)$, where $Q_i \in \mathcal{R}$ $(i = 1, \dots, g)$ are different from the ramification points. Applying Theorem 4.41, we see that the zeroes of $F(P) = \Theta(w(P) - s + \Delta)$ are exactly the points Q_i $(i = 1, \dots, g)$. Next, we compute the sum of the residues of the differential $w_{B,A}\, \mathrm{d}\ln F(P)$, to obtain

$$\sum_{i=1}^{g} \int_{P_*}^{Q_i} w_{B,A} = -\frac{\Theta(s + \Delta - w(B))}{\Theta(s + \Delta - w(A))} + c.$$

Another important application of Theorem 4.41 and the parametrization (4.82) is the following fundamental theorem, which provides an answer to the following question: is it possible to express meromorphic functions defined on $\mathcal{R}$ in terms of functions defined on $J(\mathcal{R})$? The answer is positive: first, we use the map $w : P \mapsto \int_{P_*}^{P} (w_1, \dots, w_g)$ which sends the zeroes and the poles of f (which belong to $\mathcal{R}$) into complex g-vectors in $J(\mathcal{R})$, then we use the Riemann Θ-function to express f.

THEOREM 4.42. *Let f be a meromorphic function on $\mathcal{R}$. Let $\mathcal{D} = (f) = P_1 + \dots + P_m - Q_1 - \dots - Q_m$. Let $X_1, \dots, X_{g-1}$ be any $g-1$ points in $\mathcal{R}$ which are different from the ramification points of $\mathcal{R}$ and the points P_i, Q_i $(i = 1, \dots, g)$. Then*

$$f(P) = \delta \prod_{i=1}^{m} \frac{\Theta(w(P) - w(P_i) - \Delta + \sum_{j=1}^{g-1} w(X_j))}{\Theta(w(Q) - w(Q_i) - \Delta + \sum_{j=1}^{g-1} w(X_j))},$$

where $\delta \in \mathbb{C}$ depends only on the choice of the points $X_1, \dots, X_g$.

An easy consequence of the above theorem is the following

THEOREM 4.43. *Let f be a meromorphic function defined on $\mathcal{R}$, with*

$$(f) = \sum_{i=1}^{m} (P_i - Q_i)$$

(*it is understood that the points* P_i *and* Q_i $(i = 1, \dots, m)$ *are the zeroes and the poles of* f *respectively*). *Let* $S_1, \dots, S_g$ *be* g *points on* $\mathcal{R}$ (*here some of the points* S_i *may coincide with some of the* P_i*'s or* Q_i*'s*). *Then*

$$\prod_{i=1}^{g} f(S_i) = \gamma \prod_{j=1}^{m} \frac{\Theta(I(S_1, \dots, S_g) - w(P_j) - \Delta)}{\Theta(I(S_1, \dots, S_g) - w(Q_j) - \Delta)},$$

where $\gamma \in \mathbb{C}$ *is a constant which does not depend on the choice of the points* $S_1, \dots, S_g$.

The theory we have just outlined is very useful if one has to face problems where only differentials of the first kind occur, but when we have to consider nonholomorphic differentials as well, things become a little more difficult.

We give a method, essentially due to Fay [53], to solve the Riemann inversion problem when in presence of one nonholomorphic differential. This method uses the concept of singular Riemann surface, which leads to the introduction of the generalized Jacobian variety, the generalized Abel map, and the generalized Riemann Θ-function as well. To begin the discussion, let D be the unit circle. We will denote points on D with z.

Following [53], we build a family $\mathfrak{F}$ of Riemann surfaces over D in the following way: let $A \neq B \in \mathcal{R}$. Let U_A and U_B be disjoint neighborhood of A and B respectively, and $z_A : U_A \to D$, $z_B : U_B \to D$ be local coordinates of A and B. Let

$$\begin{aligned} W = \bigl\{(P, z) \mid z \in D,\ P \in \mathcal{R} \setminus (U_A \cup U_B) \text{ or } P \in U_A(U_B) \text{ and} \\ \bigl|z_A(P)\bigr| > |z|\bigl(|z_B(P)| > |z|\bigr)\bigr\}. \end{aligned}$$

Let $\mathcal{S}$ be the surface

$$\mathcal{S}\colon \bigl\{XY = z \mid (X, Y, z) \in D^3\bigr\},$$

and define

$$\mathfrak{F} = W \cup \mathcal{S},$$

where the points $(P_A, z) \in (W \cap U_A) \times D$ are identified with $(z_A(P_A), z/z_A(P_A), z) \in \mathcal{S}$ and the points $(P_B, z) \in (W \cap U_B) \times D$ are identified with $(z_B(P_B), z/z_B(P_B), z) \in \mathcal{S}$ $(z \in D)$. We take

$$x = \frac{1}{2}(X + Y) \quad \text{and} \quad y = \frac{1}{2}(X - Y)$$

as coordinates on $\mathcal{S}$. For every fixed $z \in D$, the fiber $\mathcal{R}_z$ over z is a Riemann surface of genus $g + 1$ for which $\mathcal{R}_z \cap \mathcal{S}$ is the surface of $y = \sqrt{x^2 - z}$, having $\pm\sqrt{z}$ as ramification points. The fiber $\mathcal{R}_0$ is a curve of genus $g + 1$ with a double point corresponding to the identification of the points A and B: hence the branches of $\mathcal{R}_0 \cap \mathcal{S}$ are $y = x$ and $y = -x$, with local parameters $x = z_A/2$ and $x = z_B/2$ respectively. Now, choose a canonical basis $\alpha_1(z), \dots, \alpha_g(z), \alpha_{g+1}(z), \beta_1(z), \dots, \beta_g(z), \beta_{g+1}(z)$ for the homology group of

$\mathcal{R}_z$ as follows: $\alpha_1(z), \ldots, \alpha_g(z), \beta_1(z), \ldots, \beta_g(z)$ are the extensions on $\mathcal{R}_z$ of the canonical basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of the homology group of $\mathcal{R}$ (moreover, we may assume that $\alpha_i, \beta_i \in \mathcal{R} \setminus (U_A \cup U_B)$ $(i = 1, \ldots, g)$): then define $\alpha_{g+1}(z) = \partial U_B \times \{z\}$ and, if $|z| < 1/4$, $\beta_{g+1}(z) = l \times \{z\} \cup l_{A,z} \cup l_{B,z} \subset W$, where l is a fixed path from $z_A^{-1}(1/2)$ to $z_B^{-1}(1/2)$ in the (canonically) dissected surface $\mathcal{R}$, $l_{A,z}$ and $l_{B,z}$ are paths from $z_A^{-1}(\sqrt{z})$ to $z_A^{-1}(1/2)$ and from $z_B^{-1}(1/2)$ to $z_B^{-1}(\sqrt{z})$ respectively, which vary continuously with respect to $z \in D$. This choice seems to be a little complicated, but its significance is clear if we note that $\beta_{g+1}(0)$ is simply a path from A to B lying in $\mathcal{R}$.

We have the following

PROPOSITION 4.44. *There are $g + 1$ linearly independent holomorphic 2-forms on $\mathfrak{F}$, having residues $w_1(z), \ldots, w_{g+1}(z)$ on $\mathcal{R}_z$, for z in a sufficiently small disc D_ε centered in 0. These residues are a normalized basis of holomorphic differentials on $\mathcal{R}_z$, for $z \neq 0$. For $z = 0$, the differentials $w_1(0), \ldots, w_g(0)$ form a normalized basis of holomorphic differentials on $\mathcal{R}$, and $w_{g+1}(0)$ is the normalized differential of the third kind $w_{B,A}$, having simple poles at A and B with residues $\pm 1/2\pi\mathrm{i}$ respectively.*

In particular,

$$w_i(z) = w_i + \zeta_i z\omega + \mathcal{O}\big(z^2\big), \quad i = 1, \ldots, g,$$

where $\zeta_i \in \mathbb{C}$ is a constant $(i = 1, \ldots, g)$, ω is the normalized differential of the second kind on $\mathcal{R}$ having double poles of residue zero at A and B, w_i is the i-th differential of the first kind in $\mathcal{R}$, and $\mathcal{O}(z^2)$ is an expression involving only the holomorphic differentials on $\mathcal{R}$.

Moreover

$$w_{g+1}(z) = w_{B,A} + z\tilde{\omega} + \mathcal{O}\big(z^2\big),$$

where $\tilde{\omega}$ is a differential of the second kind on $\mathcal{R}$ having as poles the points A and B, each of order 3.

PROOF. We use without explanations various concepts of algebraic geometry, which are amply discussed by Griffiths and Harris [65]. Let $\mathcal{W}_{\mathfrak{F}}$ and $\mathcal{W}_z$ be the sheafs of holomorphic 2-forms on $\mathfrak{F}$ and on $\mathcal{R}_z$ respectively. Under the residue map $r_z : \mathcal{W}_{\mathfrak{F}} \to \mathcal{W}_z$, the quotient space $\mathcal{W}_{\mathfrak{F}}/\mathcal{R}_z$ is the sheaf consisting, for $z = 0$, of the holomorphic differentials on $\mathcal{R}$, plus the differentials having simple poles at A and B with opposite residues. It turns out that, for every $z \in D$, the cohomology group $H^0(\mathcal{R}_z, \mathcal{W}_{\mathfrak{F}}/\mathcal{R}_z)$ has dimension $g + 1$. Using methods from Fay ([53], Proposition 3.1, page 37), this fact implies that there is a sufficiently small disc D_ε centered at $z = 0$, and differential 2-forms $W_1(z), \ldots, W_{g+1}(z)$ on $\mathfrak{F}$, whose first g residues along $z = 0$ give a normalized basis $w_1, \ldots, w_g$ of holomorphic differentials on $\mathcal{R}$. For $z = 0$, the $(g + 1)$-th residue gives the differential of the third kind $w_{B,A}$ on $\mathcal{R}$, having simple poles at A and B with residues $\pm 1/2\pi\mathrm{i}$ respectively. By continuity, for $z \in D_\varepsilon$, the residue map computed at the differential forms $W_i(z)$ $(i = 1, \ldots, g + 1)$, provides a normalized basis $w_1(z), \ldots, w_{g+1}(z)$ of differentials of the first kind on $\mathcal{R}_z$.

Now, let x be the coordinate in a neighborhood $U \subset \mathcal{R}_z \cap \mathcal{S}$ of the double point; then, for every $i = 1, \dots, g+1$, set

$$w_i(z) = \sum_{j=0}^{\infty} a_{i,j}(z) x^j \, \mathrm{d}x + \sum_{n=0}^{\infty} b_{i,n}(z) \frac{x^n}{\sqrt{x^2 - z}} \, \mathrm{d}x,$$

where $a_{i,j}(z)$ and $b_{i,n}(z)$ are holomorphic functions near $z = 0$ for every $j, n \in \mathbb{N}$ and $i = 1, \dots, g+1$. Then, for $i = 1, \dots, g$,

$$w_i = w_i(0) = \sum_{j=0}^{\infty} a_{i,j}(0) x^j \, \mathrm{d}x \pm \sum_{n=0}^{\infty} b_{i,n}(0) x^{n-1} \, \mathrm{d}x,$$

where the sign of the second summand on the right-hand side depends on whether we move in a neighborhood of A or B (where, in fact, the local coordinate is x or $-x$ respectively). If we let $w_i(z) = w_i(x, z) dx$, for every $i = 1, \dots, g+1$, then $b_{i,0}(0) = 0$, $w_i(A, 0) = a_{i,0}(0) + b_{i,1}(0)$, $w_i(B, 0) = a_{i,0}(0) - b_{i,1}(0)$, etc. Moreover

$$\lim_{z \to 0} \frac{w_i(z) - w_i}{z} = \sum_{j \geqslant 0} a'_{i,j}(0) x^j \, \mathrm{d}x \pm \sum_{n \geqslant 0} \left(\frac{1}{2} b_{i,n+1}(0) + x b'_{i,n}(0) \right) x^{n-2} \, \mathrm{d}x$$

is a differential of the second kind on $\mathcal{R}$ whose only double poles are A and B, with opposite principal parts. For $i = g+1$, we have $w_{g+1}(0) = w_{B,A}$, hence $b_{g+1,0}(0) = -1/2\pi\mathrm{i}$, and thus the limit

$$\begin{aligned} \lim_{z \to 0} \frac{w_{g+1}(z) - w_{B,A}}{z} = {} & \sum_{j \geqslant 0} a'_{g+1,j}(0) x^j \, \mathrm{d}x \\ & \pm \sum_{n \geqslant 0} \left(\frac{b_{g+1,n}(0)}{2x} + x b'_{g+1,n}(0) \right) x^{n-2} \, \mathrm{d}x \end{aligned}$$

is a normalized differential of the second kind on $\mathcal{R}$, having only poles at A and B, each of order 3. The proof follows easily, if we put $\zeta_i = \frac{w_i(A,0) - w_i(B,0)}{4}$. $\square$

The following theorem is fundamental.

THEOREM 4.45. *With the $g+1$-tuple of the normalized basis of holomorphic differentials on $\mathcal{R}_z$ ordered as $(w_{g+1}(z), w_1(z), \dots, w_g(z))$, the period matrix for the Riemann surface $\mathcal{R}_z$ is the $(g+1) \times (g+1)$ matrix*

$$Z_z = \begin{pmatrix} \alpha_i + z\sigma_{gi} & z_{ij} + z\sigma_{ij} \\ \ln z + c_1 + c_2 z & \alpha_j + z\sigma_{jg} \end{pmatrix}_{1 \leqslant i,j \leqslant g} + \mathcal{O}(z^2),$$

where c_1 and c_2 are constants, $\alpha_i = \int_A^B w_i$, σ_{ij} are functions involving the holomorphic differentials on $\mathcal{R}$ together with their derivatives, and $Z = (z_{ij})$ is the $g \times g$ period matrix for $\mathcal{R}$.

PROOF. The proof follows easily from Proposition 4.44, except for the entry $z_{g+1,1}$ of Z_z. However, this follows directly from Proposition 4.44 as well, since $z_{g+1,1} - \ln z$ is a well defined holomorphic function of the variable z in the punctured disc $D_\varepsilon \setminus \{0\}$, and it must be holomorphic in the disc D_ε as well, otherwise $\Im Z_z$ would not be positive definite as $z \to 0$. □

Following Fay again [53], let Λ_z^{2g+2} be the $\mathbb{Z}$-lattice spanned by all vectors of the form

$$\int_c \bigl(w_{g+1}(z), w_1(z), \ldots, w_g(z)\bigr) \quad \bigl(c = \alpha_1(z), \ldots, \alpha_{g+1}(z), \beta_1(z), \ldots, \beta_{g+1}(z)\bigr).$$

From the above theorem, one can easily prove that two vectors $s, s_1 \in \mathbb{C}^{g+1}$ differ by a vector in Λ_z^{2g+2} if and only if, as $z \to 0$, they differ by a vector in the $\mathbb{Z}$-lattice Λ_0^{2g+1} of rank $2g+1$, generated by the rows of the matrix $\left(\begin{smallmatrix} I \\ Z_0 \end{smallmatrix}\right)$, where I is the $(g+1) \times (g+1)$ identity matrix, $Z_0 = \left(\begin{smallmatrix} C & Z \\ 0 & 0 \end{smallmatrix}\right)$, and C is the transpose of the vector $\int_A^B (w_1, \ldots, w_g)$. Equivalently, two vectors $s, s_1 \in \mathbb{C}^{g+1}$ satisfy $s - s_1 = \Lambda_z^{2g+2} t$ $(t \in \mathbb{Z}^{2g+2})$, if and only if there exists $t_0 \in \mathbb{Z}^{2g+1}$ such that $s - s_1 = \Lambda_0^{2g+1} t_0 + \mathcal{O}(z)$.

Now, we let $\mathcal{G}$ be the family of the $(g+1)$-dimensional manifolds over the disc D and denote by $\tilde{\pi}$ the canonical map $\tilde{\pi} : \mathcal{G} \to D$. If $z \in D \setminus \{0\}$, the fiber $\tilde{\pi}^{-1}(z)$ is the Jacobian variety $J(\mathcal{R}_z) = \mathbb{C}^{g+1}/\Lambda_z^{2g+2}$ of the Riemann surface $\mathcal{R}_z$, while $\tilde{\pi}^{-1}(0)$ is $J_0(\mathcal{R}) = \mathbb{C}^{g+1}/\Lambda_0^{2g+1}$.

In [72] it is shown that $J_0(\mathcal{R})$ is a manifold, and that the canonical map π is analytic. We want to describe $J_0(\mathcal{R})$.

Let P be the double point of $\mathcal{R}_0$; as already observed, P coincides with the identification of the points $A, B \in \mathcal{R}$. Every divisor $\mathcal{D} = \sum_{Q_i \neq P} n_i Q_i + nP$ on $\mathcal{R}_0$ can be lifted to a divisor $\mathcal{D} = \sum_{Q_i \neq P} n_i Q_i + n(A+B)$ on $\mathcal{R}$ (we abuse slightly the notation). Let $\mathcal{B} - \mathcal{D}$ is the divisor of a meromorphic function f on $\mathcal{R}_0$; if we lift $\mathcal{B} - \mathcal{D}$ on $\mathcal{R}$, then we must have $f(B)/f(A) = 1$, and since f is meromorphic on $\mathcal{R}$ as well, Abel's Theorem gives (see [53])

$$\frac{f(B)}{f(A)} = \exp\left\{ \int_{\mathcal{D}}^{\mathcal{B}} v_{B,A} - \sum_{i=1}^{g} \int_A^B k_i w_i \right\} \in \mathbb{C}^*, \tag{4.83}$$

where

$$(k_1, \ldots, k_g) = \int_{\mathcal{D}}^{\mathcal{B}} (w_1, \ldots, w_g) \in \mathbb{C}^g,$$

and we used the notation $\int_{\mathcal{D}}^{\mathcal{B}} (\cdot) = \sum_{i=1}^{p} \int_{D_i}^{B_i} (\cdot)$. The relation (4.83) holds even if f has a zero or a pole at A: in fact, from the identification of A with B, f must have the same zero

or pole at B. Now it is quite natural to associate to a divisor $\mathcal{B}-\mathcal{D}$ of degree 0 on $\mathcal{R}_0$, the equivalence class of the vector

$$\int_{\mathcal{D}}^{\mathcal{B}}(w_{B,A}, w_1, \dots, w_g) \in \mathbb{C}^{g+1} \quad \text{mod } \Lambda_0^{2g+1},$$

and $J_0(\mathcal{R})$ is the group of the equivalence classes of the divisors of degree 0 on $\mathcal{R}_0$, with two divisors $\mathcal{D}$ and $\mathcal{D}'$ identified if $\mathcal{D}-\mathcal{D}'$ is the divisor of a meromorphic function on $\mathcal{R}_0$, i.e. a function satisfying $f(B)/f(A)=1$.

The exact sequence of groups

$$0 \longrightarrow \mathbb{C}^* \xrightarrow{\Phi} J_0(\mathcal{R}) \xrightarrow{\Psi} J(\mathcal{R}) \longrightarrow 0$$

where, for every $r \in \mathbb{C}^*$, $\Phi(r)$ is the class of the divisor of any meromorphic function on $\mathcal{R}$ with $f(B)/f(A)=r$, and Ψ is the projection $J_0(\mathcal{R}) \to J(\mathcal{R})$, gives the fundamental identification

$$J_0(\mathcal{R}) \cong \mathbb{C}^* \times J(\mathcal{R}).$$

We call $J_0(\mathcal{R})$ the *generalized Jacobian variety of* $\mathcal{R}$. If $\tilde{s}=(z_0, z_1, \dots, z_g) \in \mathbb{C}^{g+1}$ is a point in the universal cover of $J_0(\mathcal{R})$, then $\Phi(r)$ is the class of $(\ln r, 0, \dots, 0)$ (mod Λ_0^{2g+1}), and $\Psi(\tilde{s})$ is the class of $(z_1, \dots, z_g) \in \mathbb{C}^g$ in $J(\mathcal{R})$.

The generalized Jacobian $J_0(\mathcal{R})$ is the natural setting for the solution of the Riemann inversion problem when in the presence of a differential of the third kind, as stated in the following

THEOREM 4.46. *Let A and B be two distinct points on $\mathcal{R}$, and let $\mathcal{D}$ be a divisor of degree $g+1$ not containing A and B. Then for every $r \in \mathbb{C}^*$ and $z \in \mathbb{C}^g$, there is a unique divisor $\mathcal{B}$ of degree $g+1$ (not containing A and B) such that*

$$\int_{\mathcal{D}}^{\mathcal{B}}(w_{B,A}, w_1, \dots, w_g) = (\ln r, z_1, \dots, z_g) \in \mathbb{C}^{g+1}/\Lambda_0^{2g+1}. \tag{4.84}$$

Let P_* be a fixed point on $\mathcal{R}_0$ different from the ramification points and the double point of $\mathcal{R}_0$. The map $I_0 : \mathrm{Symm}^{g+1}(\mathcal{R}_0) \to J_0(\mathcal{R})$ defined by

$$I_0(P_0, \dots, P_g) = \sum_{i=0}^{g} \int_{P_*}^{P_i} (w_{B,A}, w_1, \dots, w_g) \quad \text{mod } \Lambda_0^{2g+1}$$

is the generalized Abel map with base point P_*.

We now want to describe a method which uses Θ-functions to express meromorphic functions on the generalized Jacobian. The standard theory must be suitably adapted for our purpose.

Let $\tilde{s} = (z, s) \in \mathbb{C} \times \mathbb{C}^g$. The generalized Θ-function is defined as the map

$$\Theta_0 : \mathbb{C}^{g+1} \to \mathbb{C} : \tilde{s} \mapsto \mathrm{e}^{z/2}\Theta\left(s + \frac{1}{2}q\right) + \mathrm{e}^{-z/2}\Theta\left(s - \frac{1}{2}q\right), \tag{4.85}$$

where Θ is the Riemann Θ-function on $J(\mathcal{R})$, and

$$q = \int_A^B (w_1, \dots, w_g).$$

It is worth noting that Θ_0 has the property that if we shift the vector $\tilde{s}$ by a period vector in the lattice Λ_0^{2g+1} (i.e. by a vector $Z_0 t$, where t is a $\mathbb{Z}^{g+1}$-column), then Θ_0 is multiplied by a simple exponential.

Now, take a point $P \in \mathcal{R}$ different from the ramification points and the points A, B, then consider the function

$$F(P) = \Theta_0\bigl(\tilde{w}(P) - v\bigr),$$

where $v \in \mathbb{C}^{g+1}$,

$$\tilde{w}(P) = \int_{P_*}^P (w_{B,A}, w_1, \dots, w_g),$$

and P_* is any fixed base point. The following holds:

PROPOSITION 4.47. *If the function $F(P)$ does not vanish identically on $\mathcal{R}_0$, then it has exactly $g+1$ zeros* (*possibly with multiplicities*).

PROOF. As in the case of the classical Θ-function, we canonically dissect $\mathcal{R}_0$, and compute

$$\begin{aligned}\frac{1}{2\pi \mathrm{i}} \int_{\partial\mathcal{R}_0} \mathrm{d} \ln F(P) \\ = \frac{1}{2\pi \mathrm{i}} \int_{\partial\mathcal{R}} \mathrm{d} \ln F(P) + \operatorname{Res} \mathrm{d} \ln F(B) + \operatorname{Res} \mathrm{d} \ln F(A).\end{aligned} \tag{4.86}$$

The first integral on the right side of (4.86) is exactly g, while, as a simple computation shows,

$$\operatorname{Res} \mathrm{d} \ln F(B) = \operatorname{Res} \mathrm{d} \ln F(A) = \frac{1}{2},$$

and the proof follows easily. □

The above proposition suggests what has to be done: we have to introduce a generalized Riemann vector $\Delta_0 \in \mathbb{C}^{g+1}$, and prove results analogous to those of the classical theory.

We define Δ_0 in the following way:

$$\Delta_0 = (\Delta_*, \Delta_1) = \left(\sum_{i=1}^{g} \int_{\alpha_i} \left(\int_{P_*}^{P} w_i \right) w_{B,A}, \Delta - \frac{w(B) + w(A)}{2} \right) \in \mathbb{C} \times \mathbb{C}^g,$$

where $\Delta \in \mathbb{C}^g$ is the Riemann vector in $J(\mathcal{R})$, whose coordinates are defined in (4.81). Note that Δ_0 depends only on the choice of the base point P_* and the canonical dissection of $\mathcal{R}_0$.

The natural extension of Theorem 4.41 to the setting of singular Riemann surfaces is the following (for a proof, see [53]):

THEOREM 4.48. *For every fixed $\tilde{s} \in \mathbb{C}^{g+1}$, if the function $\Theta_0(\tilde{w}(P) - \tilde{s} + \Delta_0)$ does not vanish identically, then it has exactly $g + 1$ zeros $P_0, \dots, P_g$ such that*

$$\tilde{s} = I_0(P_0, \dots, P_g).$$

The analytic noncompact subvariety Υ_0 of $J_0(\mathcal{R})$ defined by the equation $\Theta_0(\tilde{s}) = 0$ is the Θ-divisor of the generalized Jacobian variety. In analogy to the case of the Riemann surface $\mathcal{R}$, Υ_0 admits a parametrization which is very similar to that of Eq. (4.82), i.e.,

$$\Upsilon_0\colon \left\{ \tilde{s} = \sum_{j=1}^{g} \tilde{w}(X_j) - \Delta_0 \bmod \Lambda_0^{2g+1} \right\},$$

where X_j $(j = 1, \dots, g)$ are arbitrarily chosen points on $\mathcal{R}$ different from the ramification points and A, B. It is clear that the Θ-divisor is the set of the solutions of the equation

$$e^{z/2} \Theta\left(s + \frac{1}{2} q \right) - e^{-z/2} \Theta\left(s - \frac{1}{2} q \right) = 0. \tag{4.87}$$

It is possible to retrieve the following parametric form for points on the Θ-divisor:

$$\begin{cases} s = I(P_1, \dots, P_g) - \Delta_1, \\ z = -\ln \dfrac{\Theta(s - \Delta - w(A))}{\Theta(s - \Delta - w(B))} + c = -\ln \dfrac{\Theta(s + (w(B) - w(A))/2)}{\Theta(s + (w(A) - w(B))/2)} + c. \end{cases} \tag{4.88}$$

It is clear that (4.88) follows from (4.87), up to an additive constant. We can formulate a generalized version of Theorem 4.43, as follows.

THEOREM 4.49. *Let f be a meromorphic function on $\mathcal{R}_0$ such that $(f) = \mathcal{D} - \mathcal{B}$, where $\mathcal{D} = Q_1 + \dots + Q_m$ and $\mathcal{B} = R_1 + \dots + R_m$ are two divisors of the same degree m. If*

$X_1 + \cdots + X_g$ is a divisor of degree g such that every X_j is different from the points Q_i and R_i and the ramification points of $\mathcal{R}_0$, then

$$f(P) = \delta \prod_{i=1}^{m} \frac{\Theta_0(\tilde{w}(P) - \tilde{w}(Q_i) - \Delta_0 + \sum_{j=1}^{g} \tilde{w}(X_j))}{\Theta_0(\tilde{w}(P) - \tilde{w}(R_i) - \Delta_0 + \sum_{j=1}^{g} \tilde{w}(X_j))},$$

where δ is a constant which depends on the choice of the points $X_1, \dots, X_g \in \mathcal{R}$.

Now, using methods from [146] together with Theorems 4.48 and 4.49, we can prove a generalization of Theorem 4.43, i.e.

THEOREM 4.50. *Let $P_0, \dots, P_g$ be any $(g+1)$-tuple of points of $\mathcal{R}_0$ different from the ramification points and A, B. Let f be a meromorphic function on $\mathcal{R}_0$, and let $(f) = \mathcal{D} - \mathcal{B} = Q_1 + \cdots + Q_m - R_1 - \cdots - R_m$ be the divisor of f. Then*

$$f(P_0)f(P_1)\cdots f(P_g) = \gamma \prod_{i=1}^{m} \frac{\Theta_0(I_0(P_0, \dots, P_g) - \tilde{w}(Q_i) - \Delta_0)}{\Theta_0(I_0(P_0, \dots, P_g) - \tilde{w}(R_i) - \Delta_0)}, \tag{4.89}$$

where γ does not depend on the choice of the points $P_0, \dots, P_g \in \mathcal{R}$.

We have given a general outline of basic facts concerning singular hyperelliptic Riemann surfaces. Although we have considered only surfaces with only one singular point, it is worth noting that all the above reasoning extend to the case of surfaces having more than one singular point as well. Obviously, the definitions must be suitably adapted; but see [53] for a good discussion of this topic.

Now we apply the above results to our particular case. We review developments in [96]. Let us introduce the following differential forms on the Riemann surface $\mathcal{R}$:

$$\upsilon_{r-1} = \frac{\lambda^{r-2} \mathrm{d}\lambda}{k(P)}, \quad \lambda = \pi(P),\ 1 \leqslant r \leqslant g+1.$$

Note that $\upsilon_1, \dots, \upsilon_g$ are holomorphic differentials, while $\upsilon_0 = \frac{\mathrm{d}\lambda}{\lambda k(P)}$ is not holomorphic, but is a differential of the third kind with simple poles at 0^+ and 0^- with residues

$$\frac{1}{k(0^\pm)} = \frac{\pm 1}{\sqrt{\prod_{i=0}^{2g} \lambda_i}}$$

respectively.

Let $\omega \in \Omega$, and let $P_1(t), \dots, P_g(t)$ be the finite poles of $M_{\omega \cdot t}$. Abusing notation slightly, write

$$\upsilon_{r-1}(t) = \sum_{i=1}^{g} \int_{P_*}^{P_i(t)} \upsilon_{r-1}, \quad 1 \leqslant r \leqslant g+1. \tag{4.90}$$

Here P_* is a fixed point of $\mathcal{R}$ different from 0^+ and 0^-.

Differentiating, and simplifying the notation, we obtain

$$v'_{r-1}(t) = \sum_{j=1}^{g} \frac{P_j^{r-2} P'_j}{k(P_j)} = \sum_{j=1}^{g} \frac{P_j^{r-1} P'_j}{P_j k(P_j)}.$$

We have

$$H'(P_r) = \left(\prod_{j=1}^{g} (\lambda - P_j) \right)' \Bigg|_{\lambda = P_r} = -P'_r \prod_{s \neq r} (P_r - P_s). \tag{4.91}$$

Note that

$$\prod_{s \neq r} (P_r - P_s) = \sum_{j=0}^{g-1} \theta^{(r)}_{g-j-1} P_r^j \tag{4.92}$$

where $\theta^{(r)}_{g-j-1}$ is $(-1)^{g-j-1}$ times the $(g-j-1)$-th elementary symmetric function in $(P_1, \ldots, \hat{P}_r, \ldots, P_g)$.

If we let

$$A = \begin{pmatrix} 1 & \theta^{(1)}_1 & \cdots & \theta^{(1)}_{g-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \theta^{(g)}_1 & \cdots & \theta^{(g)}_{g-1} \end{pmatrix},$$

then we can rewrite (4.92) as follows

$$A \begin{pmatrix} \frac{P_1^{g-1} P'_1}{k(P_1) P_1} & \cdots & \frac{P_g^{g-1} P'_g}{k(P_g) P_g} \\ \vdots & \ddots & \vdots \\ \frac{P'_1}{k(P_1) P_1} & \cdots & \frac{P'_g}{k(P_g) P_g} \end{pmatrix} = \operatorname{diag} \left(\frac{(-1)^g [m_-^0 - m_+^0] \prod_{r \neq s} P_s}{p(t) k(0^+)} \right)_{r=1\ldots g}.$$

Multiplying both sides on the right by $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ we obtain

$$A \begin{pmatrix} v'_{g-1} \\ \vdots \\ v'_0 \end{pmatrix} = \begin{pmatrix} \frac{-[m_-^0(t) - m_+^0(t)] \theta^{(1)}_{g-1}}{p(t) k(0^+)} \\ \vdots \\ \frac{-[m_-^0(t) - m_+^0(t)] \theta^{(g)}_{g-1}}{p(t) k(0^+)} \end{pmatrix}.$$

Multiplying by A^{-1} we have

$$\begin{pmatrix} v'_{g-1} \\ \vdots \\ v'_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -\frac{[m_-^0(t)-m_+^0(t)]}{p(t)k(0^+)} \end{pmatrix},$$

hence

$$v'_0(t) = -\frac{[m_-^0(t) - m_+^0(t)]}{p(t)k(0^+)}. \tag{4.93}$$

This implies that

$$\begin{aligned} v_{r-1}(t) &= \sum_{j=1}^{g} \int_{P_*}^{P_j(t)} v_{r-1} \\ &= \begin{cases} c_{r-1}, & r = 2, \dots, g, \\ c_0 - \displaystyle\int_0^x \frac{[m_-^0(s) - m_+^0(s)]}{p(s)k(0^+)}\, \mathrm{d}s, & r = 1. \end{cases} \end{aligned} \tag{4.94}$$

From (4.91) and (4.78), we get the following important formula describing the pole motion

$$P'_r(t) = \frac{(-1)^g k(P_r(t))[m_-^0(t) - m_+^0(t)]\prod_{i=1}^{g} P_i(t)}{p(t)k(0^+)\prod_{s\neq r}(P_r(t) - P_s(t))}, \quad 1 \leqslant r \leqslant q. \tag{4.95}$$

Note that (4.95) states that a pole $P_j(t)$ $(1 \leqslant j \leqslant g)$ has 0 derivative if and only if it attains one of the values $\lambda_{2j-1}, \lambda_{2j}$ in the resolvent interval containing it: when it reaches such a value, then it jumps from one sheet to the other in the Riemann surface $\mathcal{R}$, and begins moving in the opposite direction in the interval.

We see that the quantities $v_0(t), \dots, v_{g-1}(t)$ are known once $p(t) = p(\omega \cdot t)$, $m_+^0(t)$, and $m_-^0(t)$ are given, while $v_g(t)$ is as yet undetermined. At this point we propose to study the pole motion by introducing an appropriate Abel map which sends the pole divisor $\{P_1(t), \dots, P_g(t)\}$ into a generalized Jacobian variety. It will turn out that the lack of an explicit formula for $v_g(t)$ is related to the fact the Abel map sends the pole divisor into Υ_0. Thus, roughly speaking, $v_g(t)$ is determined implicitly by $v_0(t), \dots, v_{g-1}(t)$.

Let $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ be a homology basis on $\mathcal{R}$, satisfying the intersection relations (4.80). Since v_0 is a differential of the third kind on $\mathcal{R}$, the natural setting for studying the pole motion is the surface $\mathcal{R}_0$ obtained, as before, by building the family $\{\mathcal{R}_z, z \in D \setminus \{0\}\}$ of Riemann surfaces, each with double points $0^+, 0^- \in \mathcal{R}$, and then letting $z \to 0$. The surface $\mathcal{R}_0$ has a double point corresponding to the identification of the points $0^\pm \in \mathcal{R}$.

Now, the differentials $v_0, v_1, \dots, v_g$ are not normalized. In order to use instruments from the theory described above, we introduce the normalizing invertible $(g+1)\times(g+1)$ complex matrix

$$C_0 = \begin{pmatrix} c_{00} & c_{01} & \dots & c_{0g} \\ 0 & & & \\ \vdots & & C & \\ 0 & & & \end{pmatrix},$$

such that

$$\begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_g \end{pmatrix} = C_0 \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_g \end{pmatrix}.$$

Here the differentials $w_0, w_1, \dots, w_g$ are the normalized differentials considered before, hence $w_1, \dots, w_g$ are a basis of holomorphic differentials on $\mathcal{R}$; also, w_0 is a differential of the third kind on $\mathcal{R}$, having poles at $0^{\pm}$ with residue $\pm 1/2\pi \mathrm{i}$ respectively, and such that $\int_{\alpha_i} w_0 = 0$, for every $i = 1, \dots, g$.

We are interested in the restriction of the map I_0 (which by abuse of terminology we again call I_0) to the set $\mathrm{Symm}^g(\mathcal{R}_0)$ of unordered g-tuples $\{P_1, \dots, P_g\}$ of points in $\mathcal{R} \setminus \{0^+, 0^-\}$, where $\mathrm{Symm}^g(\mathcal{R}_0)$ is viewed as a subset of $\mathrm{Symm}^{g+1}(\mathcal{R}_0)$ by setting $P_0 = P_*$. In particular, we wish to describe the image $I_0(\mathrm{Symm}^g(\mathcal{R}_0)) \subset J_0(\mathcal{R})$.

Let Θ_0 be the generalized Riemann Θ-function. It is a simple observation that, if $\Theta_0(\tilde{s}) = 0$, then $\Theta_0(\tilde{s} + C_0^{-1}\lambda) = 0$ for all $\lambda \in \Lambda_0$. Define

$$\Upsilon = \left\{ \sum_{i=1}^{g} \int_{P_*}^{P_i} (v_0, v_1, \dots, v_g) \mid P_1, \dots, P_g \in \mathcal{R} \setminus \left\{0^+, 0^-\right\} \right\} \subset \mathbb{C}^{g+1}.$$

Then $\varsigma \in \Upsilon$ if and only if $\Theta_0(C_0^{-1}\varsigma - \Delta_0) = 0$. It follows easily that

$$\Upsilon_0 = \left\{ (\varsigma - C_0\Delta_0) \bmod \Lambda_0 \mid \varsigma \in \Upsilon \right\}.$$

Thus the pole motion $\{P_1(t), \dots, P_g(t)\} \mapsto I_0(P_1(t), \dots, P_g(t)\}$ takes place in the affine translate Υ of Υ_0, and also in Υ_0 itself via the relation $\{P_1(t), \dots, P_g(t)\} \mapsto C_0^{-1}[I_0(P_1(t), \dots, P_g(t))] - \Delta_0$.

Now recall the expression for $2\sqrt{py}$ as in (4.78), i.e.,

$$2\sqrt{p(t)y(t)} = \frac{(-1)^{g+1}[m_-^0(t) - m_+^0(t)] \prod_{i=1}^{g} P_i(t)}{k(0^+)}.$$

If we put

$$\delta(t) = \frac{(-1)^{g+1}[m_-^0(t) - m_+^0(t)]}{2k(0^+)},$$

then

$$\sqrt{p(t)y(t)} = \delta(t) \prod_{i=1}^{g} P_i(t). \tag{4.96}$$

Next, consider the function $P \mapsto \pi(P)$; it is a meromorphic function having a double pole at ∞ and zeros at 0^+ and 0^-. Let $P_0 = P_*$. It follows that (4.96) is the restriction of a well defined meromorphic function defined on $J_0(\mathcal{R})$, to Υ_0. Note that in general this restriction is not any more a meromorphic function on Υ_0. These observations, together with Theorem 4.50, provide the proof of the following important

THEOREM 4.51. *$\sqrt{p(t)y(t)}$ coincides with the restriction of a meromorphic function, defined on $J_0(\mathcal{R})$, to Υ_0.*

In particular

$$\begin{aligned}\sqrt{p(t)y(t)} &= \frac{(-1)^{g+1}[m_-^0(t) - m_+^0(t)]\prod_{i=1}^g \pi(P_i(t))}{2k(0^+)} \\ &= \delta(t)\gamma \frac{\Theta_0(C_0^{-1}[I_0(P_1(t),\ldots,P_g(t))] - \int_{P_*}^{0^+}(w_0,w_1,\ldots,w_g) - \Delta_0)}{\Theta_0^2(C_0^{-1}[I_0(P_1(t),\ldots,P_g(t))] - \int_{P_*}^{\infty}(w_0,w_1,\ldots,w_g) - \Delta_0)} \\ &\quad\times \frac{\Theta_0(C_0^{-1}[I_0(P_1(t),\ldots,P_g(t))] + \int_{P_*}^{0^+}(w_0,w_1,\ldots,w_g) - \Delta_0)}{\Theta_0^2(C_0^{-1}[I_0(P_1(t),\ldots,P_g(t))] - \int_{P_*}^{\infty}(w_0,w_1,\ldots,w_g) - \Delta_0)}.\end{aligned} \tag{4.97}$$

Here γ does not depend on the choice of $P_j(t)$ $(1 \leqslant j \leqslant g)$.

The following implicit relation holds between the components of $(w_0(t),\ldots,w_g(t)) = C_0^{-1}[I_0(P_1(t),\ldots,P_g(t))] = \sum_{i=1}^g \int_{P_*}^{P_i(t)}(w_0,\ldots,w_g)$:

$$\Theta_0\left(\sum_{i=1}^{g}\int_{P_*}^{P_i(t)}(w_0,\ldots,w_g) - \Delta_0\right) = 0.$$

Another way of retrieving formula (4.97) is presented in [156], Section 8. In that paper, formulas for the expression on $J_0(\mathcal{R})$ of all the elementary symmetric functions of g points on $\mathcal{R}$ are obtained, depending on the logarithmic derivative of a well-known Θ-quotient.

We now discuss another approach to the study of the pole motion which is based on the Baker–Akhiezer function $\varphi(t,P)$ of Eqs. (SL_ω). Fix $\omega \in \Omega$. For each $\lambda \in \mathbb{C}$ which is not a pole of $m_+(0,\lambda)$, let $\varphi_+(t,\lambda)$ be the solution of (4.24_ω) which satisfies $\varphi_+(0,\lambda) = 1, \varphi'(0,\lambda) = \frac{m_+(0,\lambda)}{p(0)}$. Similarly, define $\varphi_-(t,\lambda)$ for each $\lambda \in \mathbb{C}$ which is not a pole of $m_-(0,\lambda)$. It turns out that, for each $t \in \mathbb{R}$, these two functions glue together to form a single meromorphic function $\varphi(t,z)$ on the punctured Riemann surface $\mathcal{R} \setminus \{\infty\}$. This

function has an essential singularity at $\infty \in \mathcal{R}$. If $z = \frac{1}{\sqrt{-\lambda}}$ is a local parameter at ∞, then by (4.67):

$$\varphi(t, z) = \exp\left(\frac{-1}{z} \int_0^t \sqrt{\frac{y(s)}{p(s)}}\, \mathrm{d}s\right) f(t, z)$$

where $f(t, \cdot)$ is holomorphic on some disc in the z-plane centered at $z = 0$.

Unfortunately, one cannot apply the Abel Theorem to $\varphi(t, \cdot)$, because it is not meromorphic at ∞. However, following [35], we analyze the differential form $w = d \ln \varphi$. We note the following interesting features of w:

(a) w has a simple pole at $P_i(0)$ with residue -1 $(1 \leqslant i \leqslant g)$.
(b) w has a simple pole at $P_i(t)$ with residue 1 $(1 \leqslant i \leqslant g)$.
(c) w has a pole of order 2 at ∞, and

$$w = \left(\int_0^t \sqrt{\frac{y(s)}{p(s)}}\, \mathrm{d}s\right) \frac{\mathrm{d}z}{z^2} + \cdots$$

near $z = 0$.
(d) φ is single-valued on $\mathcal{R} \setminus \{\infty\}$, and hence $\int_{\alpha_j} w = 2\pi \mathrm{i} m_j$ and $\int_{\beta_j} w = 2\pi \mathrm{i} n_j$ where $m_j, n_j \in \mathbb{Z}$ $(1 \leqslant j \leqslant g)$.

Following the developments in [35], we let η be the differential of the second kind on $\mathcal{R}$ which has a pole of order 2 at ∞ and which is normalized so that $\int_{\alpha_j} \eta = 0$ $(1 \leqslant j \leqslant g)$. As usual, let $w_1, \ldots, w_g$ be the normalized basis of holomorphic differentials on $\mathcal{R}$. Let $w_{P_i(0), P_i(t)}$ be the differential of the third kind on $\mathcal{R}$ having simple poles at $P_i(0)$ and $P_i(t)$ with residues -1 and $+1$ respectively $(1 \leqslant i \leqslant g)$, and normalized in such a way that

$$\int_{\alpha_j} w_{P_i(0), P_i(t)} = 0 \quad (1 \leqslant i, j \leqslant g).$$

This normalization and Riemann's bilinear relation [112] ensure that

$$\int_{\beta_j} w_{P_i(0), P_i(t)} = \int_{P_i(0)}^{P_i(t)} w_j \mod \Lambda^{2g} \quad (1 \leqslant i, j \leqslant g).$$

It is clear now that

$$w = \int_0^t \sqrt{\frac{y(s)}{p(s)}}\, \mathrm{d}s\, \eta + \sum_{i=1}^{g} w_{P_i(0), P_i(t)} + \sum_{j=1}^{g} \gamma_j w_j,$$

where $\gamma_1, \ldots, \gamma_g$ are complex constants. Since $\int_{\alpha_j} w = 2\pi \mathrm{i} m_j$, we see that $\gamma_j = 2\pi \mathrm{i} m_j$ $(1 \leqslant j \leqslant g)$. Since $\int_{\beta_k} w = 2\pi \mathrm{i} n_k$, we see that

$$2\pi \mathrm{i} n_k = -\int_0^t \sqrt{\frac{y(s)}{p(s)}}\,\mathrm{d}s\, U_k + \sum_{i=1}^{g} \int_{P_i(0)}^{P_i(t)} w_k + \sum_{i=1}^{g} 2\pi \mathrm{i} m_j z_{jk},$$

where $U_k = -\int_{\beta_k} \eta$ $(1 \leqslant k \leqslant g)$. We thus find

$$\sum_{i=1}^{g} \int_{P_i(0)}^{P_i(t)} w_k = 2\pi \mathrm{i}\left(n_k - \sum_{j=1}^{g} m_j z_{jk}\right) + \left(\int_0^t \sqrt{\frac{y(s)}{p(s)}}\,\mathrm{d}s\right) U_k \quad (1 \leqslant k \leqslant g).$$

Let us consider the standard Abel map $I : \mathrm{Symm}^g(\mathcal{R}) \to J(\mathcal{R})$. We see that the pole motion $t \to \{P_1(t), \ldots, P_g(t)\}$ is mapped into $J(\mathcal{R})$, and assumes a very suggestive form, i.e.,

$$I\big(P_1(t), \ldots, P_g(t)\big) = c + \left(\int_0^t \sqrt{\frac{y(s)}{p(s)}}\,\mathrm{d}s\right) U \mod \Lambda^{2g} \tag{4.98}$$

where c is a constant vector and $U = (U_1, \ldots, U_g)$.

This approach, however, does not provide a solution of our difficulty concerning the implicit nature of the pole motion. In fact, from (4.79) and Theorem 4.40, we see that the function $\sqrt{\frac{y(t)}{p(t)}}$ depends on the product $\prod_{i=1}^{g} P_i(t)$. Thus (4.98) gives the pole motion implicitly. At this point one can use the change of variable

$$\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} = \prod_{i=1}^{g} P_i(t),$$

as in [2]. With respect to this new variable, the motion $\tilde{t} \to \{P_1(\tilde{t}), \ldots, P_g(\tilde{t})\}$ defines a motion along a straight line in $J(\mathcal{R})$ whose velocity is given by

$$\frac{(-1)^{g+1}[m_-^0(\tilde{t}) - m_+^0(\tilde{t})]}{2p(\tilde{t})k(0^+)}.$$

Again, the implicit relation seen earlier between v_g and $(v_0, \ldots, v_{g-1})$ is reflected here in the relation between t and $\tilde{t}$.

Theorem 4.40 ensures that, if Hypotheses 4.38 hold, then there exist relations of algebro-geometric type for the coefficients $p(t), q(t), y(t)$ and for the Weyl m-functions $m_\pm$ satisfying the Riccati equation (4.25). Going backwards, we seek to determine if a fixed algebro-geometric configuration determines the coefficients of a Sturm–Liouville operator.

To this end, we fix the Riemann surface $\mathcal{R}$ of genus g of the algebraic relation

$$w^2 = -(\lambda - \lambda_0)(\lambda - \lambda_1)\cdots(\lambda - \lambda_{2g})$$

where $\lambda_i \in \mathbb{R}^+$ $(0 \leqslant i \leqslant 2g)$ and are all distinct. Let $\tilde{P}_1, \ldots, \tilde{P}_g$ be points on $\mathcal{R}$ such that $\lambda_{2i-1} \leqslant \tilde{P}_i \leqslant \lambda_{2i}$ $(1 \leqslant i \leqslant g)$. Define

$$H_0(\lambda) = \prod_{i=1}^{g}\bigl(\lambda - \pi(\tilde{P}_i)\bigr).$$

Let $\tilde{\Omega}$ be a compact metric space, let $\{\tilde{\tau}_t \mid t \in \mathbb{R}\}$ be a continuous one-parameter group of homeomorphisms of $\tilde{\Omega}$, and let $\tilde{\nu}$ be a $\{\tilde{\tau}_t\}$-ergodic measure on $\tilde{\Omega}$ such that $\tilde{\Omega}$ is the topological support of $\tilde{\nu}$. Let $p : \tilde{\Omega} \to \mathbb{R}$ and $\mathcal{M}_1 : \tilde{\Omega} \to \mathbb{R}$ be positive bounded continuous functions. Suppose further that the function $\tilde{\omega} \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_1(\tilde{\tau}(\tilde{\omega}))|_{t=0} : \tilde{\Omega} \to \mathbb{R}$ is defined and continuous. Fix $\tilde{\omega} \in \tilde{\Omega}$, and write $p(t) = p(\tilde{\tau}_t(\tilde{\omega}))$, $\mathcal{M}_1(t) = \mathcal{M}_1(\tilde{\tau}_t(\tilde{\omega}))$.

Let $(P_1(t), \ldots, P_g(t))$ be the solution of the following system of first-order differential equations

$$\frac{-P_r'(t)\prod_{s\neq r}(P_r(t) - P_s(t))}{k(P_r(t))} = \frac{(-1)^{g+1}\mathcal{M}_1(t)\prod_{i=1}^{g} P_i(t)}{p(t)k(0^+)} \qquad (1 \leqslant r \leqslant g) \tag{4.99}$$

with initial condition $P_r(0) = \tilde{P}_r$ for every $r = 1, \ldots, g$ (i.e. the analogue of (4.95)), and $k(P)$ is the meromorphic function on $\mathcal{R}$ defined in (4.69). Note that (4.99) implies that the motion of the points P_r $(1 \leqslant r \leqslant g)$ has the following properties:

(1) $P_r'(t)$ attains the value 0 only at the points t such that $P_r(t)$ is a ramification point;
(2) when $P_r(t)$ reaches a ramification point, say at $t = \tilde{t}$, then $P_r(t)$ jumps from a sheet to the other in the Riemann surface $\mathcal{R}$, and begins to move towards the opposite ramification point in the resolvent interval containing it (this can be seen by introducing a parameter $z = \sqrt{\lambda - P_r(\tilde{t})}$ in a neighborhood of the ramification point $P_r(\tilde{t})$ of $\mathcal{R}$).

As before, let $c_i = \pi^{-1}[\lambda_{2i-1}, \lambda_{2i}] \subset \mathcal{R}$ $(1 \leqslant i \leqslant g)$. The product $\tilde{D} = c_1 \times c_2 \times \cdots \times c_g$ is a real analytic g-torus which we view as an embedded submanifold of $\mathrm{Symm}^g(\mathcal{R}_0)$. The differential equations (4.99) define a one-parameter group of homeomorphisms $\{\tau_t\}$ on $\tilde{\Omega} \times \tilde{D}$, as follows: if $\tilde{\omega} \in \tilde{\Omega}$ and $\tilde{d} = (\tilde{P}_1, \ldots, \tilde{P}_g) \in \tilde{D}$, then

$$\tau_t(\tilde{\omega}, \tilde{d}) = \bigl(\tilde{\tau}_t(\tilde{\omega}), d(t)\bigr),$$

where $d(t) = \{P_1(t), \ldots, P_g(t)\}$ is obtained by solving (4.99) with initial value $\tilde{d}$.

Fix $\tilde{\omega} \in \tilde{\Omega}$, $\tilde{d} \in \tilde{D}$. Define a (positive) function $y(t)$ such that

$$2\sqrt{p(t)y(t)} = \frac{(-1)^{g+1}\mathcal{M}_1(t)\prod_{i=1}^{g} P_i(t)}{k(0^+)}. \tag{4.100}$$

Put

$$H(t,\lambda) = \prod_{i=1}^{g} \big[\lambda - \pi\big(P_i(t)\big)\big].$$

Consider the nonstandard Abel Jacobi coordinates (4.90). We repeat the computations after (4.90) with $\mathcal{M}_1(t)$ instead of $m_-^0(t) - m_+^0(t)$. The implicit dependence of $v_g(t)$ with respect to $v_0(t), \dots, v_{g-1}(t)$ is a general fact and does not depend on the particular choice of the surface and the points on it. Thus $v_g(t)$ is a transcendental function of $v_0(t), v_1(t) \dots, v_{g-1}(t)$ such that

$$\varsigma(t) = \sum_{i=1}^{g} \int_{P_*}^{P_i(t)} (v_0, v_1, \dots, v_g) \in \Upsilon,$$

or equivalently, $\Theta_0(C_0^{-1}\varsigma(t) - \Delta_0) = 0$.

Let $D = I_0(\tilde{D}) \subset \Upsilon$. We can identify $\tilde{\Omega} \times \tilde{D}$ with the product space $\tilde{\Omega} \times D \subset \tilde{\Omega} \times \Upsilon$. Let $(\tilde{\omega}, \tilde{d}) \in \tilde{\Omega} \times \tilde{D}$, then let

$$\Omega = \operatorname{cls}\left\{\left(\tilde{\tau}_t(\tilde{\omega}), I_0(0) + \left(\sum_{i=1}^{g} \int_{P_*}^{P_i(t)} (v_0, v_1, \dots, v_g)\right)\right) \,\middle|\, t \in \mathbb{R}\right\},$$

where $I_0(0) = (\sum_{i=1}^{g} \int_{P_*}^{P_i(0)} (v_0, v_1, \dots, v_g))$. The flow $(\tilde{\Omega} \times \tilde{D}, \{\tau_t\})$ induces a flow on Ω which by abuse of notation we denote by $(\Omega, \{\tau_t\})$. Let $\tilde{\omega} \in \tilde{\Omega}$ be a point such that the orbit $\{\tilde{\tau}_t(\tilde{\omega}) \mid \tilde{\omega} \in \tilde{\Omega}\}$ is dense in $\tilde{\Omega}$ (this holds for $\tilde{\nu}$-a.a. $\tilde{\omega} \in \tilde{\Omega}$), then let $\tilde{\pi} : \tilde{\Omega} \times \tilde{D} \to \tilde{\Omega}$ be the projection. The flow $(\Omega, \{\tau_t\})$ is an extension of the flow $(\tilde{\Omega}, \{\tilde{\tau}_t\})$, in the sense that the projection $\tilde{\pi}$ restricted to Ω is a flow homeomorphism of Ω onto $\tilde{\Omega}$. There exists a $\{\tau_t\}$-ergodic measure ν on Ω which is the lift of the $\{\tilde{\tau}_t\}$-ergodic measure $\tilde{\nu}$ on $\tilde{\Omega}$, that is, $\tilde{\pi}(\nu) = \tilde{\nu}$. We give a short proof of this fact. First, we use the Birkhoff ergodic theorem together with a construction of Krylov–Bogoliubov type similar to that used in the proof of Lemma 4.18, to prove that the set $\mathcal{S}$ of all the $\{\tau_t\}$-invariant measures ι on Ω such that $\tilde{\pi}(\iota) = \tilde{\nu}$ is nonempty. Moreover, one can see that $\mathcal{S}$ is compact and convex in the weak-* topology. By the Krein–Mil'man theorem, $\mathcal{S}$ is the hull of its extreme points. It is easy to show that every extreme point ν of $\mathcal{S}$ is $\{\tau_t\}$-ergodic and projects into $\tilde{\nu}$.

If necessary, redefine Ω to be topological support of the measure ν: $\Omega = \operatorname{Supp}\nu$. Fix $\omega = (\tilde{\omega}, d) \in \Omega$. Put

$$k_r(t) = k\big(P_r(t)\big),$$

and define a polynomial $Q(t,\lambda)$ of degree g in λ and with coefficients depending on t, such that

$$Q_i(t) = Q\big(t, \pi\big(P_i(t)\big)\big) = \sqrt{p(t)y(t)}\, k_i(t) \quad (i = 1, \dots, g). \tag{4.101}$$

We also require that the highest term coefficient q_g is

$$q_g(t) = -\frac{(p(t)y(t))'}{4y(t)} \tag{4.102}$$

Define

$$M\big(d(t), P\big) = \frac{\sqrt{p(t)y(t)}\,k(P) + Q(t,\lambda)}{H(t,\lambda)}, \quad \lambda = \pi(P). \tag{4.103}$$

It is clear that

$$M\big(d(t), \sigma(P)\big) = \frac{Q(t,\lambda) - \sqrt{p(t)y(t)}\,k(P)}{H(t,\lambda)},$$

where σ is the hyperelliptic involution.

Define

$$h_+(t, P) = M\big(d(t), P\big) \quad \text{and} \quad h_-(t, P) = M\big(d(t), \sigma(P)\big), \tag{4.104}$$

and note that

$$h_+(t,0) - h_-(t,0) = -\mathcal{M}_1(t). \tag{4.105}$$

Put

$$\mathcal{M}_2(t) = h_+(t,0) + h_-(t,0) = \frac{2Q(t,0)}{H(t,0)}. \tag{4.106}$$

Finally, let

$$q(t) = y(t)\left(\sum_{i=0}^{2g} \lambda_i - 2\sum_{i=1}^{g} P_i(t)\right) + \frac{q_g^2(t)}{p(t)} + q_g'(t). \tag{4.107}$$

Corresponding to every $\omega \in \Omega \subset \tilde{\Omega} \times \Upsilon$, there is a one parameter family of matrices

$$\begin{pmatrix} 0 & 1/p(t) \\ q(t) - \lambda y(t) & 0 \end{pmatrix},$$

where y and q are defined as in (4.100) and (4.107) respectively.

We are ready to prove the following important result

THEOREM 4.52. *Let $\omega \in \Omega$, $\lambda = \pi^{-1}(P) \in \mathbb{C}$ with $\Im\lambda \neq 0$ and let $p(t), y(t)$ and $q(t)$ be continuous functions such that the relations* (4.100) *and* (4.107) *hold. Then $h_\pm(t, P)$ satisfy the Riccati equation*

$$h_\pm' + \frac{1}{p}h_\pm^2 = q - \lambda y. \tag{4.108}$$

PROOF. We will give the proof only for h_+. We want to prove that, differentiating (4.103) with respect to t, we obtain

$$\begin{aligned}
&\big((\sqrt{py})'k + Q'\big)H - (\sqrt{py}\,k + Q)H' \\
&\quad + \frac{1}{p}(\sqrt{py}\,k + Q)^2 - (q - \lambda y)H^2 = 0.
\end{aligned} \tag{4.109}$$

We rearrange (4.109) in the following way

$$\begin{aligned}
&Q'H - QH' + yk^2 + \frac{1}{p}Q^2 - (q - \lambda y)H^2 \\
&\quad = k\bigg(\sqrt{py}H' - (\sqrt{py})'H - \frac{2}{p}\sqrt{py}Q\bigg).
\end{aligned} \tag{4.110}$$

We claim that

$$\begin{aligned}
F(t,\lambda) = {}& \sqrt{p(t)y(t)}\,H'(t,\lambda) - \big(\sqrt{p(t)y(t)}\big)'H(t,\lambda) \\
&- \frac{2}{p(t)}\sqrt{p(t)y(t)}\,Q(t,\lambda) = 0,
\end{aligned} \tag{4.111}$$

Such a F is a polynomial in λ of degree g. It is easy to observe that $F(t,\lambda)$ has the points $\pi(P_i(t))$ as roots for every $i = 1,\dots,g$, hence it has g roots. Moreover the λ^g coefficient of $F(t,\lambda)$ is

$$-\big(\sqrt{p(t)y(t)}\big)' + \frac{2}{p(t)}\sqrt{p(t)y(t)}\,\frac{(p(t)y(t))'}{4y(t)} = 0,$$

hence $F(t,\lambda) = 0$. To conclude the proof, we examine the remaining term in (4.110), i.e.

$$\begin{aligned}
L(t,\lambda) = {}& Q'(t,\lambda)H(t,\lambda) - Q(t,\lambda)H'(t,\lambda) + y(t)k^2(P) \\
&+ \frac{1}{p(t)}Q^2(t,\lambda) - \big(q(t) - \lambda y(t)\big)H^2(t,\lambda), \quad \pi(P) = \lambda.
\end{aligned} \tag{4.112}$$

$L(t,\lambda)$ is polynomial in λ of degree at most $2g+1$. It is easy to observe that for every $i = 1,\dots,g$, $\pi(P_i(t))$ is a root of $L(t,\lambda)$. As before we compute the coefficient α of λ^{2g+1} and β of λ^{2g}: we have

$$\alpha = y - y = 0$$

$$\beta = y(t)\Big(\sum \lambda_i - 2\sum P_i(t)\Big) + \frac{1}{p(t)}q_g^2(t) + q_g'(t) - q(t) = 0,$$

hence $L(t,\lambda)$ has degree at most $2g-1$. If we show that every $\pi(P_i(t))$ is a double root of $L(t,\lambda)$, then the proof is complete.

We observe that $H^2(t,\lambda)$ certainly has all the points $\pi(P_i(t))$ as double zeros, so we consider only

$$V(t,\lambda) = Q'(t,\lambda)H(t,\lambda) - Q(t,\lambda)H'(t,\lambda) + y(t)k^2(P) + \frac{1}{p(t)}Q^2(t,\lambda) = H(t,\lambda)N(t,\lambda), \quad \pi(P) = \lambda \tag{4.113}$$

where

$$N(t,\lambda) = Q'(t,\lambda) - Q(t,\lambda)\frac{H'(t,\lambda)}{H(t,\lambda)} + y(t)\frac{k^2(P)}{H(t,\lambda)} + \frac{Q^2(t,\lambda)}{p(t)H(t,\lambda)}. \tag{4.114}$$

In view of the fact that $F(t,\lambda) = 0$, we can write

$$H'(t,\lambda) = \frac{(p(t)y(t))'}{2p(t)y(t)}H(t,\lambda) + \frac{2}{p(t)}Q(t,\lambda) \tag{4.115}$$

and substituting this expression in (4.113) we have

$$V(t,\lambda) = Q'(t,\lambda)H(t,\lambda) - Q(t,\lambda)\left(\frac{(p(t)y(t))'}{2p(t)y(t)}H(t,\lambda) + \frac{1}{p(t)}Q(t,\lambda)\right) + y(t)k^2(P). \tag{4.116}$$

Differentiating V with respect to t we have

$$\begin{aligned} V'(t,\lambda) &= Q''(t,\lambda)H(t,\lambda) + Q'(t,\lambda)H'(t,\lambda) \\ &\quad - Q'(t,\lambda)\left(\frac{(p(t)y(t))'}{2p(t)y(t)}H(t,\lambda) + \frac{1}{p(t)}Q(t,\lambda)\right) \\ &\quad - Q(t,\lambda)\left(\frac{(p(t)y(t))'}{2p(t)y(t)}H(t,\lambda) + \frac{1}{p(t)}Q(t,\lambda)\right)' + y'(t)k^2(P), \end{aligned} \tag{4.117}$$

and

$$\begin{aligned} V'(t,P_r) &= Q_r'(t)H_r'(t) - \frac{2}{p(t)}Q_r(t)Q_r'(t) - \frac{(p(t)y(t))'}{2p(t)y(t)}H_r'(t)Q_r(t) \\ &\quad + \frac{p'(t)}{p^2(t)}Q_r^2(t) + y'(t)k_r^2(t), \end{aligned} \tag{4.118}$$

since $H_r(t) = H(t,\pi(P_r(t))) = 0$.

Substituting in (4.118) the formulas for H_r' and Q_r obtained from (4.99), (4.100) and (4.101), we see that $V'(t,P_r) = 0$, and we have proved that H divides $V' = HN' + H'N$,

and so H divides $H'N$. Now, $H'(t, P_r) = -P_r'(t)\prod_{s\neq r}(P_r(t) - P_s(t))$, and $H_r'(t) = 0$ if and only if $P_r'(t) = 0$. If $P_r' \neq 0$ for every r then H divides N too, and the proof is complete.

Now, suppose that there is an index j and a value $\tilde{t} \in \mathbb{R}$ such that $P_j'(\tilde{t}) = 0$. In this case we must have necessarily $\sigma(P_j(\tilde{t})) = P_j(\tilde{t})$. In fact, if $\sigma(P_r(\tilde{t})) \neq P_r(\tilde{t})$ for every r, then we see that $P_r'(\tilde{t}) \neq 0$, since the (restricted) Abel map I_0 is a diffeomorphism of $\mathrm{Symm}^g(\mathcal{R}_0)$ onto Υ and π is invertible in a neighborhood of each P_r. If then $\sigma(P_j(\tilde{t})) = P_j(\tilde{t})$ for some $j \in \{1, \dots, g\}$ and $\tilde{t}$, we may approximate the divisor $d(\tilde{t})$ with an arbitrarily near divisor $d_\varepsilon(\tilde{t}) = P_{1,\varepsilon}(\tilde{t}) + \cdots + P_{g,\varepsilon}(\tilde{t})$ such that $\sigma(P_{r,\varepsilon}(\tilde{t})) \neq P_{r,\varepsilon}(\tilde{t})$ for every r. For this divisor, the conclusion of the proof above holds, and, passing to the limit, the same conclusion holds for $d(\tilde{t})$. The proof is complete. $\square$

From Theorem 4.52 we obtain this important

THEOREM 4.53. *The real flow $(\Omega, \{\tau_t\})$ has the property that the spectrum of the family of two-dimensional nonautonomous differential systems of Sturm–Liouville type*

$$\begin{aligned}\begin{pmatrix} u \\ v \end{pmatrix}' &= \begin{pmatrix} 0 & \frac{1}{p(\omega\cdot t)} \\ q(\omega\cdot t) - \lambda y(\omega\cdot t) & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \\ &= \omega(t,\lambda)\begin{pmatrix} u \\ v \end{pmatrix} \quad (\omega \in \Omega)\end{aligned} \tag{SL_ω}$$

does not depend on the choice of $\omega \in \Omega$ and has the form

$$\Sigma(L) = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \cdots \cup [\lambda_{2g}, \infty). \tag{4.119}$$

Moreover, $\beta(\lambda) = 0$, for all $\lambda \in \Sigma(L)$.

PROOF. For $\lambda \in \mathbb{C}$, define a flow $\{\hat{\tau}_t\}$ on $\Omega \times \mathbb{C}^2$ by setting

$$\hat{\tau}_t(\omega, u) = \big(\omega\cdot t, \Phi_\omega(t)u\big),$$

where, for every $\omega \in \Omega$, $\Phi_\omega(t)$ is the fundamental matrix solution for Eq. (SL_ω) such that $\Phi_\omega(0) = \mathrm{Id}$. Clearly, this flow is induced on $\Omega \times \mathbb{P}(\mathbb{C})$ as well.

Given $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$, let $P \in \pi^{-1}(\lambda)$ such that $\Im\lambda\Im k(P) > 0$; since $\omega = (\tilde{\omega}, d) \in \tilde{\Omega} \times \Upsilon$, define $m_+(\tilde{\omega}, d, \lambda) = M(d, P)$ and $m_-(\tilde{\omega}, d, \lambda) = M(d, \sigma(P))$.

By Theorem 4.52, both m_+ and m_- satisfy the Riccati equation (4.25), hence the sections

$$S^\pm(\lambda) = \big\{\big(\lambda, m_\pm(\tilde{\omega}, d, \lambda)\big) \mid (\tilde{\omega}, d) \in \Omega\big\} \subset \Omega \times \mathbb{P}(\mathbb{C})$$

are invariant under the flow $(\Sigma, \{\hat{\tau}_t\})$ for every λ with $\Im\lambda \neq 0$. By continuity, we can extend $S^\pm(\lambda)$ to the real line, keeping them still invariant, and hence we can define $m_\pm$ for every λ with $\Im\lambda = 0$.

Now, take $\lambda \in (\lambda_0, \lambda_1) \cup \cdots \cup (\lambda_{2g}, \infty)$; then it easy to observe that $\Im m_+(\tilde{\omega}, d, \lambda) > 0 > \Im m_-(\tilde{\omega}, d, \lambda)$. It follows that $S^+(\lambda)$ and $S^-(\lambda)$ are distinct. Clearly $\Omega \times \mathbb{P}(\mathbb{R})$ is invariant

under the flow $(\Sigma, \{\hat{\tau}_t\})$. By using Liouville's formula, it is easy to show that every solution of (SL_ω) is bounded, for every $\omega \in \Omega$. Thus $\beta(\lambda) = 0$ and λ belongs to the spectrum of the full-line operator (SL_ω). On the other hand, if $\lambda \notin \Sigma(L)$, then the sections $S^{\pm}(\lambda)$ are real, hence the rotation number $\alpha(\lambda)$ of (SL_ω) is constant. It is clear that the only possibility is that λ belongs to the resolvent set (in fact equations (SL_ω) admit an exponential dichotomy; see Theorem 4.21(iii)).

Thus we proved that the spectrum has the form $\Sigma(L) = [\lambda_0, \lambda_1] \cup \cdots \cup [\lambda_{2g}, \infty)$ for every $\omega \in \Omega$, and that $\beta(\lambda) = 0$ for all $\lambda \in \Sigma(L)$. $\square$

A more detailed study of the flow $(\Omega, \{\tau_t\})$ has been carried out in [95]. It is possible to obtain more information concerning the structure of the flow $(\Omega, \{\tau_t\})$ if the set of endpoints of the intervals which determine the spectrum $\Sigma(L)$ satisfy a certain condition, namely that if $\mathcal{R}$ is the Riemann surface of the algebraic relation $w^2 = -(\lambda - \lambda_0)(\lambda - \lambda_1) \cdots (\lambda - \lambda_{2g})$ and $c_i = \pi^{-1}([\lambda_{2i-1}, \lambda_{2i}])$ $(i = 1, \ldots, g)$, then the vectors $\zeta_i = \int_{c_i} (v_0, \ldots, v_{g-1})$ $(i = 1, \ldots, g)$ are linearly independent in $\mathbb{R}^g$. This condition is satisfied in an open dense subset of the space of endpoints, and there are examples for which the above condition is not fulfilled, at least when g is odd. In any case, if the above condition is satisfied, then one can show that for "almost" every pair of functions $(p(t), \mathcal{M}(t))$, the flow $(\Omega, \{\tau_t\})$ generated by $(p, \mathcal{M})$ laminates into closed subsets Ω_h, where h varies in a quotient group $\mathcal{H}$ of the g-torus. Moreover, each set Ω_h is the topological support of an ergodic measure ν_h on Ω. It turns out that, in general, it is difficult to actually determine $\mathcal{H}$, and hence an exact parametrization of the sets Ω_h. However, if $p(t)$ and $\mathcal{M}(t)$ are Bohr almost periodic, then it is possible to determine the "size" of the quotient group $\mathcal{H}$ by looking at the frequency module of the hull of the pair $(p, \mathcal{M})$ (see [95] for more information on this topic).

5. Genericity of exponential dichotomy

As was stated in Section 1, we will consider the question of the genericity of the exponential dichotomy property for cocycles generated by quasi -periodic, two-dimensional linear systems. Thus, we will consider minimal quasi-periodic flows of Kronecker-type generated on the torus $\mathbb{T}^k$ by a vector of frequencies γ, where $\gamma \in \Gamma \subset \mathbb{R}^k$, the set of frequencies whose components are rationally independent. To be precise we study the minimal flow $\{\tau_t\}$ on $\mathbb{T}^k$ given by $\tau_t(\psi) = \psi + \gamma t$ where $\psi = (\psi_1, \ldots, \psi_k) \in \mathbb{T}^k$ and $\gamma = (\gamma_1, \ldots, \gamma_k) \in \Gamma$. Let us agree to abbreviate exponential dichotomy by E.D. We have that, by the robustness properties of the E.D. (see [29,130]), the set $\mathcal{C}_{\mathrm{ED}} = \{A \in C_0 \mid$ the family of systems $x' = A(\tau_t(\psi))x$ $(\psi \in \mathbb{T}^k)$ admits an E.D.$\}$ is open in the corresponding topology. Here recall that C_0 is the set defined by $C_0 = \{A : \mathbb{T}^k \to \mathrm{sl}(2, \mathbb{R}) \mid A$ is continuous$\}$ in which we consider the sup norm $\|A\| = \sup_{\psi \in \mathbb{T}^k} |A(\psi)|$, where in turn $|\cdot|$ is the sup norm on $\mathrm{sl}(2, \mathbb{R})$ given by $|B| = \sup\{|Bx|_{\mathbb{R}^2}$ with $|x|_{\mathbb{R}^2} = 1\}$. What we will show is the density of the set $\mathcal{C}_{\mathrm{ED}}$ for a residual subset of frequencies $\gamma \in \Gamma$. It will be apparent from the proof that these frequencies are of "Liouville-type". By this we mean rather vaguely that the frequencies are "sufficiently close" to rational numbers; the result we discuss below gives no estimate on the degree of "closeness".

Let $\mathcal{D} = \{(\gamma, A) \in \mathbb{R}^k \times C_0 \mid \text{ the system } x' = A(\tau_t(\psi))x \text{ admits an E.D. over } \mathbb{T}^k\}$. We will prove that the set $\mathcal{D}$ is open and dense in $\mathbb{R}^k \times C_0$. We have already observed that the set $\mathcal{C}_{\mathrm{ED}}$ is open. That $\mathcal{D}$ is open is a consequence of a theorem of Sacker and Sell [137] which is a variant of the results concerning roughness of exponential dichotomies mentioned above [29,130]. What is of interest here is the density result. If we fix a frequency $\gamma \in \Gamma$, we could ask if the set $\mathcal{D}_\gamma = \{A \in C_0 \mid \text{ the system } x' = A(\tau_t(\psi))x \text{ admits an E.D. over } \mathbb{T}^k\}$ is open and dense in C_0. While the openness follows from [29] (also [130,137]), we have that the density of $\mathcal{D}_\gamma$ cannot be proved for all $\gamma \in \Gamma$, but only for a residual set of frequencies.

We will state and prove the exact result below; before doing so, recall that in the Introduction we already noted that if we consider as a compact metric space Ω the circle and as the flow the periodic flow given by the rigid rotation on the circle, we have that the density fails to hold. There are also some other examples in which the density property fails, depending on the topology of the compact metric space Ω and on the nature of the flow (see [75]).

The main result of this section is the following:

THEOREM 5.1. *Consider the set $\mathcal{D} = \{(\gamma, A) \in \mathbb{R}^k \times C_0 \mid$ the system $x' = A(\tau_t(\psi))x$ admits an E.D. over $\mathbb{T}^k\}$. Then the set $\mathcal{D}$ is open and dense in $\mathbb{R}^k \times C_0$.*

We have said that the openness of the set $\mathcal{D}$ in $\mathbb{R}^k \times C_0$ follows from a theorem of Sacker and Sell [137] which we will state in a form convenient for our purposes in Theorem 5.3 below. To prove the density we will proceed by contradiction: supposing that there is an open set $V \subset \mathbb{R}^k \times C_0$ such that, if $(\gamma, A) \in V$, then the corresponding family of systems does not have an E.D. over $\mathbb{T}^k$, we will arrive at a contradiction.

We will give the scheme of the proof of this fact. Before doing so, we recall here some general facts about the properties of the rotation number and the Lyapunov exponent of the systems considered. Consider then the family of equations

$$x' = A\big(\tau_t(\psi)\big)x, \quad x \in \mathbb{R}^2 \tag{5.1}$$

where $t \in \mathbb{R}$, $\psi \in \mathbb{T}^k$, $\tau_t(\psi) = \psi + \gamma t$, $\gamma \in \mathbb{R}^k$ and $A \in C_0$. We can write the coefficient matrix as

$$A = \begin{pmatrix} a & -b+c \\ b+c & -a \end{pmatrix}$$

where $a, b, c : \mathbb{T}^k \to \mathbb{R}$ are continuous functions. As already explained in Section 2, after passing to polar coordinates (r, θ), we can define the rotation number $\alpha = \alpha(\psi, \gamma, A)$ as the following limit

$$\alpha = \lim_{t \to \infty} \frac{\theta(t)}{t}$$

where θ is the solution of the equation for the polar angle given by

$$\theta' = b(\psi \cdot t) - a(\psi \cdot t)\sin 2\theta + c(\psi \cdot t)\cos 2\theta$$

where we set $\psi \cdot t = \psi + \gamma t$. If we consider a vector of frequencies $\gamma \in \Gamma$, we have that the limit defining α exists and depends neither on the choice of the initial value $\theta(0)$ nor on the choice of the element $\psi \in \mathbb{T}^k$ ([89] and Theorem 4.19). On the other hand, if we take a frequency vector $\gamma \notin \Gamma$, that is a vector whose components satisfy linear relations with coefficients in $\mathbb{Q}$, then the rotation number does not depend on the initial value $\theta(0)$, but may depend on $\psi \in \mathbb{T}^k$. In this case we have that the rotation number is constant on the closure of the orbit passing through ψ, for each $\psi \in \mathbb{T}^k$.

As already mentioned the rotation number has good continuity properties which we now discuss and which will be useful in the proof of the result. If we fix a vector of frequencies $\gamma \in \mathbb{R}^k$, then using the basic Krylov–Bogoliubov method for the construction of invariant measures, one can prove that α is "very" continuous when the matrix A varies. In fact, consider a topological space $\mathcal{E}$ and a continuous map $\mathcal{E} \to C_0$: $E \mapsto A_E$. We have then that $(\psi, E) \mapsto \alpha(\psi, A_E) : \mathbb{T}^k \times \mathcal{E} \to \mathbb{R}$ is jointly continuous with respect to $\psi \in \mathbb{T}^k$ and $E \in \mathcal{E}$. Moreover, there is a sort of continuity with respect to the frequency: suppose that $\gamma \in \Gamma$, and consider a sequence of frequencies $\{\gamma_n\} \subset \mathbb{R}^k$ such that $\gamma_n \to \gamma$ as $n \to \infty$. Observe that it is not required that γ_n belongs to Γ. Then one has, considering a continuous map $E \mapsto A_E$: $\mathcal{E} \to C_0$, that

$$\alpha(\psi, \gamma_n, A_E) \to \alpha(\psi, \gamma, A_E) \equiv \alpha(\psi, A_E) \tag{5.2}$$

when the convergence is uniform on compact subsets of $\mathbb{T}^k \times \mathcal{E}$. Observe that the last equality follows from the fact that, when $\gamma \in \Gamma$, α does not depend on the frequency. The convergence (5.2) can be proved using the Krylov–Bogoliubov method already mentioned. See [113] and [89].

The following theorem relates the presence of an E.D. for a (real) perturbation of the family (5.1) to the monotonicity of the rotation number.

THEOREM 5.2. *Let $A \in C_0$ and consider the family of equations*

$$x' = \big[A\big(\tau_t(\psi)\big) + JE\big]x \tag{5.3}$$

where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $E \in \mathbb{R}$ and $\psi \in \mathbb{T}^k$. Let $I = (E_1, E_2) \subset \mathbb{R}$ be an open interval. Suppose that $\gamma \in \Gamma$, then the family of Eqs. (5.3) *admits an E.D. over $\mathbb{T}^k$ for all $E \in I$ if and only if the rotation number $\alpha(E)$ is constant on I.*

The proof of the theorem can be found in [59]; see Theorem 4.21. Remember that the rotation number also depends on the ergodic measure defined in our compact metric space, which in the present case is the normalized Lebesgue measure on $\mathbb{T}^k$.

The rotation number of Eq. (5.3), for E fixed and $\psi \in \mathbb{T}^k$, defines a function $E \to \alpha(E)$ which is a monotone, nondecreasing nonnegative function which increases exactly on the spectrum of the corresponding linear operator $L_\psi := J^{-1}[\frac{\mathrm{d}}{\mathrm{d}t} - A(\tau_t(\psi))]$ for each $\psi \in \mathbb{T}^k$. This operator, called the AKNS operator, is viewed as un unbounded self-adjoint operator on the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^2)$. There is an important relation between the dichotomy property and the spectral properties of the operator L_ψ: a complex number E belongs to the resolvent of the operator L_ψ if and only if the corresponding differential system

$x' = [A(\tau_t(\psi)) + JE]x$ admits an E.D. See [79,59] and Proposition 4.15 together with the succeeding discussion.

In the proof of Theorem 5.1, we will perturb our family of equations and arrive at a family of periodic systems. For this reason we recall here some facts and properties of periodic ordinary differential operators which we will use later.

Consider a summable function $a:[0,T) \to \mathrm{sl}(2,\mathbb{R})$; we can view $a(\cdot)$ as a T-periodic function on the whole real axis by extending it in the natural way. Consider then the corresponding periodic differential operator $L = J^{-1}[\frac{\mathrm{d}}{\mathrm{d}t} - a(t)]$ on the space $L^2(\mathbb{R},\mathbb{C}^2)$. Using Floquet theory and extending the arguments of [104], one can obtain the following facts:

1) The spectrum Σ of the self-adjoint operator L is a union (finite or countably infinite) of nondegenerate closed intervals.
2) Consider the rotation number $\alpha(E)$ of the periodic system $x' = [a(t) + JE]x$. Let $I \subset \mathbb{R}$ be a resolvent interval of L; i.e., I is a maximal subinterval of $\mathbb{R} \setminus \Sigma$. Then there exists a unique integer l such that $E \in \operatorname{cls} I$ if and only if $\alpha(E) = \frac{\pi l}{T}$, in particular $\alpha(E)$ is a integer multiple of $\frac{\pi}{T}$.
3) Let $\Phi_E(T)$ be the period matrix of the system $x' = [a(t) + JE]x$ and let $I \subset \mathbb{R}$ be a resolvent interval of L. We have then that for each $E \in I$, there are two linearly independent real eigenvectors $v_\pm$ of $\Phi_E(T)$. (We refer to this as the hyperbolic case). If $\alpha(E) = \frac{2\pi m}{T}$, that is, l is even, with $l = 2m$, $m \in \mathbb{Z}$, then the corresponding eigenvalues $B_\pm$ verify $0 < B_- < 1 < B_+$.

   If E is an endpoint of the interval I, then-up to constant multiple- $\Phi_E(T)$ admits a unique eigenvector v.The corresponding eigenvalue is -1 if the integer l is odd, and is $+1$ if l is even. We refer to this as the parabolic case.
4) Consider E in the interior of the spectrum Σ such that $\alpha(E) = \frac{\pi l}{T}$ for $l \in \mathbb{Z}$. Then we have that $\Phi_E(T) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ if l is odd, while $\Phi_E(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if $l = 2m$ is even. In either case we say that E defines a *closed gap* of L. We refer to this as the elliptic case.
5) Consider a sequence $\{a_n\}$ of T-periodic, $\mathrm{sl}(2,\mathbb{R})$-valued functions such that $\int_0^T |a_n(s) - a(s)|\,\mathrm{d}s \to 0$ as $n \to \infty$. Suppose also that I_n and I are resolvent intervals of the corresponding periodic operators L_n and L where $L_n = J^{-1}[\frac{\mathrm{d}}{\mathrm{d}t} - a_n(t)]$ and $L = J^{-1}[\frac{\mathrm{d}}{\mathrm{d}t} - a(t)]$. Let α_n be the rotation number associated to the operator L_n and suppose that $\alpha(E) = \frac{\pi l}{T}$ if $E \in I$ and $\alpha_n(E) = \frac{\pi l}{T}$ if $E \in I_n$ where l is a fixed integer and $n = 1, 2, \ldots$. Then $I_n \to I$, in the sense that the sequences of endpoints of I_n converge to the endpoints of I, and moreover one has the following result. Let $I_n = (b_n, c_n)$ be a resolvent interval for each $n \geqslant 1$, and suppose that $\overline{E}$ defines a closed gap of the operator L. Suppose also that $\alpha(\overline{E}) = \frac{\pi l}{T} = \alpha_n(E)$ where $E \in I_n, n = 1, 2, \ldots$. Then $b_n \to \overline{E}$ and $c_n \to \overline{E}$ as $n \to \infty$.

Before passing to the proof of Theorem 5.1, we recall some facts about the upper Lyapunov exponent $\beta = \beta(\psi, A)$ of the family of equations $x' = A(\psi + \gamma t)x$. This object is defined by

$$\beta = \lim_{t\to\infty} \frac{1}{t} \log\left\| \Phi_\psi(t) \right\| \tag{5.4}$$

where $\Phi_\psi(t)$ is the fundamental matrix of the equation defined by $\psi \in \mathbb{T}^k$. It is known that, if $\gamma \in \Gamma$, the limit (5.4) exists and takes on a constant value for Lebesgue-a.a. $\psi \in \mathbb{T}^k$. For fixed $A \in C_0$, if the vector of frequencies $\gamma \in \mathbb{Q}^k$, then the limit defining β exists for all $\psi \in \mathbb{T}^k$ and the function $\psi \to \beta(\psi)$ is continuous. This follows from Floquet theory for periodic systems (remember that $\gamma \in \mathbb{Q}^k$). On the other hand, if $\gamma \notin \mathbb{Q}^k$, the limit (5.4) need not exist for *all* $\psi \in \mathbb{T}^k$. So, when we consider the frequency vector $\gamma \in \Gamma$, we refer to this a.e.-constant limit as the upper Lyapunov exponent of the family $x' = A(\psi + \gamma t)x$. Moreover, β is an upper semi-continuous function. More precisely, let $\mathcal{E}$ be a topological space, and let $E \to A_E; \mathcal{E} \to C_0$ be a continuous function. Let $\gamma \in \Gamma$; then the function $E \to \beta(A_E)$ is upper semi-continuous, but is not in general continuous. In our case $A(\cdot) \in \mathrm{sl}(2,\mathbb{R})$, and therefore we have two “ordinary” Lyapunov exponents $\beta_1 = -\beta_2$ with $\beta_2 \geqslant 0$, and $\beta = \beta_2$.

We recall now the robustness properties of the E.D. in a form convenient for our case. Consider the space $\mathcal{L} = L^\infty(\mathbb{R}, \mathrm{sl}(2,\mathbb{R}))$ and consider the weak-* topology in $\mathcal{L}$: a generalized sequence $\{A_n\}$ converges to A in $\mathcal{L}$ if and only if

$$\int_{-\infty}^{\infty} A_n(t)\varphi(t)\,\mathrm{d}t \to \int_{-\infty}^{\infty} A(t)\varphi(t), \quad \forall \varphi \in L^1(\mathbb{R}).$$

By the Alaoglu theorem, norm-bounded subsets of the space $\mathcal{L}$ are precompact in the weak-* topology (they are also metrizable), so, when we consider the hull of $A \in \mathcal{L}$ given by $Y = \mathrm{cls}\{A_s(\cdot) \mid s \in \mathbb{R}\}$ where $A_s(\cdot) = A(\cdot + s)$ is the translation with respect to the time, we obtain a compact and time-translation invariant subset of $\mathcal{L}$. Observe that there is no need to restrict attention to weak-* compact, translation invariant sets Y which are the hull of some $A \in \mathcal{L}$, but rather we can consider Y to be any weak-* compact, translation invariant subset of $\mathcal{L}$.

Now we state the robustness properties of the E.D. concept as follows. Suppose that the family of equations

$$x' = y(t)x, \quad y \in Y \tag{5.5}$$

admits an E.D. over the compact set Y. According the definition this means that there exist constants $c, \eta > 0$ and a projection valued function $P = P_y : Y \to \mathrm{Proj}(\mathbb{R}^2)$ where $\mathrm{Proj}(\mathbb{R}^2) = \{Q \in \mathbb{M}_2 \mid Q^2 = Q\}$, such that, if $\Phi_y(t)$ is the fundamental matrix solution of (5.5), then the following inequalities hold:

$$\begin{aligned} &\left\|\Phi_y(t)P_y\Phi_y(s)^{-1}\right\| \leqslant c\mathrm{e}^{-\eta(t-s)}, \quad t \geqslant s, \\ &\left\|\Phi_y(t)(I - P_y)\Phi_y(s)^{-1}\right\| \leqslant c\mathrm{e}^{\eta(t-s)}, \quad t \leqslant s. \end{aligned}$$

The perturbation theorem for the E.D. [137] states the following

THEOREM 5.3. *Let $Y_0 \subset \mathcal{L}$ a fixed norm-bounded weak-* compact, translation invariant set. Let $Y \subset Y_0$ be a weak-* compact, translation invariant set such that Eqs.* (5.5) *admit*

an E.D. over Y. There exists a weak- neighborhood U of Y in Y_0 so that, if $\tilde{Y} \subset U$ is another weak-* compact and translation invariant set, then the corresponding Eqs.* (5.5) *have an E.D. over $\tilde{Y}$.*

Observe that we consider neighborhoods with respect to a distance defined by a metric compatible with the weak-* topology considered in the space $Y_0 \subset \mathcal{L}$.

From Theorem 5.3 there follows the openness in $\mathbb{R}^k \times C_0$ of the set $\{(\gamma, A) \in \mathbb{R}^k \times C_0 \mid x' = A(\psi + \gamma t)x$ has an E.D. over $\mathbb{T}^k\}$. See [44] for more details.

5.1. *Proof of Theorem 5.1 (Density result)*

In this subsection we will sketch the proof of Theorem 5.1; as already mentioned, the proof is a fairly lengthy argument by contradiction. We start from the following

HYPOTHESIS 5.4. There exists an open subset $V \subset \mathbb{R}^k \times C_0$ such that, if the pair $(\gamma, A) \in V$, then the corresponding differential system

$$x' = A(\psi + \gamma t)x$$

does not have an E.D. over $\mathbb{T}^k$.

Remember that, if the frequency vector $\gamma \in \Gamma$, Hypothesis 5.4 means that E.D. fails for *all* $\psi \in \mathbb{T}^k$. If the vector $\gamma \in \mathbb{R}^k \setminus \Gamma$, Hypothesis 5.4 means that for at least one point $\psi \in \mathbb{T}^k$ the E. D. property does not hold. Further, if there is no E.D. for (at least) one point $\psi \in \mathbb{T}^k$, then E.D. fails to hold for each point in the orbit closure $\mathrm{cls}\{\tau_t(\psi) \mid t \in \mathbb{R}\}$ of the point as well.

We can consider w.l.g. an open set $V \subset \mathbb{R}^k \times C_0$ of the following type: $V = V_1 \times V_2$ where $V_1 \subset \mathbb{R}^k$ and $V_2 \subset C_0$ are open sets. Consider now $\Gamma_1 = \Gamma \cap V_1$. Using a result of [44], see Theorem 5.5 below, we have that there exists a residual subset W_* of $\Gamma_1 \times C_0$ such that, if the pair $(\gamma, A) \in W_*$, then the Lyapunov exponent $\beta(\gamma, A)$ of the corresponding family $x' = A(\psi + \gamma t)x$ equals zero. Let us note that there are recent results of Bochi and Viana [19] which generalize this statement; for our purposes the result of [44] is quite sufficient.

Recall here that the presence of an E.D. for a family of equations implies that the corresponding Lyapunov exponent is positive, but that the positivity of the Lyapunov exponent is not enough to guarantee the presence of an E.D. The exact result (here we are considering the continuous case) proved in [44] is the following

THEOREM 5.5. *Suppose that $0 \leqslant r < 1$ where $r \in \mathbb{R}$ and let $C^r = C^r(\mathbb{T}^k, \mathrm{sl}(2, \mathbb{R}))$ be the set of C^r- mappings from $\mathbb{T}^k$ to $\mathrm{sl}(2, \mathbb{R})$ with the usual topology. There is a residual subset W_* of $\Gamma \times C^r$ with the following property*: *if $w = (\gamma, A) \in W_*$, then either equations $x' = A(\psi + \gamma t)x$ admit an E.D. or the Lyapunov exponent $\beta(w)$ equals zero.*

According to Theorem 5.5, we can then find a frequency $\gamma_* \in \Gamma \cap V_1$ and a function $A_* \in V_2$, such that the corresponding Lyapunov exponent $\beta(\gamma_*, A_*)$ equals zero. From

now on we fix γ_* and A_*. What we are going to do now is to perturb the frequencies of our system of equations to obtain a certain rational frequency vector and a corresponding family of periodic linear systems. It is a fact that the Lyapunov exponent $\beta = \beta(\gamma, A)$ is a semicontinuous function with respect to the coefficient matrix A and with respect to various topologies; in the proof we apply the following result. For a proof we refer to [44].

LEMMA 5.6. *Consider the sequence $\{\gamma_n\} \subset \mathbb{Q}^k$ of frequency vectors with rationally dependent components such that $\gamma_n \to \gamma_*$. Let $A_n \to A_*$ as $n \to \infty$ in C_0. Indicate with $\beta(\psi, \gamma_n, A_n)$ the Lyapunov exponent of the periodic system $x' = A_n(\psi + \gamma_n t)x$. Then $\beta(\psi, \gamma_n, A_n) \to \beta(\gamma_*, A_*) = 0$ where the convergence is uniform with respect to $\psi \in \mathbb{T}^k$.*

After these considerations and remarks we can proceed with the proof of Theorem 5.1; we are considering a pair $(\gamma_*, A_*) \in \Gamma_1 \times V_2$ such that the corresponding system does not have an E.D. and the related Lyapunov exponent $\beta(\gamma_*, A_*) = 0$. Consider now the family of equations

$$x' = \left[A_*(\psi + \gamma_* t) + JE\right]x, \quad E \in \mathbb{R},$$

and its corresponding rotation number $\alpha_*(E)$; we have then that the function $E \to \alpha_*(E)$ is continuous and it strictly increases at $E = 0$ by the monotonicity of the rotation number and by the fact that for $E = 0$ there is no E.D. There exists $E_0 > 0$ such that, if $|E| < 2E_0$, then $A_* + JE \in V_2$. The rotation number is strictly increasing at $E = 0$: precisely, for each $\delta > 0$, $\alpha(\delta) - \alpha(0) > 0$ and $\alpha(0) - \alpha(-\delta) > 0$.

Now, let $\gamma_n \in \mathbb{Q}^k$ be a sequence of frequency vectors with rational coefficients such that $\gamma_n \to \gamma_*$. We choose $\gamma_n = (\frac{p_1^{(n)}}{q_1^{(n)}}, \ldots, \frac{p_k^{(n)}}{q_k^{(n)}})$ and require that each pair $q_i^{(n)}$, $q_j^{(n)}$ of denominators be relatively prime for $1 \leqslant i < j \leqslant k$. This condition implies that, if $T_n = q_1^{(n)} q_2^{(n)} \cdots q_k^{(n)}$, then the flow $\psi + \gamma_n t$ is T_n-periodic. Let $\rho > 0$ a positive number. As proved in [77] (see also [44] and [45]) it is the case that there exists a number $N_0 \geqslant 1$ such that, to each $n \geqslant N_0$, there correspond an integer m and a continuous function $E_n : \mathbb{T}^k \to \mathbb{R}$ such that

(i) The function E_n is constant along each γ_n-orbit, that is

$$E_n(\psi) = E_n(\psi + \gamma_n t) \quad (\psi \in \mathbb{T}^k, 0 \leqslant t \leqslant T_n).$$

(ii) $\|E_n\| < \rho$.

(iii) The rotation number $\alpha_n(\psi)$ of the periodic system

$$x' = \left[A_*(\psi + \gamma_n t) + JE_n(\psi)\right]x \tag{5.6}$$

satisfies

$$\alpha_n(\psi) = \frac{2\pi m}{T_n} \quad (\psi \in \mathbb{T}^k).$$

We obtain E_n by using the monotonicity and continuity properties of the rotation number to arrive to a family of T_n-periodic systems for which the rotation number is an even multiple of $\frac{\pi}{T_n}$. In this case, according to the discussion made in the previous subsection for the periodic differential operator L associated to a T-periodic function with value in the algebra $\mathrm{sl}(2,\mathbb{R})$, only two situations may occur (remember that we are imposing Hypothesis 5.4, so E.D. does not hold over $\mathbb{T}^k$ in the open set $V \subset \mathbb{R}^k \times C_0$).

- If the (periodic) operator $L_\psi = J^{-1}[\frac{\mathrm{d}}{\mathrm{d}t} - A_*(\psi + \gamma_n t)]$ admits a resolvent interval I_ψ wherein the rotation number $\alpha(\psi, \gamma_n, A_* + JE)$ has value $\frac{2\pi m}{T_n}$, then define $E_n(\psi)$ to be the center point of I_ψ.
- If on the other hand there is a number $E_0 \in \mathbb{R}$ which defines a closed gap of L_ψ with $\alpha(\psi, \gamma_n, A_* + JE_0) = \frac{2\pi m}{T_n}$, then set $E_n(\psi) = E_0$. One verifies that E_n is continuous.

Now we continue discussing the proof of Theorem 5.1. Choose a positive number ρ in such a way that the ball in C_0 of center A_* and radius ρ is contained in the open set V_2. Moreover determine N_0, $n \geqslant N_0$, m and E_n as above (for the construction of them see the discussion in [44]) and let $\psi^0 = (0, 0, \ldots, 0)$ be the origin of $\mathbb{T}^k$ with respect to the angular coordinates $(\psi_1, \ldots, \psi_k)$. Write $\bar{A}(\psi) = A_*(\psi) + JE_n(\psi)$ ($\psi \in \mathbb{T}^k$), so that $\bar{A}(\psi) \in V_2$ and for the corresponding periodic system of equations $x' = \bar{A}(\psi + \gamma_n t)x$ E.D. does not hold over $\mathbb{T}^k$, and the corresponding rotation number is given by $\alpha(\psi, \gamma_n, \bar{A}) = \frac{2\pi m}{T_n}$. Adapting a method of Moser [110], as done in [45], we find an $A_0 \in C_0$ with $\|A_0 - A_*\| < \rho$ and such that the corresponding system along the orbit of ψ^0 with frequency γ_n has an E.D. (e.g. such that the system $x' = A_0(\psi^0 + \gamma_n t)x$ admits an E.D.).

We discuss in more detail the method of Moser. Observe that the condition $\alpha(\psi, \gamma_n, \bar{A}) = \frac{2\pi m}{T_n}$ induces a restriction on the behavior of the period matrix $\Phi_\psi(T_n)$ of the (periodic) system $x' = \bar{A}(\psi + \gamma_n t)x$. We reformulate this restriction geometrically: we have already listed, in the previous section , the relations between the rotation number and the spectral properties of a T-periodic differential operator (see (1), ..., (5) on page 253). The facts that the rotation number $\alpha(\psi, \gamma_n, \bar{A}) = \frac{2\pi m}{T_n}$ and the absence of an E.D. for the corresponding periodic system $x' = \bar{A}(\psi + \gamma_n t)x$, imply that $\Phi_\psi(T_n)$ verifies one of the following conditions:

(i) $\Phi_\psi(T_n) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, the identity matrix. This is the elliptic case.

(ii) $\Phi_\psi(T_n)$ has eigenvalue 1 and a unique one-dimensional eigenspace (parabolic case).

We cannot have the hyperbolic case because the system does not admit an E.D. Geometrically these conditions mean that the logarithm of the periodic matrix solution $\Phi_\psi(T_n)$ lies on the surface of a certain cone C whose construction was made and used in [111] for the study of the quasi-periodic Schrödinger operator. Introduce then the following cone C: observe that each matrix $A \in \mathrm{sl}(2,\mathbb{R})$ can be written in the form $\left(\begin{smallmatrix} a & b-c \\ b+c & -a \end{smallmatrix}\right)$ where a, b, c are real numbers. Let C defined by the set $\{(a,b,c) \in \mathbb{R}^3 \mid c^2 = a^2 + b^2\}$, then C is just the set of matrices A which admit zero as an eigenvalue. We of course identify the triple $(a,b,c) \in \mathbb{R}^3$ with the matrix $\left(\begin{smallmatrix} a & b-c \\ b+c & -a \end{smallmatrix}\right)$. Now, let $D_* \subset \mathrm{SL}(2,\mathbb{R})$ be the set of all matrices Φ which are parabolic, or hyperbolic and have positive eigenvalues. There is an open set $D \subset \mathrm{SL}(2,\mathbb{R})$ which contains D_* and a single-valued, real-analytic branch $\ln : D \to \mathrm{sl}(2,\mathbb{R})$ of the matrix logarithm such that $\ln I = \ln\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Moreover $\ln(D)$ contains the cone C and its exterior $\{(a,b,c) \mid a^2 + b^2 > c^2\}$. Writing this fact in terms of the T-period matrix solution $\Phi_\psi(T)$ one has immediately that:

$\Phi_\psi(T)$ is elliptic if and only if $\ln\Phi_\psi(T) = (0, 0, 0)$;

$\Phi_\psi(T)$ is parabolic if and only if $\ln\Phi_\psi(T)$ lies on the surface of the cone C but is not the vertex $(0, 0, 0)$;

$\Phi_\psi(T)$ is hyperbolic if and only if $\ln\Phi_\psi(T)$ lies outside the cone C.

We have thus that in our case the logarithm $\ln\Phi_\psi(T_n)$ of the period matrix solution $\Phi_\psi(T_n)$ of the periodic system $x' = \bar{A}(\psi + \gamma_n t)x$ lies on the surface of the cone C. It is now intuitively reasonable that we can perturb A to a matrix function $A_0 \in C_0$ such that $x' = A_0(\psi^0 + \gamma_n t)x$ has an E.D. See [43] and [44] for details.

Let us write $\gamma = \gamma_n$ and then consider the rational frequency vector $\gamma = (\frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ for which the properties (i)–(iii) hold. Recall also that each pair q_i, q_j is relatively prime ($1 \leqslant i < j \leqslant k$). The proof of Theorem 5.1 now proceeds as follows. We extend A_0 along the various faces of a "quasi-section" $S \subset \mathbb{T}^k$ to the linear flow on $\mathbb{T}^k$ with frequency vector γ. In the end we obtain a function $A \in C_0$ with $\|A - A_*\| < \rho$ and such that $x' = A(\psi + \gamma t)x$ has an E.D. for all $\psi \in S$ and hence – by the choice of S – for all $\psi \in \mathbb{T}^k$. The presence of an E.D. contradicts the Hypothesis 5.4 and thus Theorem 5.1 is proved.

Let us explain briefly the construction of the "quasi-section" S. Let S be the following set: $S = \{\psi = (\psi_1, \dots, \psi_k) \in \mathbb{T}^k \mid \psi_k = 0 \text{ and } 0 \leqslant \psi_i \leqslant \frac{2\pi}{q_i} (1 \leqslant i \leqslant k-1)\}$. (Recall that $\psi_1, \dots, \psi_k$ are angular coordinates mod 2π on $\mathbb{T}^k$). Observe that the S lies in the hyperplane $\{\psi \in \mathbb{T}^k \mid \psi_k = 0\}$ and that $\psi^0 = (0, 0, \dots, 0)$ is one of its vertices. Moreover $S = \bigcup_{p=0}^{k-1} S_p$ where S_p is the union of all p-dimensional faces of S where each face is a p-dimensional rectangle.

If $k = 2$, the quasi-section S is simply the segment

$$S = \left\{ (\psi_1, \psi_2) \in \mathbb{T}^2 \;\middle|\; 0 \leqslant \psi_1 \leqslant \frac{2\pi}{q_1}, \psi_2 = 0 \right\}$$

and for $k = 3$ the quasi-section S is the rectangle

$$S = \left\{ (\psi_1, \psi_2, \psi_3) \in \mathbb{T}^3 \;\middle|\; 0 \leqslant \psi_1 \leqslant \frac{2\pi}{q_1}, 0 \leqslant \psi_2 \leqslant \frac{2\pi}{q_2}, \psi_3 = 0 \right\}$$

in the ψ_1–ψ_2 plane. We can thus consider $\psi^0 = (0, 0, 0)$ as the lower-left hand vertex of the rectangle S.

Moreover observe that, if $\psi \in \mathbb{T}^k$, then its orbit $\{\psi + \gamma t \mid 0 \leqslant t \leqslant T_n\}$ intersects S. If an orbit intersects S in an interior point (i.e., a point $\psi \in S \setminus \bigcup_{p=0}^{k-2} S_p$), then ψ is the unique point of intersection between the orbit and S. On the other hand, if an orbit intersects S in a noninterior point, then the orbit intersects S in other noninterior points as well. Observe that we have in $\mathbb{T}^k$ the flow defined by the rational frequency γ which permutes the various p faces whose union is S_p. We can then consider noninterior points of S as "boundary" points. Denote the set of such boundary points of S as ∂S.

In the detailed proof carried out in [44] the desired function A for which there is an E.D. for the corresponding system is obtained altering A_* many times to obtain a contradiction with Hypothesis 5.4. First a map from the boundary ∂S into the unit circle S^1 is defined, then, after proving that this map is homotopic to a constant for a suitable perturbation of A_*, this map is extended to the whole of S. Then, using Tietze's theorem, the map is

extended to the space $\mathbb{T}^k$. In the end one obtains a continuous function defined on the entire space $\mathbb{T}^k$ and belonging to the open set V_2 for which the corresponding family of systems admits an E.D.

We include here the details for $k = 3$. In this case the quasi-section is the rectangle

$$S = \left\{ (\psi_1, \psi_2, \psi_3) \in \mathbb{T}^3 \;\Big|\; 0 \leqslant \psi_1 \leqslant \frac{2\pi}{q_1}, 0 \leqslant \psi_2 \leqslant \frac{2\pi}{q_2}, \psi_3 = 0 \right\}$$

in the ψ_1–ψ_2 plane. We already noticed that we can consider $\psi^0 = (0, 0, 0)$ as the lower-left hand vertex of the rectangle S.

We choose $\gamma_n = \gamma$ and A_0 such that the system $x' = A_0(\psi^0 + \gamma t)x$ has an E.D. and we can also assume (by choosing $\gamma = \gamma_n$ even closer to γ_*) that, for each $\psi \in S$, the resolvent interval $I_m(\psi)$ (if such an interval exists) corresponding to the (constant) value $\frac{2\pi m}{T_n}$ of the rotation number has length so small that, if $E : \mathbb{T}^k \to \mathbb{R}$ is a continuous function with $E(\psi) \in I_m(\psi)$ for any $\psi \in \mathbb{T}^k$, then the function $A_0 + JE$ belongs to V_2.

Let ψ^1, ψ^2, ψ^3 be the other vertices of the rectangle S in the $\psi_1 - \psi_2$ plane, where $\psi^1 = (0, \frac{2\pi}{q_2}, 0)$, $\psi^2 = (\frac{2\pi}{q_1}, 0, 0)$. We have then that $S = S_0 \cup S_1 \cup S_2$ where $S_0 = \{\psi^0, \psi^1, \psi^2, \psi^3\}$; $S_1 = \overline{\psi^0\psi^1} \cup \overline{\psi^1\psi^3} \cup \overline{\psi^2\psi^3} \cup \overline{\psi^0\psi^2}$ and S_2 is the interior of the rectangle which has vertices $\{\psi^0, \psi^1, \psi^2, \psi^3\}$. Observe that, when we consider the flow on $\mathbb{T}^3$ generated by the rational frequency γ, the segment $\overline{\psi^0\psi^1}$ is mapped to $\overline{\psi^2\psi^3}$ and the segment $\overline{\psi^0\psi^2}$ to the segment $\overline{\psi^1\psi^3}$. Let $t_1, t_2 \in [0, T_n)$ be the unique times such that $\tau_{t_i}(\psi^0) = \psi^0 + \gamma t_i = \psi^i$ $(i = 1, 2)$. We have then that $\psi^1 = \tau_{t_1}(\psi^0)$, $\psi^2 = \tau_{t_2}(\psi^0)$, and the other vertex $\psi^3 = \tau_{t_1+t_2}(\psi^0)$.

Before defining the desired function A (which will contradict Hypothesis 5.4), we make a digression to introduce the technique we will use. Let C_T be the set of T-periodic, continuous functions $\mathcal{A} : \mathbb{R} \to \mathrm{sl}(2, \mathbb{R})$. If $\mathcal{A} \in C_T$, consider the system

$$x' = \left[\mathcal{A}(t) + JE\right]x \tag{5.7}$$

where $E \in \mathbb{R}$ and $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Write $\mathcal{A} = \left(\begin{smallmatrix} a & -b+c \\ b+c & -a \end{smallmatrix}\right)$ and after considering the polar angle $\theta = \operatorname{Arg} x$, we have that θ verifies the equation

$$\theta' = \left(E + b(t)\right) - a(t)\sin 2\theta + c(t)\cos 2\theta. \tag{5.8}$$

If $\phi \in \mathbb{R}$, let $v_\phi = \left(\begin{smallmatrix} \cos\phi \\ \sin\phi \end{smallmatrix}\right)$. If $x(t)$ is the solution of Eq. (5.7) with the initial condition $x(0) = v_\phi$, then the solution $\theta(t)$ of Eq. (5.8) such that $\theta(0) = \phi$ is a continuous determination of $\arg x(t)$. Let m a fixed integer. It can be shown that there is a unique $E_m \in \mathbb{R}$ such that the solution $\theta(t, E_m)$ of Eq. (5.8) with $E = E_m$ and verifying the initial condition $\theta(0, E_m) = \phi$ satisfies

$$\theta(T, E_m) - \phi = 2\pi m.$$

Consider now the corresponding periodic operator $L = L_m = J^{-1}[\frac{\mathrm{d}}{\mathrm{d}t} - \mathcal{A}(t)]$ acting on $L^2(\mathbb{R}, \mathbb{C}^2)$. According to the discussion carried out on page 253, one has just two possibilities: either E_m is an element of the closure of a resolvent interval I_m of L, in which case

the rotation number of the operator equals $\frac{2\pi m}{T}$ (parabolic case), or E_m defines a closed gap (elliptic case).

Define now $F^{\phi}(\mathcal{A}) = \mathcal{A} + JE_m$. Then F^{ϕ} is a nonlinear transformation from C_T into C_T which is continuous in the topology of the uniform convergence on $[0, T]$. Observe that, for each $\tilde{A} \in C_T$, v_{ϕ} is an eigenvector of the period matrix of the system $x' = F^{\phi}(\mathcal{A})(t)x$ (which is by definition $x' = [\mathcal{A}(t) + JE_m]x$). Moreover, if the rotation number of the system $x' = \mathcal{A}(t)x$ is already equal to $\frac{2\pi m}{T}$, then the transformation $F^{\phi}(\mathcal{A})$ is obtained from $\mathcal{A}$ by adding a translate JE_m such that $E_m = 0$ if the real number 0 defines a closed gap of L, and $|E_m|$ is no greater than the length of I_m if 0 is in the resolvent interval I_m of L.

We have finally that, for each $\phi \in \mathbb{R}$, the family (depending on the real parameter s) $F_s^{\phi} : C_T \to C_T$ defined by

$$F_s^{\phi}(\mathcal{A}) = \mathcal{A} + sJE_m \quad (0 \leqslant s \leqslant 1) \tag{5.9}$$

is an homotopy joining F^{ϕ} to the identity transformation on C_T. We will obtain the desired A – for which we will have an E.D. – using the functional F^{ϕ}.

Let r be a real parameter varying in the interval $[0, t_1 + t_2] = [0, t_1] \cup [t_1, t_1 + t_2]$. Let P_r be the projection defined by the presence of an E.D. for the translated equation $x' = A_0(\psi^0 + \gamma(t + r))x$ for $r \in [0, t_1 + t_2]$ for the translation invariance of the E.D. property. Extend the parameter r by letting it decrease from $t_1 + t_2$ to t_2, then from t_2 to 0. Consider a parametrization of the boundary ∂S of the section S given by the following map $\psi = \psi(r)$ with $0 \leqslant r \leqslant t_1 + t_2$.

$$\partial S := \begin{cases} \psi\colon [0, t_1] \to [\psi^0, \psi^1] \text{ is linear with } \psi(0) = \psi^0 \text{ and } \psi(t_1) = \psi^1, \\ \psi : [t_1, t_1 + t_2] \to [\psi^1, \psi^3] \text{ is linear with } \psi(t_1) = \psi^1 \\ \quad \text{and } \psi(t_1 + t_2) = \psi^3, \\ \psi : [t_1 + t_2, t_2] \to [\psi^3, \psi^2] \text{ is linear with } \psi(t_1 + t_2) = \psi^3, \\ \quad \psi(t_2) = \psi^2, \\ \psi : [t_2, 0] \to [\psi^2, \psi^0] \text{ is linear with } \psi(t_2) = \psi^2 \text{ and } \psi(0) = \psi^0. \end{cases}$$

Of course we have put $[\psi^0, \psi^1] = \overline{\psi^0\psi^1}, \ldots, [\psi^0, \psi^2] = \overline{\psi^0\psi^2}$ for the segments whose union is the boundary ∂S.

Now, we associate to each value of the parameter r a continuously determined argument $\theta(r)$ of the line in the plane $\mathbb{R}^2$ which is equal to the image of the projection P_r given by the presence of an E.D. Recall that when there is an E.D. the corresponding projection P has a one-dimensional range and a one dimensional kernel which are lines in the plane $\mathbb{R}^2$. For each $\psi \in \partial S$, let $v(\psi)$ be the unit vector in the plane $\mathbb{R}^2$ with polar angle $\theta(r)$ where r corresponds to the point ψ on the boundary ∂S. We have then that the map $v : \partial S \to S^1 : \psi \to v(\psi)$ is homotopic to a constant map. Here we indicate with S^1 the unit circle on the plane.

We define $A_1(\psi) = A_0(\psi) + JE(\psi)$ for $\psi \in \partial S$ as follows: if $\psi \in [\psi^0, \psi^1] \cup [\psi^1, \psi^3]$, let $A'_{\psi}(t) = A_0(\psi + \gamma t)$. Then let $A_{\psi}^{(1)}(t) = F^{\theta(r)}(A'_{\psi})(t)$ where r is the parameter which corresponds to the point ψ and $0 \leqslant t \leqslant T_n$. Finally set $A_1(\psi + \gamma t) = A_{\psi}^{(1)}(t)$ for $0 \leqslant t \leqslant$

T_n. We observe that A_1 is defined on a subset of the torus $\mathbb{T}^3$ containing the whole boundary ∂S. We also have that, if $\psi \in [\psi^0, \psi^1] \cup [\psi^1, \psi^3]$, then the solution $x_\psi(t) := x(t)$ of the corresponding (periodic) system $x' = A_\psi^{(1)}(t)x$ with initial condition $x_\psi(0) = v(\psi)$ has the property that $x(T_n)$ is a positive constant multiple of $v(\psi)$.

Next define $v_1(\psi) = v(\psi)$ if $\psi \in [\psi^0, \psi^1] \cup [\psi^1, \psi^3]$. If we consider $\psi \in [\psi^2, \psi^3]$, it follows that, by the properties of the flow generated on $\mathbb{T}^3$ by the rational frequency γ, there exists a unique $\tilde{\psi} \in [\psi^0, \psi^1]$ such that $\psi = \tilde{\psi} + \gamma t_2$. Define $v_1(\psi) = \frac{x_{\tilde{\psi}}(t_2)}{|x_{\tilde{\psi}}(t_2)|}$. Observe that the map $v_1(\psi)$ has values in the unit circle S^1. Moreover, if $\psi \in [\psi^0, \psi^2]$, there exists a unique $\tilde{\psi} \in [\psi^1, \psi^3]$ such that $\psi = \tilde{\psi} - \gamma t_1$ and, as before, we put $v_1(\psi) = \frac{x_{\tilde{\psi}}(-t_1)}{|x_{\tilde{\psi}}(-t_1)|}$. We have thus that the map v_1 is defined on the whole boundary ∂S with values in the unit circle S^1.

Consider now the map $f : \partial S \to C_T$ defined by $f(\psi)(t) = A_0(\psi + \gamma(t + r))$ where r is the parameter value corresponding to the point ψ on the boundary ∂S. Moreover we consider $g : \partial S \to C_T$ defined by $g(\psi)(t) = A_1(\psi + \gamma t)$. Using the contractibility of the space C_T, we have then that the maps f and g are homotopic via a homotopy $H : [0, 1] \times \partial S \to C_T : (s, \psi) \to H(s, \psi)$. This homotopy can be modified in a convenient way. First, using the transformation F^θ where $\phi = \theta$, we can arrange the homotopy in such a way that, for each $0 \leqslant s \leqslant 1$ and each $\psi \in [\psi^0, \psi^1] \cup [\psi^1, \psi^3]$, the corresponding system of equations $x' = H(s, \psi)(t)x$ has a solution $x_\psi(t)$ with $x_\psi(0) = v(\psi)$ and $x_\psi(T_n) = cv(\psi)$ where c is a positive number. Second, we can also impose some translation invariance relations.

(i) If $\psi \in [\psi^2, \psi^3]$ and $\tilde{\psi} \in [\psi^0, \psi^1]$ is the unique point such that $\psi = \tilde{\psi} + \gamma t_2$, then

$$H(s, \psi)(t) = H(s, \tilde{\psi})(t + t_2).$$

(ii) If $\psi \in [\psi^0, \psi^2]$ and if $\tilde{\psi} \in [\psi^1, \psi^3]$ is the unique point such that $\psi = \tilde{\psi} - \gamma t_1$, we arrange that

$$H(s, \psi)(t) = H(s, \tilde{\psi})(t - t_1).$$

Now, using the translation relations one can show that v and v_1 are homotopic as maps from the boundary ∂S to the unit circle S^1. We have hence that v_1 is homotopic to a constant map too and admits a continuous extension to all of S. Indicate with $v_1 : S \to S^1$ such an extension. When $\psi \in \operatorname{int} S$, let θ_ψ be a continuous determination of the polar angle of the unit vector $v_1(\psi)$. We now extend, using Tietze's theorem, the function A_1 to a continuous map (again called A_1) defined on all the torus. Then we redefine $A_1(\psi + \gamma t)$ to be $F^{\theta(\psi)}(A_\psi^{(1)})(t)$ ($\psi \in S,\ 0 \leqslant t \leqslant T_n$). One can arrange that $A_1 \in V_2$. For each $\psi \in S$ (and hence for each ψ in the torus) one has a unit vector $v_1(\psi) \in \mathbb{R}^2$ varying continuously with ψ such that, if $x_\psi(t)$ is the solution of the system $x' = A_1(\psi + \gamma t)x$ with $x_\psi(0) = v_1(\psi)$, then $x_\psi(T_n) = cv_1(\psi)$ where $c > 0$. Now, if $c > 1$ for all $\psi \in \mathbb{T}^3$, then the corresponding system of equations has an E.D. over $\mathbb{T}^3$.

The last thing to do is alter the function A_1 for the last time, using a triangularization technique [50], to arrange that the corresponding Lyapunov exponent $\beta(\psi) > 0$ for all

$\psi \in \mathbb{T}^3$. Equivalently, $c > 1$ for all $\psi \in \mathbb{T}^3$. Now, putting everything together, we have arrived at a contradiction with Hypothesis 5.4 and then Theorem 5.1 is finally proved.

For $k > 3$, an analogous construction can be carried out. One constructs A starting from A_0 in an inductive way, modifying A_0 first on S_0, then on $S_1, S_2, \ldots, S_{k-1}$. There is no essential difficulty in passing from S_p to S_{p+1} for $p \geqslant 2$, for the following reason. Suppose that A has been defined on S_p. One wishes to define A along one of the $p+1$-faces $\mathcal{F}$ making up S_{p+1}. The boundary $\partial\mathcal{F}$ of $\mathcal{F}$ is a topological p-sphere which is a union of p-faces. When one carries out the appropriate analogue of the construction outlined above, one arrives at a crucial step in which one must extend a continuous map $v_1 : \partial\mathcal{F} \to S^1$ to a continuous map (again called v_1) from $\mathcal{F}$ to S^1. If $p \geqslant 2$ this is always possible because v_1 is homotopic to a constant map; see, e.g. Dugundji [37].

As one can see from the arguments given above, it seems quite difficult to obtain the same result considering the Hölder space $C^r(\mathbb{T}^k)$ with $r \geqslant 1$ instead of C_0. In view of the results obtained and the techniques used in the paper [45], we conjecture that Theorem 5.1 can be generalized to the case of $0 < r < 1$ but not to the case $r \geqslant 1$. We have already stated the first result on page 255 (Theorem 5.5), the second is the following [45]:

THEOREM 5.7. *Let $C^r = C^r(\mathbb{T}^k, \mathrm{sl}(2,\mathbb{R}))$ There is a dense subset $P_r \subset W_r = \Gamma \times C^r$ with the following property*: *if $w = (\gamma, A) \in P_r$, then the Lyapunov exponent $\beta = \beta(w)$ of the corresponding family of equations $x' = A(\psi + \gamma t)x$ is strictly positive.*

Both of these results hold for $0 \leqslant r < 1$, and this, together with the nature of their proofs, leads one to suspect that Theorem 5.1 also holds when $0 \leqslant r < 1$.

Observe that, using the Baire category theorem, we can give a "fixed-frequency" version of Theorem 5.1:

COROLLARY 5.8. *There is a residual subset $\Gamma_* \subset \Gamma$ such that, if $\gamma \in \Gamma_*$, then the set $\mathcal{D}_\gamma = \{A \in C_0 \mid x' = A(\psi + \gamma t)x$ has an E.D. over $\mathbb{T}^k\}$ is open and dense in C_0.*

Note first that the set Γ is a Baire space and thus it admits a metric with respect to which it is a complete metric space.

PROOF. The openness of the set $\mathcal{D}_\gamma$ in C_0 follows from Theorem 5.3 for all $\gamma \in \mathbb{R}^k$. It remains to prove the density result. Take a countable dense subset of C_0 given by $\{A_n \mid n = 1, 2, \ldots\}$ and indicate with $B_{j,n}$ the open ball in C_0 centered in A_n with radius $1/j$ where $j = 1, 2, \ldots$. Let $\Gamma_{n,j} = \{\gamma \in \Gamma \mid \text{for each} A \in B_{n,j}$ the system $x' = A(\psi + \gamma t)$ does not have an E.D. over $\mathbb{T}^k\}$. For Theorem 5.3, it follows that each $\Gamma_{n,j}$ is closed in Γ. Moreover Theorem 5.1 implies that each $\Gamma_{n,j}$ has no interior in Γ. If we put $\Gamma_* = \Gamma \setminus \bigcup\{\Gamma_{n,j} \mid n, j \geqslant 1\}$ we have that Γ_* is residual in Γ and it verifies the condition of corollary. □

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CHAPTER 3

Complex Differential Equations

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Contents

HANDBOOK OF DIFFERENTIAL EQUATIONS
Ordinary Differential Equations, volume 4
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Preface

This presentation is mainly devoted to considering growth and value distribution of meromorphic solutions of complex differential equations, both in the complex plane as well as in the unit disc. Our main emphasis is in results and considerations not to be found in the key references of the present field, namely [98] and [107] below. Therefore, results achieved during the last fifteen years or so play a special role, hence in particular recent investigations concerning linear differential equations in the unit disc. Conjectures and open problems as well as comments about defectively understood specific areas have been scattered in the text, hopefully prompting additional interest.

The book reference list [63,74,77,99,141] and [189] includes items known to us that cover, at least partially, our present point of view. Of course, these references include a lot of material not to be found in this presentation, due to lack of space. Additional book references of more specialized type can be found in the reference lists of specific chapters below.

After a short introduction into the local theory of complex differential equations, we first collect essentials about growth and value distribution of solutions of linear differential equations in the complex plane, followed by a survey of the very recent investigations about solutions of linear differential equations in the unit disc. The next chapter, non-linear differential equations in a complex domain, takes about one third of the whole presentation, including Malmquist type theorems, Riccati, Schwarzian, Painlevé and Briot–Bouquet differential equations, algebraic differential equations in general and their connections to differential fields. The final chapter offers a short presentation of algebroid solutions of complex differential equations, a field that definitely needs more studies.

1. Local theory of complex differential equations

In this section, we recall basic results of the local theory of complex differential equations. These results serve as a device to the existence of local solutions to a given differential equation in the complex plane, usually under given initial condition(s). In addition to the local existence, the global existence of solutions in a domain $G \subset \mathbf{C}$ may often be deduced. The solutions to be looked for are typically analytic or meromorphic or some slight extensions of these such as algebroid functions. For the convenience of the reader, we have included some proofs, not immediately available, in this section.

1.1. *Elementary background from several complex variables*

Let $\Omega \subset \mathbb{C}^n$ be an open set, equipped with the topology obtained from the standard Euclidean topology by identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. For a point $(z_1, \dots, z_n) \in \Omega$, let $x_1, \dots, x_{2n}$ denote its corresponding real coordinates, with the proviso that $z_j = x_{2j-1} + ix_{2j}$ for $j = 1, \dots, n$. Consider now a function $f : \Omega \to \mathbb{C}$ continuously differentiable with respect to $x_1, \dots, x_{2n}$. Defining the complex partial derivatives by

$$\frac{\partial f}{\partial z_j} := \frac{1}{2}\left(\frac{\partial f}{\partial x_{2j-1}} - \mathrm{i}\frac{\partial f}{\partial x_{2j}}\right), \tag{1.1.1}$$

$$\frac{\partial f}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial f}{\partial x_{2j-1}} + \mathrm{i}\frac{\partial f}{\partial x_{2j}}\right), \tag{1.1.2}$$

the total differential

$$\mathrm{d}f := \sum_{j=1}^{2n} \frac{\partial f}{\partial x_j}\,\mathrm{d}x_j \tag{1.1.3}$$

may be written in the form

$$\mathrm{d}f = \partial f + \bar{\partial} f := \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\,\mathrm{d}z_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}\,\mathrm{d}\bar{z}_j. \tag{1.1.4}$$

Recall now

DEFINITION 1.1.1. A continuously differentiable function $f : \Omega \to \mathbb{C}$ is *analytic* (*holomorphic*) in Ω, if $\bar{\partial} f = 0$ in Ω.

REMARK. If $n = 1$ and we denote $f = u + \mathrm{i}v$, $z = x + \mathrm{i}y$, Definition 1.1.1 reduces to the familiar Cauchy–Riemann equations in the classical sense.

In the local theory, the following several variables counterpart to the classical Taylor expansion is frequently needed. Using the notion of a *polydisc*

$$B(\beta, r) := \prod_{j=1}^{n} B(\beta_j, r_j), \tag{1.1.5}$$

where $\beta_j \in \mathbb{C}$, $r_j > 0$ for $j = 1, \ldots, n$ and $B(\beta_j, r_j) \subset \mathbb{C}$ is the open disc of radius r_j, centered at β_j, we prove the following theorem, see [85], p. 27:

THEOREM 1.1.2. *Suppose $f : B(\beta, r) \to \mathbb{C}$ is analytic. Then f may be uniquely represented as an absolutely converging power series*

$$f(z_1, \ldots, z_n) = \sum_{k_1,\ldots,k_n=0}^{\infty} a_{k_1,\ldots,k_n}(z_1 - \beta_1)^{k_1} \cdots (z_n - \beta_n)^{k_n}, \tag{1.1.6}$$

converging uniformly in compact subsets of $B(\beta, r)$. If $|f(z_1, \ldots, z_n)| \leqslant M$ in $\prod_{j=1}^{n} B(\beta_j, \rho_j)$, where $0 < \rho_j < r_j$, $j = 1, \ldots, n$, then

$$|a_{k_1,\ldots,k_n}| \leqslant \frac{M}{r_1^{k_1} \cdots r_n^{k_n}}. \tag{1.1.7}$$

By Theorem 1.1.2, it is not too difficult to prove the following theorem (due to Hartogs), which is of frequent (implicit) use in practical considerations of complex differential equations. For the proof of this theorem, see [85], pp. 28–29.

THEOREM 1.1.3. *If $f:\Omega\to\mathbb{C}$ is analytic in each of the complex variables $z_1,\dots,z_n$ separately, when all other variables are given arbitrary fixed values, then f is analytic in the sense of Definition* 1.1.1.

1.2. *Basic local existence and uniqueness results*

The classical existence result for solutions of first order differential equations reads as follows:

THEOREM 1.2.1. *Suppose $f:\Omega\to\mathbb{C}$ in a domain $\Omega\subset\mathbb{C}^2$ is analytic and $(z_0,w_0)\in\Omega$. Then there exists $R>0$ such that the initial value problem*

$$w'(z)=f\bigl(z,w(z)\bigr),\quad w(z_0)=w_0, \tag{1.2.1}$$

admits a unique analytic solution $w(z)$ in $B(z_0,R)$ such that $(z,w(z))\in\Omega$ for all $z\in B(z_0,R)$.

There are several classical ways to prove this theorem. We recall here three of the most standard of these proofs: (1) the method of successive approximations, (2) the Banach fixed point method and (3) the Cauchy majorant method. See [74,77] for more details.

The previous theorem may easily be extended to the corresponding existence and uniqueness result for solutions of systems of first order differential equations:

THEOREM 1.2.2. *Suppose that $f_j:\Omega\to\mathbb{C}$, $j=1,\dots,n$, are analytic in a domain $\Omega\subset\mathbb{C}^{n+1}$ and that $(z_0,w_{1,0},\dots,w_{n,0})\in\Omega$. Suppose also that for some $r>0$,*

$$\overline{B}(z_0,w_{1,0},\dots,w_{n,0};r):=\overline{B}(z_0,r)\times\prod_{j=1}^{n}\overline{B}(w_{j,0},r)\subset\Omega \tag{1.2.2}$$

and that $|f_j(z,w_1,\dots,w_n)|\leqslant M$, $j=1,\dots,n$, for all $(z,w_1,\dots,w_n)\in\overline{B}(z_0,w_{1,0},\dots,w_{n,0};r)$. Then the initial value problem

$$w_j'(z)=f_j\bigl(z,w_1(z),\dots,w_n(z)\bigr),\quad w_j(z_0)=w_{j,0},\quad j=1,\dots,n, \tag{1.2.3}$$

admits a unique analytic solution $(w_1(z),\dots,w_n(z))$ in some neighborhood U of z_0 such that

$$B\left(z_0,r\left(1-\exp\left(-\frac{1}{(n+1)M}\right)\right)\right)\subset U. \tag{1.2.4}$$

REMARK. The previous theorem has been formulated for a proof by the Cauchy majorant method. An interested reader may find a complete proof in [74], pp. 19–22.

A slight modification of the majorant method may be applied to prove that local solutions of a system of linear differential equations remain analytic as long as all coefficients are analytic, see [74], pp. 23–25:

THEOREM 1.2.3. *Let $b_{jk}(z)$, $a_j(z)$, $j,k=1,\dots,n$, be analytic functions in $B(z_0,r)$. Then the initial value problem*

$$\begin{cases} w_j' = \sum_{k=1}^{n} b_{jk}(z)w_k + a_j(z), \quad j=1,\dots,n, \\ w_1(z_0)=w_{1,0},\dots,w_n(z_0)=w_{n,0} \end{cases} \tag{1.2.5}$$

possesses a unique solution $(w_1(z),\dots,w_n(z))$ analytic in $B(z_0,r)$.

Theorem 1.2.3 has two immediate corollaries of frequent use in practical considerations. To formulate these consequences, let $p_1(z),\dots,p_n(z),a(z)$ be analytic in $B(z_0,r)$ and consider the initial value problem

$$\begin{cases} w^{(n)} + p_1(z)w^{(n-1)} + \cdots + p_n(z)w = a(z), \\ w(z_0)=w_{0,0},\, w'(z_0)=w_{1,0},\dots,w^{(n-1)}(z_0)=w_{n-1,0}. \end{cases} \tag{1.2.6}$$

Clearly, Eq. (1.2.6) may be represented as the system of linear differential equations,

$$\begin{cases} w_j' = w_{j+1}, \quad j=1,\dots,n-1, \\ w_n' = -p_n(z)w_1 - \cdots - p_1(z)w_n + a(z). \end{cases} \tag{1.2.7}$$

Therefore, an immediate consequence of Theorem 1.2.3 is

COROLLARY 1.2.4. *The initial value problem* (1.2.6) *possesses a unique solution $w(z)$ analytic in $B(z_0,r)$.*

In particular, this implies the following important global consequence:

COROLLARY 1.2.5. *If the coefficients of a linear differential equation* (1.2.6) *are entire functions, then all solutions of* (1.2.6) *are entire functions as well.*

In the case of non-linear differential equations, the conclusion corresponding to Corollary 1.2.4, i.e. proving that all local solutions remain meromorphic as long as the coefficients are analytic, is a complicated problem in general. Standard cases where global existence of meromorphic solutions can be proved, are the Riccati differential equation and the Painlevé differential equations. Concerning the Riccati differential equation, we get

THEOREM 1.2.6. *Suppose that the coefficients a_0, a_1, a_2 of the Riccati differential equation*

$$w' = a_0(z) + a_1(z)w + a_2(z)w^2 \tag{1.2.8}$$

are analytic in $B(z_0, r)$. Then all local solutions of (1.2.8) *in a simply connected domain $\Omega \subset B(z_0, r)$ admit a meromorphic continuation into $B(z_0, r)$.*

The basic idea to prove this theorem is to fix three distinct complex numbers c_1, c_2, c_3, and consider the following pair of differential equations:

$$\begin{cases} u' = a_0(z)v + a_1(z)u, \\ v' = -a_2(z)u. \end{cases} \tag{1.2.9}$$

By Theorem 1.2.3, (1.2.9) possesses a unique solution (u_j, v_j), $j = 1, 2, 3$, analytic in $B(z_0, r)$ and satisfying the initial conditions

$$u_j(z_0) = c_j, \quad v_j(z_0) = 1, \quad j = 1, 2, 3. \tag{1.2.10}$$

By elementary computation, $w_j = u_j/v_j$, $j = 1, 2, 3$, are three distinct solutions of (1.2.8), meromorphic in $B(z_0, r)$. To complete the proof, it suffices to recall that an arbitrary local solution w of a Riccati differential equation may be represented in terms of these three solutions, or actually in terms of their restrictions to the domain to be considered, see [9], Proposition 2.1. By the standard uniqueness theorem of meromorphic functions, w possesses the required meromorphic continuation.

COROLLARY 1.2.7. *Suppose the coefficients of the Riccati differential equation* (1.2.8) *are entire functions. Then all local solutions of* (1.2.8) *admit a meromorphic continuation into $\mathbb{C}$.*

Concerning the Painlevé differential equations, see Section 4.5.

Cauchy majorant method may easily be applied to prove the classical fact that local solutions depend analytically on initial values and parameters. For simplicity, we restrict ourselves to formulate the result for one parameter only. However, it is no difficulty indeed to permit several (finitely many) parameters in the assertion.

THEOREM 1.2.8. *Consider a parameter-depending initial value problem*

$$w' = f(z, w, \lambda), \quad w(z_0, \lambda_0) = w_0, \tag{1.2.11}$$

where f is analytic in a domain $\Omega \subset \mathbb{C}^3$, containing the point (z_0, w_0, λ_0). Then (1.2.11) *admits a unique analytic solution $w(z, \lambda)$ in a domain $\Delta \subset \mathbb{C}^2$ such that $(z, w(z), \lambda) \in \Omega$ for all $(z, \lambda) \in \Delta$. In particular, the dependence on the parameter λ is analytic in a neighborhood of λ_0.*

For the proof, see [74], pp. 26–27.

1.3. *Some remarks on the behavior of local solutions*

This section is devoted to considering some complements for the preceding basic existence and uniqueness results. We also consider analytic continuation of local solutions along curves on the complex plane and, secondly, possibilities of obtaining some information of solutions around such points in $\mathbb{C}^2$, where $f(z,w)$ has a singularity. Concerning the proofs in this section, see [22].

THEOREM 1.3.1. *Suppose $f:\Omega\to\mathbb{C}$ is analytic in a domain $\Omega\subset\mathbb{C}^2$ and $(z_0,w_0)\in\Omega$. Let further U be a neighborhood of z_0. Then the differential equation*

$$w' = f(z,w) \tag{1.3.1}$$

cannot possess a solution $w:U\setminus\{z_0\}\to\mathbb{C}$ such that w would be analytic in $U\setminus\{z_0\}$ with an essential singularity at z_0.

PROOF. Fix first $\overline{B}(z_0,w_0;r)\subset\Omega$ such that $\overline{B}(z_0,r)\subset U$, and assume that $|f(z,w)|\leqslant M$ in $\overline{B}(z_0,w_0;r)$. Assume then, contrary to the assertion, that w is analytic in $U\setminus\{z_0\}$, with an essential singularity at z_0. By the Casorati–Weierstraß theorem, we may construct a sequence $(z_j)_{j\in\mathbb{N}}$ of distinct points in $B(z_0,r)$ converging to z_0 such that $(w(z_j))_{j\in\mathbb{N}}$ is a sequence of distinct points in $B(w_0,r)$ converging to w_0. Moreover, we may assume that $z_j\neq z_0$, $w(z_j)=w_j\neq w_0$, and that $(z_j,w_j)\in\overline{B}(z_0,w_0;\frac{r}{2})$. By the local existence theorem, the initial value problem

$$w'(z) = f\bigl(z,w(z)\bigr),\quad w(z_j)=w_j \tag{1.3.2}$$

has a unique analytic solution in some neighborhood U_j of z_j such that $B(z_j,\frac{r}{2}(1-\exp(-\frac{1}{2M}))\subset U_j$. Therefore, $w(z)$ equals with this local solution determined by the existence theorem. If $|z_j-z_0|<\frac{r}{2}(1-\exp(-\frac{1}{2M}))$, we conclude that $w(z)$ must be analytic at z_0, a contradiction. □

Actually, the above theorem is a corollary of the following theorem, originally due to Painlevé:

THEOREM 1.3.2. *Suppose that $f:\Omega\to\mathbb{C}$ is analytic in a domain $\Omega\subset\mathbb{C}^2$ and that $(z_0,w_0)\in\Omega$. Let $w(z;z_0,w_0)$ be the unique analytic solution of the initial value problem*

$$w' = f(z,w),\quad w(z_0)=w_0. \tag{1.3.3}$$

Moreover, let $\gamma:[0,1]\to\Omega$ be a Jordan curve such that $\gamma(0)=z_0$, $\gamma(1)=a\neq z_0$, and assume that $w(z;z_0,w_0)$ permits an analytic continuation along the curve γ such that $(\gamma(t),w(\gamma(t);z_0,w_0))\in\Omega$ for all $t\in[0,1)$. Finally, suppose that for a sequence $(z_j)_{j\in\mathbb{N}}$ on the curve γ such that $z_j\to a$, we have $\lim_{j\to\infty}w(z_j;z_0,w_0)=\alpha\in\mathbb{C}$ with $(a,\alpha)\in\Omega$. Then $w(z;z_0,w_0)$ permits an analytic continuation along the curve γ over the end-point a and we have $w(z;z_0,w_0)=w(z;a,\alpha)$ locally in some neighborhood of a.

PROOF. We now fix $r > 0$ such that $\overline{B}(a, \alpha; 2r) \subset \Omega$, assuming that $|f(z, w)| \leqslant M$ in $\overline{B}(a, \alpha; 2r)$. We also choose, as we may by our assumptions, $(\zeta, \omega) \in \Omega$ such that

$$\zeta \in \gamma, \ \zeta \neq a, \quad |\zeta - a| < R := r\left(1 - \exp\left(-\frac{1}{2M}\right)\right) \tag{1.3.4}$$

and that for $\omega := w(\zeta; z_0, w_0)$ we have $|\omega - \alpha| < r$. Clearly, we then have $w(z; z_0, w_0) = w(z; \zeta, \omega)$ in some neighborhood of ζ by the basic existence and uniqueness theorem and by our assumption on the analytic continuation of $w(z; z_0, w_0)$. Since $\overline{B}(\zeta, \omega; r) \subset \overline{B}(a, \alpha; 2r)$ by the triangle inequality, the local solution $w(z; \zeta, \omega)$ must be analytic at least in $B(\zeta, R)$. Since $w(z_j; z_0, w_0) \to \alpha$ as $j \to \infty$, we must have $w(a; \zeta, \omega) = \alpha$. By the uniqueness of local solutions, $w(z; a, \alpha) = w(z; \zeta, \omega)$, implying the second assertion. $\square$

In order to continue our considerations on the analytic continuations of local solutions, we turn our attention to the poles of $f(z, w)$. We say that (z_0, w_0) is a *pole* of $f(z, w)$, provided that $1/f(z, w)$ is analytic in a neighborhood of (z_0, w_0) and that $1/f(z_0, w_0) = 0$. As an example, $f(z, w) := 1/(z + w)$ has a pole at every point $(z_0, -z_0)$, $z_0 \in \mathbb{C}$.

PROPOSITION 1.3.3. *Suppose that $f(z, w)$ has a pole at $(z_0, w_0) \in \mathbb{C}^2$ and that $1/f(z_0, w)$ does not vanish identically. Then the differential equation $w' = f(z, w)$ admits, locally around z_0, a multivalent solution $\widetilde{w}(z)$ such that $\widetilde{w}(z_0) = w_0$.*

PROOF. Consider the initial value problem

$$z'(w) = \frac{1}{f(z(w), w)}, \quad z(w_0) = z_0. \tag{1.3.5}$$

By Theorem 1.2.1, this initial value problem admits a unique analytic solution $z(w)$ locally around z_0. Since $1/f(z, w)$ does not vanish identically, $z(w)$ cannot be a constant. Moreover,

$$z'(w_0) = \frac{1}{f(z(w_0), w_0)} = \frac{1}{f(z_0, w_0)} = 0. \tag{1.3.6}$$

Therefore, the Taylor expansion of $z(w)$ around w_0 takes the form

$$z(w) = z_0 + c_k(w - w_0)^k + \cdots, \quad c_k \neq 0, \ k > 1. \tag{1.3.7}$$

Clearly, by (1.3.7), $z(w)$ admits, locally around z_0, a multivalent inverse function $w(z)$ satisfying the assertion. $\square$

REMARK. The solution $\widetilde{w}$, $\widetilde{w}(z_0) = w_0$ constructed in Proposition 1.3.3 is unique in the following sense: If $\widetilde{w}(z) \to w_0$ as $z \to z_0$ along a Jordan curve γ, and if $z \in \gamma$ is sufficiently close to z_0, then $\widetilde{w}(z)$ must locally coincide around z with a branch of the multivalued solution of $w' = f(z, w)$ in Proposition 1.3.3. Indeed, since $\widetilde{w}'(z) = f(z, \widetilde{w}(z))$

along the curve γ, and (z_0, w_0) is a pole of $f(z, w)$, then $\widetilde{w}'(z) \neq 0$ for any $z \in \gamma$ sufficiently close to z_0. Therefore, $\widetilde{w}(z)$ admits an analytic local inverse around z, satisfying the differential equation in (1.3.5). Clearly, this inverse function may be continued along the curve γ, approaching to z_0. By Theorem 1.3.2, the inverse function must coincide with the $z(w)$ constructed in the proof of Proposition 1.3.3.

REMARK. Before proceeding, we introduce a couple of standard notations. In the case of Proposition 1.3.3, we say that the multi-valued function $w(z)$ has an *algebraic branch point* at z_0. Similarly, we say that z_0 is an *algebraic pole* of w at z_0, if w is multivalent around z_0, $t(z_0) := 1/w(z_0) = 0$, and if the local inverse $z(t)$ of $t(z) := 1/w(z)$ around 0 admits a local expansion of type

$$z(t) = z_0 + c_k t^k + \cdots, \quad c_k \neq 0,\ k > 1. \tag{1.3.8}$$

To complete this section, we consider the equation (1.3.1) in the special case where $f(z, w)$ is either a polynomial or a rational function in w.

THEOREM 1.3.4. *Considering*

$$w' = P(z, w) = \sum_{j=0}^{n} p_j(z) w^j, \quad n \geqslant 2, \tag{1.3.9}$$

assume that the coefficients $p_j(z)$ are analytic in a domain $\Omega \subset \mathbb{C}$. If $a \in \Omega$ and $p_n(a) \neq 0$, for $n > 2$, and if $w(z)$ is a solution of (1.3.9) *analytic on a Jordan curve $\gamma : [0, 1) \to \Omega$ from z_0 to a, then either* (i) *$w(z)$ permits a meromorphic continuation over a, or* (ii) *$w(z)$ has an algebraic pole at a.*

PROOF. Suppose first that $w(z)$ remains bounded on γ as $z \to a$. Then there is a sequence (z_j) on γ converging to a such that $w(z_j)$ converges to some $\alpha \neq \infty$ as $z_j \to a$. By Theorem 1.3.2, $w(z)$ permits an analytic continuation over a. Otherwise, we may find a sequence (z_j) on γ such that $w(z_j) \to \infty$ as $z_j \to a$. By the transformation $v := 1/w$ we obtain

$$v' = Q(z, v) = -\frac{1}{v^{n-2}} \big(p_0(z) v^n + \cdots + p_n(z) \big). \tag{1.3.10}$$

If now $n = 2$, then $Q(z, v)$ is analytic around $(a, 0)$. By the first part of the proof, $v(z)$ has an analytic continuation over a, hence w has a meromorphic continuation over a with a pole at a. Finally, if $n > 2$, then $Q(z, v)$ has a pole at $(a, 0)$ and $1/Q(a, v)$ does not vanish identically. By Proposition 1.3.3, the case (ii) follows. $\square$

THEOREM 1.3.5. *Consider now*

$$w' = \frac{P(z, w)}{Q(z, w)}, \tag{1.3.11}$$

where $P(z,w)$, $Q(z,w)$ are polynomials in w with analytic coefficients in a domain $\Omega \subset \mathbb{C}$. Assume that $a \in \Omega$, that $\deg_w Q \geqslant 1$ *and that $Q(a,w)$ does not vanish identically. Moreover, we assume that $P(a,w)$ and $Q(a,w)$ have no common factors. If now $w(z)$ is a solution of* (1.3.11) *analytic on a Jordan curve $\gamma : [0,1) \to \mathbb{C}$ from z_0 to a in Ω, then either* (i) *$w(z)$ permits a meromorphic continuation over a, or* (ii) *$w(z)$ has an algebraic branch point at a, or* (iii) *$w(z)$ has an algebraic pole at a.*

PROOF. Since $Q(a,w)$ does not vanish identically, it has finitely many zeros as a polynomial of w, say $w_1, \dots, w_k$. By the non-existence of common factors, $P(a,w_j) \neq 0$ for all $j = 1, \dots, k$. Given $\varepsilon > 0$, consider the set

$$H(\varepsilon) := \bigl\{ w \mid |w| \leqslant 1/\varepsilon, |w - w_j| \geqslant \varepsilon, j = 1, \dots, k \bigr\}, \tag{1.3.12}$$

and fix ε_0 sufficiently small. Assume first that for every ε, $0 < \varepsilon < \varepsilon_0$, there exists some $t_\varepsilon \in [0,1)$ such that $w(\gamma(t)) \in \mathbb{C} \setminus H(\varepsilon)$ for all $t \in [t_\varepsilon, 1)$. By continuity of w, we may conclude that only one of the inequalities

$$\bigl| w\bigl(\gamma(t)\bigr) - w_j \bigr| < \varepsilon, \quad j = 1, \dots, k, \quad \bigl| w\bigl(\gamma(t)\bigr) \bigr| > 1/\varepsilon \tag{1.3.13}$$

holds for all $t \in [t_\varepsilon, 1)$, provided ε_0 is small enough. Therefore, we must have either $w(z) \to \infty$ or $w(z) \to w_j$ for some $j = 1, \dots, k$ as $z \to a$ along the curve γ. If $w(z) \to w_j$, we have $Q(a, w_j) = 0$, $P(a, w_j) \neq 0$, while $Q(a,w)$ does not vanish identically. Therefore, by Proposition 1.3.3, we have the case (ii) at hand. If then $w(z) \to \infty$, we substitute $w = 1/v$, resulting in

$$v' = -\frac{P_1(z,v)}{Q_1(z,v)} = v^2 \frac{P(z,1/v)}{Q(z,1/v)}, \tag{1.3.14}$$

and we have two subcases to consider. If $Q_1(a,0) \neq 0$, then $v(z)$ permits an analytic continuation over a, and $v(a) = 0$. Therefore, w has a pole at a, permitting a meromorphic continuation over a. If, on the other hand, $Q_1(a,0) = 0$, then clearly $P_1(a,0) \neq 0$. Moreover, $Q_1(a,v)$ cannot vanish identically, since $Q(a,w)$ does not vanish identically. By Proposition 1.3.3 again, v has an algebraic branch point at a with $v(a) = 0$. But this implies that w has an algebraic pole at a, and we have the case (iii).

It remains to consider the case that for some sequence (t_j) converging to 1 as $j \to \infty$, $w(\gamma(t_j)) \in H(\varepsilon)$. Since $H(\varepsilon)$ is a compact set, we may assume, by taking a subsequence, if needed, that $w(\gamma(t_j))$ converges to some point $w_0 \neq w_1, \dots, w_k, \infty$. Therefore, $P(z,w)/Q(z,w)$ is analytic at (a, w_0). By Theorem 1.3.2, w permits an analytic continuation over the point a. □

1.4. *Local behavior of solutions of $w' = P(z,w)/Q(z,w)$*

In this section, we consider the local behavior of solutions of the differential equation

$$w' = \frac{P(z,w)}{Q(z,w)}, \tag{1.4.1}$$

where $P(z,w)$, $Q(z,w)$ both are analytic in both variables. If the initial value $w(z_0)=w_0$ is to be satisfied, and if $Q(z_0,w_0)\neq 0$, resp. $Q(z_0,w_0)=0$ and $P(z_0,w_0)\neq 0$, we may apply Theorem 1.3.2, resp. Proposition 1.3.3, provided that $Q(z_0,w)/P(z_0,w)$ does not vanish identically. If then $P(z_0,w_0)=Q(z_0,w_0)=0$, the situation becomes more complicated. We may assume that $(z_0,w_0)=(0,0)$, hence we have to consider the differential equation (1.4.1) under the initial value $w(z_0)=w_0$ in the case of

$$\begin{aligned} P(z,w) &= \alpha z+\beta w+\sum_{j+k\geqslant 2} p_{jk}z^j w^k,\\ Q(z,w) &= \gamma z+\delta w+\sum_{j+k\geqslant 2} q_{jk}z^j w^k. \end{aligned} \tag{1.4.2}$$

The basic idea to handle this initial value problem is to normalize the equation by a suitable linear transformation. Indeed, it appears that the linear terms in (1.4.2) are essential for the behavior of solutions of (1.4.1) around $z=0$. In this section, we restrict ourselves to considering two special cases only, leaving aside more general considerations. Concerning the role of non-linear terms, it should perhaps be mentioned, however, that they may have a substantial influence in some cases. As an example, the non-vanishing solutions of

$$w'=\frac{-w+zw^2}{z} \tag{1.4.3}$$

may be expressed, locally around a point $z\neq 0$ close to the origin, in the form

$$w(z)=\frac{1}{z(C-\log z)},\quad C\in\mathbb{C}, \tag{1.4.4}$$

while for

$$w'=-\frac{w}{z} \tag{1.4.5}$$

we have the solutions

$$w(z)=\frac{C}{z},\quad C\in\mathbb{C}. \tag{1.4.6}$$

1.4.1. *The linear case* This section is devoted to considering the non-singular linear case

$$w'=\frac{\alpha w+\beta z}{\gamma w+\delta z},\quad \alpha,\beta,\gamma,\delta\in\mathbb{C},\ \alpha\delta-\beta\gamma\neq 0. \tag{1.4.7}$$

It appears that (1.4.7) is, essentially, equivalent to the following pair of linear differential equations:

$$\begin{cases} w'=\alpha w+\beta z,\\ z'=\gamma w+\delta z. \end{cases} \tag{1.4.8}$$

Indeed, let $w(z)$ be a meromorphic solution of (1.4.8), satisfying the initial condition $w(z_0) = w_0 \neq \infty$. Then, obviously, $\gamma w(z) + \delta z$ does not vanish identically. Fixing now $t_0 \in \mathbb{C}$ arbitrarily, we may solve, at least locally, $z(t)$ from the linear differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \gamma w(z) + \delta z, \quad z(t_0) = z_0. \tag{1.4.9}$$

But then, considering $w = w(t) := w(z(t))$, we have

$$w(t_0) = w\big(z(t_0)\big) = w(z_0) = w_0 \tag{1.4.10}$$

and

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\alpha w(z) + \beta z}{\gamma w(z) + \delta z}\big(\gamma w(z) + \delta z\big) = \alpha w(z) + \beta z. \tag{1.4.11}$$

Conversely, let $(w(t), z(t))$ be a meromorphic solution of (1.4.9), satisfying the finite initial conditions $w(t_0) = w_0$, $z(t_0) = z_0$ for an arbitrarily given t_0. However, we have to assume that $(z_0, w_0) \neq (0, 0)$, since $z_0 = w_0 = 0$ would result in $w(t) \equiv 0$, $z(t) \equiv 0$ by Theorem 1.2.1. Since $\alpha\delta - \beta\gamma \neq 0$, we conclude from (1.4.9) that $|w'(t_0)|^2 + |z'(t_0)|^2 > 0$. Assuming that $z'(t_0) \neq 0$, we infer that the inverse function $t(z)$ is locally defined around z_0. But then

$$\widetilde{w}(z) := w\big(t(z)\big) \tag{1.4.12}$$

satisfies

$$\widetilde{w}(z_0) = w\big(t(z_0)\big) = w(t_0) = w_0 \tag{1.4.13}$$

and

$$\widetilde{w}'(z) = w'\big(t(z)\big) = \frac{\mathrm{d}w}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}z} = \frac{\alpha w + \beta}{\gamma w + \delta}. \tag{1.4.14}$$

Therefore, it means no essential restriction to consider the pair (1.4.8) of differential equations. Denoting

$$\Phi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{1.4.15}$$

we may represent (1.4.9) in vector notation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} w \\ z \end{pmatrix} = \Phi \begin{pmatrix} w \\ z \end{pmatrix}. \tag{1.4.16}$$

Let now Γ be a 2×2-matrix of complex numbers such that $\det \Gamma \neq 0$, to apply the linear transformation

$$\begin{pmatrix} w \\ z \end{pmatrix} = \Gamma \begin{pmatrix} \omega \\ \zeta \end{pmatrix} \tag{1.4.17}$$

to (1.4.16). This results in

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \omega \\ \zeta \end{pmatrix} = \Gamma^{-1}\Phi\Gamma \begin{pmatrix} \omega \\ \zeta \end{pmatrix}. \tag{1.4.18}$$

By an elementary fact from linear algebra, we may choose Γ to satisfy

$$\Gamma^{-1}\Phi\Gamma = \begin{pmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{pmatrix}, \tag{1.4.19}$$

where $\mu = 0$, if $\lambda_1 \neq \lambda_2$. Therefore, two possibilities essentially cover the linear case:

(1) If $\lambda_1 = \lambda_2$, then (1.4.18) takes the form

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \lambda_1\omega + \mu\zeta, \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}t} = \lambda_1\zeta, \tag{1.4.20}$$

and so

$$\frac{\mathrm{d}\omega}{\mathrm{d}\zeta} = \frac{\omega + \alpha\zeta}{\zeta}, \quad \alpha \neq 0. \tag{1.4.21}$$

Now, any local solution of (1.4.21) at a point $\zeta \neq 0$ near the origin takes the form

$$\omega(\zeta) = \zeta(\alpha \log \zeta + C), \quad C \in \mathbb{C}, \tag{1.4.22}$$

where we may take the branch of the logarithm to satisfy $\log 1 = 0$. If now γ is a Jordan curve ending at $\zeta = 0$ with the representation $\zeta = \rho \exp(\mathrm{i}\varphi(\rho))$, then $\zeta \log \zeta = (\rho \cos\varphi + \mathrm{i}\sin\varphi)(\log\rho + i\varphi(\rho))$, from which we compute that

$$|\zeta \log \zeta|^2 = (\rho \log \rho)^2 + \big(\rho\varphi(\rho)\big)^2. \tag{1.4.23}$$

Combining now (1.4.22) and (1.4.23), we observe that $\lim_{\zeta\to 0}\omega(\zeta) = 0$ if and only if $\lim_{\rho\to 0}\rho\varphi(\rho) = 0$.

(2) If $\lambda_1 \neq \lambda_2$, then (1.4.18) reduces to

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \lambda_1\omega, \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}t} = \lambda_2\zeta, \tag{1.4.24}$$

hence

$$\frac{\mathrm{d}\omega}{\mathrm{d}\zeta} = \beta\frac{\omega}{\zeta}, \quad \beta \neq 0. \tag{1.4.25}$$

In this case, we get

$$\omega(\zeta) = C\zeta^{\beta} = C\mathrm{e}^{\beta \log \zeta}, \quad C \in \mathbb{C}. \tag{1.4.26}$$

Let again γ be a curve ending at $\zeta = 0$, represented by $\zeta = \rho \exp(\mathrm{i}\varphi(\rho))$. Denoting $\beta = \beta_1 + \mathrm{i}\beta_2$, where β_1, β_2 are real numbers, we obtain

$$\mathrm{e}^{\beta \log \zeta} = \mathrm{e}^{\beta \log(\rho \exp(\mathrm{i}\varphi))} = \mathrm{e}^{(\beta_1 + \mathrm{i}\beta_2)(\log \rho + \mathrm{i}\varphi(\rho))}, \tag{1.4.27}$$

and therefore

$$|\mathrm{e}^{\beta \log \zeta}| = \mathrm{e}^{\beta_1 \log \rho - \beta_2 \varphi(\zeta)}. \tag{1.4.28}$$

To consider the behavior of $\omega(\zeta)$ as $\zeta \to 0$, let us first assume that $\beta_2 = 0$. Then we have $\lim_{\zeta \to 0} \omega(\zeta) = 0$, if $\beta < 0$, and $\lim_{\zeta \to 0} \omega(\zeta) = \infty$, if $\beta > 0$. Supposing now that $\beta_2 \neq 0$, then we have $\lim_{\zeta \to 0} \omega(\zeta) = 0$, provided that $\beta_1 > 0$. On the other hand, if $\beta_1 \leqslant 0$, we then may always find a curve γ such that $\omega(\zeta)$ doesn't converge to 0 as $\zeta \to 0$ along γ.

1.4.2. *A non-linear case: a glimpse at the Briot–Bouquet theory* As an example of a non-linear case, we first consider

$$zw' = \lambda w + p_{10} z + \sum_{j,k=1}^{\infty} p_{jk} z^j w^k, \quad \lambda \in \mathbb{C} \tag{1.4.29}$$

with constant coefficients. This is the most simple case of what are usually called as Briot–Bouquet equations.

THEOREM 1.4.1. *If* $\lambda \in \mathbb{C} \setminus \mathbb{N}$, *then* (1.4.29) *admits a local analytic solution in a neighborhood of* $z = 0$. *The solution satisfying the initial condition* $w(0) = 0$ *is unique.*

PROOF. For a proof based on the Cauchy majorant method, see [74], pp. 55–56. □

This theorem is a special case of more general systems of differential equations having a singular point of regular type at $z = 0$. We may write such a system in the form

$$zy_j' = f_j(z, y_1, \dots, y_n), \quad j = 1, \dots, n, \tag{1.4.30}$$

where the right-hand sides $f_j(z, y_1, \dots, y_n) = f_j(z, \mathbf{y})$ are analytic around $(z, \mathbf{y}) = (0, \mathbf{0})$, and assume that the initial conditions $f_j(0, \mathbf{0}) = 0$ are satisfied for $j = 1, \dots, n$. Systems of this type are called Briot–Bouquet systems. Of course, we may write (1.4.30) in vectorial form as

$$zY' = F(z, Y), \tag{1.4.31}$$

where $Y = {}^t(\mathbf{y}) \in \mathbb{C}^n$ and $F(z,Y) = {}^t(f_1(z,Y),\dots,f_n(z,Y))$. Denoting $K := (k_1,\dots,k_n) \in (\mathbb{N}\cup\{0\})^n$, $|K| := k_1+\dots+k_n$, $A_{jK} := {}^t(a_{jK}^{(1)},\dots,a_{jK}^{(n)}) \in \mathbb{C}^n$, $||Y|| := \max_j |y_j|$ and $Y^K := y_1^{k_1}\cdots y_n^{k_n}$, the power series expansion

$$F(z,Y) = \sum_{j+|K|\geqslant 1} A_{jK} z^j Y^K \tag{1.4.32}$$

converges in some domain around the origin, say $|z| < r_0$, $||Y|| < R_0$. Then we obtain the following

THEOREM 1.4.2. *If none of the eigenvalues of the matrix*

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(0,\mathbf{0}) & \cdots & \frac{\partial f_1}{\partial y_n}(0,\mathbf{0}) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial y_1}(0,\mathbf{0}) & \cdots & \frac{\partial f_n}{\partial y_n}(0,\mathbf{0}) \end{pmatrix}$$

is a natural number, then Eq. (1.4.31) *admits a unique solution of the form*

$$Y(z) = \sum_{j=1}^{\infty} C_j z^j, \quad C_j \in \mathbb{C}^n, \tag{1.4.33}$$

analytic in a neighborhood of the origin.

PROOF. See [97], pp. 261–263. □

2. Linear differential equations in the complex plane

Despite of the extensive literature on linear differential equations in a real domain as well as in abstract spaces such as Banach spaces etc., the global theory of linear differential equations in the complex plane has a surprisingly short research history. Of course, global theory may be understood in a variety of different ways. In this connection, it will be meant to studying properties of solutions defined in the whole complex plane. Moreover, we assume that the solutions under consideration are meromorphic (or algebroid) in the plane. In this sense, the birth of Nevanlinna theory in 1920's was a decisive step towards such global investigations. Indeed, Nevanlinna theory has been the key method in this field since 1929 when F. Nevanlinna considered [136] the differential equation $f'' + A(z)f = 0$ in the case of a polynomial coefficient $A(z)$. A more continuous activity had to wait until 1950's, started by Wittich, see e.g. [189] and Frei [44]. In late 1960's and in 1970's, most of the research made in this field was due to S. Bank and former students of Wittich (J. Nikolaus, G. Frank, E. Mues). The papers [14,15] by S. Bank and the present author in 1982 prompted extensive activity in the field of linear differential equations, having been continued ever since. In this section, we concentrate ourselves to presenting recent

developments in the field during the last fifteen years, as there are two relevant sources [98], [107] covering previous investigations. Also, the book [76] by E. Hille contains a lot of classical material about complex differential equations.

For the convenience of the reader, we recall here some of the key notations and basic results of the Nevanlinna theory, due to its importance to understand a lot of the material presented below.

Given a non-constant meromorphic function, denote by $n(r, f)$ the number of poles of f, resp. by $n(r, \frac{1}{f-a}) = n(r, a, f)$ the number of a-points of f in the disc $|z| \leqslant r$, each such point being counted according to its multiplicity. The respective *counting functions* are now obtained by logarithmic integration:

$$N\left(r, \frac{1}{f-a}\right) := \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t}\, \mathrm{d}t + n(0, a, f) \log r,$$

resp.

$$N(r, f) := \int_0^r \frac{n(t, f) - n(0, f)}{t}\, \mathrm{d}t + n(0, f) \log r.$$

The corresponding *proximity functions* will be defined as the following integrals over the boundary $|z| = r$:

$$m\left(r, \frac{1}{f-a}\right) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| \mathrm{d}\theta$$

and

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| \mathrm{d}\theta,$$

where $\log^+ a := \max(0, \log a)$ for $a \geqslant 0$. The *characteristic function* of f is nothing but the sum of the proximity function and the counting function:

$$T(r, f) := m(r, f) + N(r, f).$$

Using this notion, the *order* of f is defined as

$$\rho(f) := \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

Observe that this definition extends the classical order notion for entire functions, where $\log M(r, f)$ is used instead of $T(r, f)$; these two notions are not identical, but close enough to result in the same order value (in the complex plane, but not necessarily in the unit disc, say).

The celebrated *first main theorem* in Nevanlinna theory now tells an almost complete balance between the number of a-points and the affinity of f towards the value $a \in \mathbb{C}$:

THEOREM 2.0.3. *For a non-constant meromorphic function f and for an arbitrary complex value a,*

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + \mathrm{O}(1).$$

Here the bounded error term slightly depends on the value a.

The characteristic function may be used to separate rational functions from the non-rational ones: A meromorphic function f is rational if and only if $T(r, f) = \mathrm{O}(\log r)$.

Although the first main theorem is one of the great achievements in function theory during the last century, the really deep component in the Nevanlinna theory is the *lemma of the logarithmic derivative*:

THEOREM 2.0.4. *Let f be a non-constant meromorphic function, and k be any natural number. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f),$$

where $S(r, f) = \mathrm{o}(T(r, f))$ outside of a possible exceptional set of finite linear measure as $r \to \infty$. If f is of finite order, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = \mathrm{O}(\log r)$$

without an exceptional set.

The logarithmic derivative lemma is the key device to prove the *second main theorem*, which is a far-reaching refinement of the great Picard theorem. We give here a simplified version only:

THEOREM 2.0.5. *Let f be a non-constant meromorphic function, let $q \geqslant 2$ and let $a_1, \ldots, a_q$ be distinct complex numbers. Then*

$$m(r, f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) \leqslant 2T(r, f) + S(r, f).$$

Defining now the *deficiency* for $a \in \widehat{\mathbb{C}}$ by

$$\delta(a, f) := \liminf_{r\to\infty} \frac{m(r, 1/(f-a))}{T(r, f)},$$

the second main theorem immediately implies the fact that $\delta(a, f) = 0$ except for at most countably many exceptional values, and that

$$\sum_{a \in \widehat{\mathbf{C}}} \delta(a, f) \leqslant 2.$$

In addition to the deficiency, a *ramification index* $\vartheta(a, f)$ will also be applied in what follows. This is used to obtain some information of multiple values of f. To this end, we just count a-points, resp. poles, of f in such a way that its multiplicity will be reduced by one, using notation $N_1(r, \frac{1}{f-a})$, resp. $N_1(r, f)$, for the corresponding integrated counting functions. We then define

$$\vartheta(a, f) := \liminf_{r \to \infty} \frac{N_1(r, 1/(f-a))}{T(r, f)}, \qquad \vartheta(\infty, f) := \liminf_{r \to \infty} \frac{N_1(r, f)}{T(r, f)}.$$

Later on, at beginning of Section 4 below, we recall a few more important background results from Nevanlinna theory, frequently applied in considering complex differential equations.

For more details, several standard textbooks may be found. In our present list of references, [98] and [107] may be useful.

2.1. *Homogeneous linear differential equations with polynomial coefficients*

In studying global properties of solutions of linear differential equations in the complex plane, the non-trivial model case is

$$f'' + A(z) f = 0. \tag{2.1.1}$$

In this section, we assume that the coefficient $A(z) = c_n z^n + \cdots + c_0$ is a non-constant polynomial of degree n, $c_n \neq 0$. Then it is a classical result that certain symmetrically spaced radii from the origin essentially describe the zeros of non-trivial solutions of (2.1.1), see e.g. Gundersen [52], pp. 279–280, and [76], Chapter 7.4:

PROPOSITION 2.1.1. *Consider a differential equation* (2.1.1) *with a non-constant polynomial coefficient* $A(z)$ *of degree* n*. Denote* $\alpha = \frac{n+2}{2}$*. For* $0 < \delta < \frac{\pi}{2\alpha}$ *and* $j = 0, \ldots, n+1$, *consider the sectors*

$$S_j(\delta) := \left\{ z;\ \frac{j\pi - \arg c_n}{\alpha} + \delta \leqslant \arg z \leqslant \frac{(j+2)\pi - \arg c_n}{\alpha} - \delta \right\}. \tag{2.1.2}$$

If now a non-trivial solution of (2.1.1) *has infinitely many zeros in a sector* $S_j(\delta)$*, then for any* $\varepsilon > 0$*, all but finitely many of these zeros lie in the sector*

$$V_j(\varepsilon) := \left\{ z;\ \left| \arg z - \frac{2(j+1)\pi - \arg c_n}{n+2} \right| < \varepsilon \right\}. \tag{2.1.3}$$

Moreover, the non-integrated counting function for the zeros of f in the sector $V_j(\varepsilon)$ is

$$n_j(r, 1/f) = \big(1 + \mathrm{o}(1)\big)\frac{\sqrt{c_n}}{n\alpha} r^{\alpha}. \tag{2.1.4}$$

This proposition enables us to obtain fairly good estimates for the growth and zero distribution of non-trivial solutions f of (2.1.1). By the Wiman–Valiron theory, see e.g. [98], all non-trivial solutions are of order $\rho(f) = \alpha := \frac{n+2}{2}$. To describe their growth and value distribution, we denote by $p(f)$ the number of sectors $V_j(\varepsilon)$ with the property that for some $\varepsilon > 0$, the non-trivial solution f of (2.1.1) has only finitely many zeros in $V_j(\varepsilon)$. Clearly, $0 \leqslant p(f) \leqslant n + 2$. We now obtain, see [52], Theorems 5 and 6, the following asymptotic equalities:

THEOREM 2.1.2. *Given a non-trivial solution f of* (2.1.1), *and using the notations described above, $p(f)$ is an even number and*

$$n(r, 1/f) = \big(1 + \mathrm{o}(1)\big)\frac{2\alpha - p(f)}{\pi\alpha}\sqrt{|c_n|}\, r^{\alpha}, \tag{2.1.5}$$

$$N(r, 1/f) = \big(1 + \mathrm{o}(1)\big)\frac{2\alpha - p(f)}{\pi\alpha^2}\sqrt{|c_n|}\, r^{\alpha}, \tag{2.1.6}$$

$$T(r, f) = \big(1 + \mathrm{o}(1)\big)\frac{4\alpha - p(f)}{2\pi\alpha^2}\sqrt{|c_n|}\, r^{\alpha}, \tag{2.1.7}$$

$$\log M(r, f) = \big(1 + \mathrm{o}(1)\big)\frac{\sqrt{|c_n|}}{\alpha}\, r^{\alpha}. \tag{2.1.8}$$

Moreover, we have for the deficiency of zeros of f;

$$\delta(0, f) = \Delta(0, f) = \frac{p(f)}{4\alpha - p(f)}. \tag{2.1.9}$$

REMARK. Equation (2.1.1) with $\deg A = n$ possesses $m \geqslant 2$ pairwise linearly independent solutions $f_1, \dots, f_m$ such that $p(f_j) \geqslant 2$ for $j = 1, \dots, m$ and $\sum_{j=1}^{m} p(f_j) = 2(n + 2)$. Moreover, any non-trivial solution f of (2.1.1) that is not a constant multiple of some f_j, $j = 1, \dots, m$, has $p(f) = 0$. Hence, most of the non-trivial solutions f have $p(f) = 0$. See [52], pp. 281–283, for more details. Finally, observe that the maximum modulus of f is independent of $p(f)$.

Because of the symmetric spacing of the critical rays described in Proposition 2.1.1, and the fact that $p(f)$ is even, it is not surprising that the number of non-real zeros of a non-trivial solution f is maximal in most cases, see [52]:

THEOREM 2.1.3. *Given two linearly independent solutions f_1, f_2 of* (2.1.1), *at least one of f_1, f_2 has the property that its sequence of non-real zeros has exponent of convergence equal to* $(n + 2)/2$. *Moreover, if the coefficient polynomial $A(z)$ in* (2.1.1) *is of degree $n = 2 + 4k$ for a non-negative integer k, then any non-trivial solution f either has finitely*

many zeros only, or the exponent of convergence of the non-real zero-sequence of f *is* $(n+2)/2$.

Combining [52], Theorem 3 and Lemma 5, see also [147], Theorem 1.1, the following conclusion appears:

THEOREM 2.1.4. *If Eq.* (2.1.1) *admits a non-trivial solution* f *with infinitely many real zeros, then* $A(z)$ *is a real polynomial* (*i.e. real on the real axis*), *and* f *is a constant multiple of a real solution of* (2.1.1).

By the preceding result, it makes sense to call special attention to the case when $A(z)$ is a real polynomial. In this direction, Eremenko and Merenkov [41] proved the following interesting results recently:

THEOREM 2.1.5. *For each* $n \geqslant 1$, *there exists Eq.* (2.1.1) *with a real polynomial* $A(z)$ *of degree* n *and a solution* f *of* (2.1.1) *such that all zeros of* f *are real. Concerning different values of* n,

(a) *if* $n = 0 \pmod 4$, *then the zero-set of* f *is either finite or it is unbounded from above and below,*
(b) *if* $n = 2 \pmod 4$, *then* f *has finitely many zeros only and*
(c) *if* n *is odd, then* f *has infinitely many zeros all of which lie on a ray of the real axis.*

The proof of this result in [41] makes use of the graph theory (Speiser graphs).

EXAMPLE. (See [52].) The classical Airy differential equation

$$f'' - zf = 0$$

admits a solution whose zeros are all real and negative, namely the standard Airy function $Ai(z)$, compare Theorem 2.1.5(c). Clearly, $p(Ai) = 2$. Therefore, by Theorem 2.1.2,

$$N(r,0,Ai) = \big(1+\mathrm{o}(1)\big)\frac{4}{9\pi}r^{3/2}, \qquad T(r,Ai) = \big(1+\mathrm{o}(1)\big)\frac{8}{9\pi}r^{3/2}$$

and

$$\log M\big(r,Ai\big) = \big(1+\mathrm{o}(1)\big)\frac{2}{3}r^{3/2}.$$

For an analysis of a generalized Airy differential equation

$$f'' - z^n f = 0,$$

see [54].

REMARK. Proposition 2.1.1 depicts what is called the *finite oscillation property*, see [7], p. 266: There exist finitely many rays $\arg z = \theta_j$, $j = 1, \dots, m$, such that for any $\varepsilon > 0$, all but finitely many zeros of any non-trivial solution of a linear differential equation

$$f^{(n)} + p_{n-1}(z) f^{(n-1)} + \cdots + p_0(z) f = 0 \tag{2.1.10}$$

lie in the union of the sectors $|\arg z - \theta_j| < \varepsilon$, $j = 1, \dots, m$. On the other hand, we say that a linear differential equation possesses the *global oscillation property*, if there is a non-trivial solution f such that for any ray $\arg z = \theta$ and any $\varepsilon > 0$, f has infinitely many zeros in the sector $|\arg z - \theta| < \varepsilon$. By Proposition 2.1.1, the finite oscillation property holds for the basic second order differential equation (2.1.1) with a polynomial coefficient $A(z)$. However, the finite oscillation property does not hold in general in the higher order case (2.1.10). For instance, equation

$$f^{(n)} + z^2 f'' + z f' + f = 0, \quad n > 2,$$

possesses the global oscillation property, see [6], p. 37. In fact, given any Eq. (2.1.10) with polynomial coefficients, it can be shown that it has either the finite oscillation property, or the global oscillation property, see [8], Theorem 1.

Concerning homogeneous linear differential equations of the form (2.1.10) with $n \geqslant 2$ and with polynomial coefficients such that $p_0(z)$ does not vanish identically, it is well known that all solutions f of Eq. (2.1.10) are entire functions of finite rational order. Moreover, denoting $d_j := \deg p_j$, if p_j does not vanish, and $d_j = -\infty$ otherwise, we have that

$$\rho(f) \leqslant \lambda := 1 + \max_{0 \leqslant j \leqslant n-1} \frac{d_j}{n-j} \tag{2.1.11}$$

for all solutions of (2.1.10). These are classical results usually proved by the Wiman–Valiron theory and, to determine the possible rational orders, the Newton–Puiseux diagram, see [98]. However, the list of possible rational orders may be obtained by simple arithmetic with the degrees of polynomial coefficients of (2.1.10), as shown by Gundersen, Steinbart and Wang in [56]. To recall this reasoning, we first determine a strictly decreasing finite sequence $s_1, \dots, s_p$ of non-negative integers by defining

$$s_0 = n, \tag{2.1.12}$$

$$s_1 := \min\left\{ j;\ \frac{d_j}{n-j} = \max_{0 \leqslant m \leqslant n-1} \frac{d_m}{n-m} \right\} \leqslant n-1 \tag{2.1.13}$$

and then, inductively,

$$s_{j+1} := \min\left\{ k;\ \frac{d_k - d_{s_j}}{s_j - m} = \max_{0 \leqslant m \leqslant s_j} \frac{d_m - d_{s_j}}{s_j - m} > -1 \right\}. \tag{2.1.14}$$

Clearly, this process terminates after a finite number $p \leqslant n$ of steps. We then define, for $j = 1, \ldots, p$,

$$\alpha_j := 1 + \frac{d_{s_j} - d_{s_{j-1}}}{s_{j-1} - s_j} > 0, \tag{2.1.15}$$

where, of course, $d_{s_0} = 0$. We are now ready to state

THEOREM 2.1.6. *Concerning the growth of solutions of* (2.1.10), *the following assertions hold*:

(a) *Equation* (2.1.10) *possesses at most s_p linearly independent polynomial solutions.*

(b) *If f is a transcendental solution, then $\rho(f) = \alpha_j$ for some j, $1 \leqslant j \leqslant p$. Moreover, if $s_1 \geqslant 1$ and $p \geqslant 2$, then*

$$\alpha_1 > \alpha_2 > \cdots > \alpha_p \geqslant \frac{1}{s_1 - s_p} \geqslant \frac{1}{s_1} \geqslant \frac{1}{n-1},$$

and if $s_1 = 0$, then all non-trivial solutions f are of order $\rho(f) = 1 + d_0/n$.

(c) *For any $j = 1, \ldots, p$, there exist at most s_j linearly independent solutions f of order $\rho(f) < \alpha_j$. Also, there always exists a solution f of order $\rho(f) = \lambda = 1 + \max_{0 \leqslant j \leqslant n-1} \frac{d_j}{n-j}$.*

(d) *If $\{f_1, f_2, \ldots, f_n\}$ is any fundamental set of solutions of* (2.1.10), *then*

$$\sum_{j=1}^{n} \rho(f_j) \geqslant n + d_0.$$

For a collection of examples showing the sharpness of the preceding theorem, see [56], pp. 1240–1246.

The frequency of zeros of solutions of (2.1.10) with polynomial coefficients has been somewhat studied since 1950's; we refer here to a number of papers by Pöschl, Wittich, Frank, Brüggemann and Steinmetz. However, the recent papers by Wang [186] and Steinbart [168] offer the most complete understanding of this topics until now. To describe these recent results, we first recall the real numbers $s_0, \ldots, s_p$ and $\alpha_1, \ldots, \alpha_p$ defined by the degrees of the polynomial coefficients of (2.1.10) as described above. Observe that we then have

$$d_{s_j} - d_{s_{j-1}} = (\alpha_j - 1)(s_{j-1} - s_j).$$

Let now A_j be the leading coefficient of $p_j = A_j z^{d_j} + \cdots$, and define

$$A^{\star}_{k,m} := \begin{cases} A_k & \text{if } d_k = d_{s_{m-1}} + (\alpha_m - 1)(s_{m-1} - k), \\ 0 & \text{otherwise} \end{cases} \tag{2.1.16}$$

and

$$A^{\sharp}_{k} := \begin{cases} A_k & \text{if } \frac{d_k}{n-k} = \max_{0 \leqslant j \leqslant n-1} \frac{d_j}{n-j}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.17}$$

We then define the following polynomials corresponding to (2.1.16) and (2.1.17) as

$$H_m(t) := \sum_{k=0}^{s_m-1} A^{\star}_{k,m} t^k, \quad 1 \leqslant m \leqslant p, \tag{2.1.18}$$

and

$$H^{\sharp}(t) := t^n + \sum_{k=0}^{n-1} A^{\sharp}_k t^k. \tag{2.1.19}$$

We may now express the key results in [186] and [168] as follows:

THEOREM 2.1.7. *Concerning the frequency of zeros of solutions of* (2.1.10), *the following assertions hold*:

(a) *If f is a non-trivial solution such that $\rho(f)$ is not an integer, then f has infinitely many zeros and $\rho(f) = \lambda(f)$.*

(b) *If f is a transcendental solution of order $\rho(f) = \alpha_m \in \mathbb{N}$ for some m, $1 \leqslant m \leqslant p$, and if all non-zero roots of equation $H_m(t) = 0$ are simple, then either $\rho(f) = \lambda(f)$, or f has finitely many zeros only.*

(c) *Suppose at least one of the coefficients of* (2.1.10) *is non-constant. Then the following properties are equivalent*:

(1) *Equation* (2.1.10) *admits a fundamental solution set $\{f_1, \ldots, f_n\}$ such that $\lambda(f_j) < \rho(f_j)$, $j = 1, \ldots, n$.*

(2) *Equation* (2.1.10) *admits a fundamental solution set $\{f_1, \ldots, f_n\}$ such that*

$$\max_{1 \leqslant j \leqslant n} \lambda(f_j) < \lambda = 1 + \max_{0 \leqslant j \leqslant n-1} \frac{d_j}{n-j}.$$

(3) *The polynomial $H^{\sharp}(t)$ takes the form $H(t) = (t+b)^n$ for some non-zero constant b.*

For some examples illuminating the preceding theorem, see [168], pp. 368–370.

REMARK. It seems quite likely that multiple roots of $H_m(t) = 0$, resp. of $H^{\sharp}(t) = 0$, correspond to subsets of the fundamental solution set $\{f_1, \ldots, f_n\}$ such that the solutions f in these subsets satisfy $\lambda(f) < \rho(f)$. As far as we know, there is no explicit proof for this claim yet. See, however, Theorem 2.5.5 below.

2.2. *Second order homogeneous linear differential equations with entire coefficients*

Looking at the non-trivial model case (2.1.1) with a transcendental entire coefficient $A(z)$, it immediately follows from the logarithmic derivative lemma that $\rho(f) = \infty$. Therefore,

the key problem here is to consider the frequency of zeros of solutions of (2.1.1). This topic has been under active research since 1982, see [15].

The key idea to consider the zero frequency of solutions of (2.1.1) with $A(z)$ transcendental entire, is to make use of the product E of two linearly independent solutions f_1, f_2. By the classical Abel identity, we know that the Wronskian $W(f_1, f_2)$ is a non-zero constant. Therefore, we may normalize it to be equal to 1. Then it is routine to check that

$$4A = \left(\frac{E'}{E}\right)^2 - \frac{1}{E^2} - 2\frac{E''}{E}. \tag{2.2.20}$$

Since the zero-sets of f_1 and f_2 are distinct, we have for the exponent of convergence of the zeros of E that

$$\lambda(E) = \max\bigl(\lambda(f_1), \lambda(f_2)\bigr).$$

Then the basic results from [15], complemented with an improvement due to Shen [154] and Rossi [146] for the case $\rho(A) = 1/2$, may be written as

THEOREM 2.2.1. *Concerning the frequency of zeros of solutions of* (2.1.1) *with* $A(z)$ *transcendental entire*, *the following assertions hold*:

(a) *Suppose* $\rho(A) < \infty$ *but is not a natural number. If* $\rho(A) \leqslant 1/2$, *then* $\lambda(E) = \infty$; *if* $\rho(A) < 1$, *then* $\lambda(E) > 1$; *and if* $\rho(A) \geqslant 1$, *then* $\lambda(E) \geqslant \rho(A)$.

(b) *Suppose next that* $\lambda(A) < \rho(A)$ (*which implies by the Hadamard factorization that* $\rho(A)$ *is a natural number or infinite*). *Then the exponent of convergence* $\lambda(f)$ *of the zero-sequence of an arbitrary non-trivial solution* f *satisfies* $\lambda(f) \geqslant \rho(A)$.

(c) *If* $\bar{\lambda}(A) < \rho(A)$, *then* $\lambda(E) \geqslant \rho(A)$.

(d) *There exist two linearly independent zero-free solutions* f_1, f_2 *of* (2.1.1) *in the complex plane if and only if* $A(z)$ *is of the form*

$$-4A(z) = h'(z)^2 + \phi'(z)^2 - 2\phi''(z)$$

where ϕ *a non-constant entire function and* h *a primitive of* $\exp(\phi)$.

(e) *If* $\lambda(E) = \max(\lambda(f_1), \lambda(f_2)) < \infty$, *then* $\lambda(f) = \infty$ *for all solutions* f *of* (2.1.1) *not being of the form* αf_1 *or* αf_2 *for some* $\alpha \in \mathbb{C}$.

PROOF. As a typical example of applying the product E of two linearly independent solutions of (2.1.1), we prove the last assertion of (a). If $\lambda(E) < \rho(A)$, then (2.2.20) easily implies that $\rho(E) \leqslant \rho(A)$. Since $\rho(A)$ is not a natural number, the Hadamard product representation implies that $\lambda(E) = \rho(A)$. $\square$

EXAMPLE. As an example of the case when $\rho(A)$ is a natural number, we collect here a few facts related to equation

$$f'' + \bigl(e^{P(z)} + Q(z)\bigr)f = 0, \tag{2.2.21}$$

where P is polynomial of degree $\deg P = \lambda > 0$ and Q is an entire function of order $\rho(Q) < \lambda$. Then the following properties are valid:

(1) If (2.2.21) admits a non-trivial solution f such that $\lambda(f) < \lambda$, then f has no zeros, $Q = -\frac{1}{16}(P')^2 + \frac{1}{4}P''$ is a polynomial and there is another zero-free solution of (2.2.21), linearly independent of f.

(2) If $Q(z) \equiv 0$, then $\lambda(f) = \infty$ for all non-trivial solutions f of (2.2.21).

(3) In the special case of $f'' + (e^z - K)f = 0$, $K \in \mathbb{C}$, a non-trivial solution f with $\lambda(f) < \infty$ exists if and only if $K = q^2/16$ for an odd integer q. In this case, the equation admits two linearly independent solutions such that for their product E, $\lambda(E) = 0$, if $|q| = 1$ and $\lambda(E) = 1$ otherwise.

Theorem 2.2.1, together with a collection of examples to be found in the literature, prompted the following conjecture, known as the Bank–Laine conjecture (BL-conjecture for short), implicitly contained in [15]:

CONJECTURE. *Let $A(z)$ be transcendental entire of finite order of growth ρ. If f_1, f_2 are two linearly independent solutions of* (2.1.1) *such that* $\max(\lambda(f_1), \lambda(f_2) < \infty$, *then ρ is a natural number.*

In its full generality, the conjecture remains open. Due to the wide interest this problem has implied, a few remarks follow. Recalling (2.2.20), it is not difficult to see that all zeros of f_1, f_2 and $E := f_1 f_2$ are simple, and $E'(z_0) = \pm 1$ at all zeros z_0 of E, provided the Wronskian $W(f_1, f_2)$ has been normalized to $W(f_1 f_2) = 1$. This condition is known as the BL-condition and any entire function E satisfying the BL-condition is called a BL-function. To resolve the BL-conjecture, it is sufficient to show that whenever E is a BL-function of finite order, then the corresponding coefficient function $A(z)$ of equation (2.1.1), determined by (2.2.20), is of an integer order of growth. A number of examples of BL-functions fall in two classes of functions: (1) Entire functions of type $P(z)e^{Q(z)}$, where P is a polynomial with simple zeros only, and Q is a (non-constant) polynomial and (2) entire functions of type $P(e^{\alpha z})e^{\beta z}$, where P is a polynomial with simple zeros only, and α, β are constants. However, not all known BL-functions of finite order belong in these two function classes as shown by Langley, see [112,113] and [114]. It has also been shown recently by Drasin and Langley, see [35], that there exists a Bank–Laine function E, associated with a transcendental coefficient function $A(z)$ determined by (2.2.20), such that for any given positive integer n, $n \leqslant \rho(E) \leqslant \lambda(E) < \infty$. For more details about the BL-conjecture, see [109].

The BL-conjecture has been very much in the background to direct research concerning Eq. (2.1.1) during the last quarter of century. A special situation having attracted a lot of attention is the case when $A(z)$ is a periodic function. The key problem here has been to explain when such an equation admits a non-trivial solution f with a few zeros in the sense of $\lambda(f) < \infty$. The first paper in this line was [16], to our knowledge. Recently, Chiang [29]

improved the results in [16]. To recall the key results in these papers, suppose that $A(z)$ is of period $2\pi\mathrm{i}$. Then we may write

$$A(z) = B(\mathrm{e}^z), \qquad B(\zeta) = \sum_{j=k}^{l} K_j \zeta^j = g(\zeta) + h(1/\zeta),$$

where $K_k K_l \neq 0$, $l \geqslant k$, and $g(\zeta), h(1/\zeta)$ are entire functions. Making the transformation $\zeta \mapsto 1/\zeta$, if needed, we may assume that $l \geqslant 0$.

THEOREM 2.2.2. (See [29], Proposition 1.) *Given a differential equation* (2.1.1), *where* $A(z) = B(\mathrm{e}^z)$ *is a rational function of* e^z, *write* $B(\zeta) = \sum_{j=k}^{l} K_j \zeta^j$, $K_l K_k \neq 0$, *and assume that* $\lambda(f) < \infty$ *for a non-trivial solution* f. *Then the following assertions hold*:

(1) *If* l *is odd and positive, then* $k = 0$ *and* f *has the following representation*:

$$f(z) = \psi(\mathrm{e}^{z/2}) \exp\left(\sum_{j=0}^{l} d_j \mathrm{e}^{jz/2} + dz \right),$$

where d, d_j *are constants and* $\psi(\zeta)$ *is a polynomial with simple zeros only. Moreover,* $d_j = 0$ *for all* $j > 0$ *even.*

(2) *If* l *is even and positive, then* k *is even as well, and*

$$f(z) = \psi(\mathrm{e}^z) \exp\left(\sum_{j=-k/2}^{l/2} d_j \mathrm{e}^{jz} + dz \right),$$

where again d, d_j *are constants and* $\psi(\zeta)$ *is a polynomial with simple zeros only.*

By the presentations for $f(z)$ in Theorem 2.2.2, it follows that f is zero-free if and only if $\lambda(f) = 0$. In this special situation, more detailed representations for $f(z)$ may be obtained, see [29], Theorems 1, 2 and 3.

Continuing to the more general case when $A(z)$ is a non-constant periodic entire function of period $2\pi\mathrm{i}$ and transcendental in e^z, we obtain [16]:

THEOREM 2.2.3. *Let* $A(z)$ *be a non-constant periodic entire function such that* $A(z) = B(\mathrm{e}^z)$, *where* $B(\zeta)$ *is analytic in* $\mathbb{C} \setminus \{0\}$, *and assume that* f *is a non-trivial solutions of* (2.1.1) *such that* $\lambda(f) < \infty$. *Then* f *has the representation*

$$f(z) = \mathrm{e}^{dz} H(\mathrm{e}^{z/q}) \exp\bigl(g(\mathrm{e}^{z/q})\bigr),$$

where (1) d *is a complex number,* (2) $q = 1$ *or* $q = 2$ *depending whether* $f(z)$ *and* $f(z + 2\pi\mathrm{i})$ *are linearly dependent or not,* (3) $H(\zeta)$ *and* $g(\zeta)$ *are analytic in* $\mathbb{C} \setminus \{0\}$, *and* (4) $g(\zeta)$ *has at most a pole at* $\zeta = \infty$, *resp.* $\zeta = 0$, *if and only if* $B(\zeta)$ *has at most a pole at* $\zeta = \infty$, *resp.* $\zeta = 0$. *Moreover, letting* ρ_∞, *resp.* ρ_0, *denote the local order at* ∞, *resp. at* 0, *we have* $\rho_0(H) = \rho_\infty(H) = 0$, $\rho_0(g) = \rho_0(B)$ *and* $\rho_\infty(g) = \rho_\infty(B)$.

For additional results concerning the case $A(z) = B(\mathrm{e}^z)$, where $A(z)$ is transcendental in e^z, see [30].

2.3. *Higher order homogeneous linear differential equations*

We now proceed to considering higher order homogeneous linear differential equations

$$L(f) := f^{(n)} + a_{n-1}(z) f^{(n-1)} + \cdots + a_0(z) f = 0 \tag{2.3.22}$$

with entire coefficients. Since (2.3.22) may easily be reduced to

$$f^{(n)} + a_{n-2}(z) f^{(n-1)} + \cdots + a_0(z) f = 0, \tag{2.3.23}$$

a substantial part below will be formulated for (2.3.23). Contrary to the normalized second-order case of (2.1.1), it may now appear, even in the second order case, that two solutions may be of substantially different growth. The first classical result in this direction is due to Frei in [44]:

THEOREM 2.3.1. *Let a_j be the last transcendental function in the coefficient sequence $a_0, \dots, a_{n-1}$ of* (2.3.22). *Then Eq.* (2.3.22) *possesses at most j linearly independent solutions of finite order*.

REMARK. Combining Theorem 2.3.1 with (2.1.11) we conclude that all solutions of (2.3.22) are of finite order of growth if and only if all coefficients of (2.3.22) are polynomials, see Wittich [190].

REMARK. In the special case of $n = 2$, Theorem 2.3.1 implies that all non-trivial solutions are of infinite order, provided a_0 is transcendental and a_1 rational. More generally, the same conclusion follows in a variety of situations, where a_0 is dominant. A nice example of this type has been proved by Kwon in [106]:

THEOREM 2.3.2. *Given two constants $\alpha, \beta > 0$, and a set E in the real axis of strictly positive upper density, suppose that the entire functions a_1, a_0 satisfy the growth conditions*

$$\bigl|a_1(z)\bigr| \leqslant \exp\bigl(\mathrm{o}(1)|z|^\beta\bigr), \qquad \bigl|a_0(z)\bigr| \geqslant \exp\bigl(\bigl(1 + \mathrm{o}(1)\bigr)\alpha|z|^\beta\bigr)$$

as $|z| \to \infty$, $|z| \in E$. Then all non-trivial solutions of

$$f'' + a_1(z) f' + a_0(z) f = 0$$

are of infinite order.

For another result of this type, see Theorem 2.3.6 below.

Making use of the notion of an iterated order,

$$\rho_j(z) := \limsup_{r\to\infty} \frac{\log_j T(r, f)}{\log r}, \tag{2.3.24}$$

where $\log_1(r) := \log r$, $\log_j(r) := \log\log_{j-1}(r)$, we may perform a more detailed analysis of the growth of solutions of (2.3.22), see [102]. To this end, we define the *finiteness degree of growth* $i(f)$ of a meromorphic function f as $i(f) = 0$ for rational functions, $i(f) := \min\{j \in \mathbb{N}; \rho_j(f) < \infty\}$, and $i(f) = \infty$ otherwise. Concerning the linear differential operator $L(f)$ with entire coefficients, we define

$$\begin{aligned}
&\delta(L) := \max\bigl\{i(f); L(f) = 0\bigr\},\\
&\gamma_j(L) := \max\bigl\{\rho_j(f); L(f) = 0\bigr\}, \quad j \in \mathbb{N},\\
&p(L) := \max\bigl\{i(a_j); j = 0, \dots, n-1\bigr\}.
\end{aligned}$$

Provided $0 < p := p(L) < \infty$, we also define

$$\kappa(L) := \max\bigl\{\rho_p(a_j); j = 0, \dots, n-1\bigr\}.$$

Using these notations, we have

THEOREM 2.3.3. *If $0 < p < \infty$, then $\delta(L) = p + 1$ and $\gamma_{p+1}(L) = \kappa(L)$. Moreover, if a_j be the last coefficient in the sequence $a_0, \dots, a_{n-1}$ such that $i(a_j) = p$, then* (2.3.22) *possesses at most j linearly independent solutions f such that $i(f) \leqslant p$.*

PROOF. The key idea of the proof makes use of the standard order reduction procedure of linear differential equations, combined with the generalized logarithmic derivative lemma, see [102], Lemma 1.3: Given a meromorphic function f such that $i(f) = p \geqslant 1$ and $\rho_p(f) = \rho$, then for any $\varepsilon > 0$ and any natural number k, $m(r, f^{(k)}/f) = \mathrm{O}(\exp_{p-2} r^{\rho+\varepsilon})$ outside of a possible exceptional set of finite linear measure. □

In the general case of (2.3.22) with $n \geqslant 2$, the preceding theorems leave open the problem of determining the situations when all solutions of (2.3.22) are, say, of infinite order. Such a situation typically appears, if one of the coefficients is dominating in the sense of growth, see [110]:

THEOREM 2.3.4. *Let $a_0(z), \dots, a_{n-1}(z)$ be entire functions such that for some integer s, $1 \leqslant s \leqslant n-1$, we have $\rho_p(a_j) < \rho_p(a_s) \leqslant \infty$ for all $j \neq s$. Then every transcendental solution f of* (2.3.22) *satisfies $\rho_p(f) \geqslant \rho_p(a_s)$.*

However, one may obtain similar results if one of the coefficients is dominant just almost everywhere only, in certain sense. Recently, a number of papers of such type have been written. Prototypes of such results may be found in [53]:

THEOREM 2.3.5. *Let* $a_0(z) \not\equiv 0$, $a_1(z)$ *be entire functions such that for some real constants* $\alpha > 0$, $\beta > 0$ *and* $\theta_1 < \theta_2$ *we have*

$$|a_1(z)| \geqslant \exp\big((1+\mathrm{o}(1))\alpha|z|^\beta\big)$$

and

$$|a_0(z)| \leqslant \exp\big(\mathrm{o}(1)|z|^\beta\big)$$

as $z \to \infty$ *in the sector* $S(0) := \{\theta = \arg z;\ \theta_1 \leqslant \theta \leqslant \theta_2\}$. *Given* $\varepsilon > 0$ *small enough, consider the sector* $S(\varepsilon) := \{\theta = \arg z;\ \theta_1 + \varepsilon \leqslant \theta \leqslant \theta_2 - \varepsilon\}$. *If* f *is a non-trivial solution of*

$$f'' + a_1(z)f' + a_0(z)f = 0$$

of finite order, then the following conditions hold:

(i) *There exists a constant* $b \neq 0$ *such that* $f(z) \to b$ *as* $z \to \infty$ *in* $S(\varepsilon)$. *More precisely,*

$$|f(z) - b| \leqslant \exp\big(-(1+\mathrm{o}(1))\alpha|z|^\beta\big).$$

(ii) *For each integer* $k \geqslant 1$,

$$|f^{(k)}(z)| \leqslant \exp\big(-(1+\mathrm{o}(1))\alpha|z|^\beta\big)$$

as $z \to \infty$ *in* $S(\varepsilon)$.

This result, combined with the classical Phragmén–Lindelöf principle, applies now to prove

THEOREM 2.3.6. *Let* $a_0(z) \not\equiv 0$, $a_1(z)$ *be entire functions, and let* $\alpha > 0$ *and* $\beta > 0$ *be constants such that* $\rho(a_0) < \beta$. *Suppose that for any given* $\varepsilon > 0$, *there exist two finite collections of real numbers that satisfy* $\phi_1 < \theta_1 < \phi_2 < \theta_2 < \cdots < \phi_n < \theta_n < \phi_{n+1} = \phi_1 + 2\pi$, *and* $\sum_{j=1}^{n}(\phi_{j+1} - \phi_j) < \varepsilon$, *such that*

$$|a_1(z)| \geqslant \exp\big((1+\mathrm{o}(1))\alpha|z|^\beta\big)$$

as $z \to \infty$ *in each of the sectors* $\phi_j \leqslant \arg z \leqslant \theta_j$, $j = 1, \ldots, n$. *Then all non-trivial solutions* f *of*

$$f'' + a_1(z)f' + a_0(z)f = 0$$

are of infinite order of growth.

Applying a certain type of nested induction process, see [110], the preceding theorems may be extended to (2.3.22), in the form that the dominating coefficient may be any one of the intermediate coefficients:

THEOREM 2.3.7. *Let $\theta_1 < \theta_2$ be given to fix the sector $S(0)$ as in Theorem* 2.3.5, *and let $\delta > 0$ be any real number such that $n\delta < 1$. Suppose that $a_0(z) \not\equiv 0, a_1(z), \ldots, a_{n-1}(z)$ are entire functions such that for real constants $\alpha > 0$, $\beta > 0$, we have for some s, $1 \leqslant s \leqslant n-1$, that*

$$\left|a_s(z)\right| \geqslant \exp\bigl((1+\delta)\alpha|z|^\beta\bigr),$$
$$\left|a_j(z)\right| \leqslant \exp\bigl(\delta\alpha|z|^\beta\bigr)$$

for all $j = 0, \ldots, s-1, s+1, \ldots, n-1$ whenever $|z|$ is large enough in the sector $S(0)$. Given $\varepsilon > 0$ small enough, if f is a transcendental solution of finite order ρ of Eq. (2.3.22), *then the following conditions hold*:

(i) *There exists j, $0 \leqslant j \leqslant s-1$, and a constant $b_j \neq 0$ such that $f^{(j)} \to b_j$ as $z \to \infty$ in the sector $S(\varepsilon)$. More precisely,*

$$\left|f^{(j)}(z) - b_j\right| \leqslant \exp\bigl(-(1-n\delta)\alpha|z|^\beta\bigr)$$

in $S(\varepsilon)$, provided $|z|$ is large enough.

(ii) *For each integer $m \geqslant j+1$,*

$$\left|f^{(m)}(z)\right| \leqslant \exp\bigl(-(1-n\delta)\alpha|z|^\beta\bigr)$$

in $S(3\varepsilon)$ for all $|z|$ large enough.

THEOREM 2.3.8. *Let $a_0(z) \not\equiv 0, a_1(z), \ldots, a_{n-1}(z)$ be entire functions, let $\alpha > 0$, $\beta > 0$ be given constants, let $\delta > 0$ be a real number such that $n\delta < 1$ and let s be an integer such that $1 \leqslant s \leqslant n-1$. Suppose that $\rho(a_s) < \beta$ for all $j \neq s$ and that for any given $\varepsilon > 0$, there exist two finite collections of real numbers that satisfy $\phi_1 < \theta_1 < \phi_2 < \theta_2 < \cdots < \phi_m < \theta_m < \phi_{m+1} = \phi_1 + 2\pi$ such that $\sum_{j=1}^{m}(\phi_{j+1} - \phi_j) < \varepsilon$ and*

$$\left|A_s(z)\right| \geqslant \exp\bigl((1+\delta)\alpha|z|^\beta\bigr)$$

as $z \to \infty$ in each of the sectors $\phi_j \leqslant \arg z \leqslant \theta_j$, $j = 1, \ldots, m$. Then every transcendental solution f of (2.3.22) *is of infinite order.*

Another result in the same line is due to Yang in [193], as a generalization of the corresponding theorem when $n = 2$:

THEOREM 2.3.9. *Let E be a set in the positive real axis of strictly positive upper density, and suppose that for some constants $0 \leqslant \beta < \alpha$ and $\mu > 0$, we have*

$$\left|a_0(z)\right| \geqslant \exp\bigl(\alpha|z|^\mu\bigr)$$

and

$$\left|a_j(z)\right| \leqslant \exp\bigl(\beta|z|^\mu\bigr),$$

for $j = 1, \ldots, n-1$ *as* $|z| \to \infty$, $|z| \in E$. *Then all non-trivial solutions of* (2.3.22) *are of infinite order and, in fact,* $\rho_2(f) \geqslant \mu$.

Proceeding now to the zero distribution of solutions, the present research status is much more scattered than it is in the special second order case (2.1.1), even in the reduced case of (2.3.23). The product $E := f_1 \cdots f_n$ still remains important, but no counterpart of the identity (2.2.20) is available in this more general situation. However, Bank and Langley were able to prove in [18] several results corresponding to previous ones in the case of (2.1.1):

THEOREM 2.3.10. *Suppose that* $n \geqslant 3$ *in Eq.* (2.3.23) *and that the coefficients* $a_0(z), \ldots, a_{n-2}(z)$ *are entire functions of finite order. Moreover, suppose that* (2.3.23) *admits a solution base* $f_1, \ldots, f_n$ *each with* $\lambda(f_j) < \infty$. *Then their product* E *is of finite order as well.*

THEOREM 2.3.11. *Suppose that* $n \geqslant 3$ *in Eq.* (2.3.23) *and that the coefficients* $a_0(z), \ldots, a_{n-2}(z)$ *are entire functions. Moreover, assume that* (i) a_0 *is transcendental of order* $\rho(a_0) < 1/2$ *and* (ii) $\rho(a_j) < \rho(a_0)$ *for* $j = 1, \ldots, n-2$, *if* $\rho(a_0) > 0$, *while* $a_1, \ldots, a_{n-2}$ *are polynomials, if* $\rho(a_0) = 0$. *Then Eq.* (2.3.23) *cannot admit two linearly independent solutions* f_1, f_2 *each with* $\lambda(f_j) < \infty$.

THEOREM 2.3.12. *Suppose that* $n \geqslant 3$ *in Eq.* (2.3.23) *and that the coefficients* $a_0(z), \ldots, a_{n-2}(z)$ *are entire functions of finite order such that* $\rho(a_j) < \rho := \rho(a_0)$ *for all* $j = 1, \ldots, n-2$. *Moreover, suppose that Eq.* (2.3.23) *admits a solution base* $f_1, \ldots, f_n$ *each with* $\lambda(f_j) < \infty$. *Then* ρ *is a natural number and* $\rho(E) = \rho$.

THEOREM 2.3.13. *Suppose that* $n \geqslant 3$ *in Eq.* (2.3.23) *and that the coefficients* $a_0(z), \ldots, a_{n-2}(z)$ *are entire functions. Moreover, assume that* (i) a_0 *is transcendental of order* $\rho(a_0) < 1/2$ *and* (ii) $\rho(a_j) < \rho(a_0)$ *for* $j = 1, \ldots, n-2$, *if* $\rho(a_0) > 0$, *while* $a_1, \ldots, a_{n-2}$ *are polynomials, if* $\rho(a_0) = 0$. *Then, given two linearly independent solutions* f_1, f_2 *of* (2.3.23), *we have that the quotient* f_1/f_2 *is of infinite order.*

A natural question is now whether somewhat similar results could be obtained, provided the dominant coefficient is some of the $a_1, \ldots, a_{n-2}$ instead of a_0. Most important results of this type are due to Langley [115] and to Hellerstein, Miles and Rossi [72]. We first give a growth result from [72]:

THEOREM 2.3.14. *Considering linear differential equation* (2.3.22), *suppose that for some* k, $0 \leqslant k \leqslant n-1$, $\rho(a_j) < \rho(a_k) \leqslant 1/2$ *for all* $j \neq k$. *Then every solution of* (2.3.22) *is either a polynomial or an entire function of infinite order.*

Concerning zero distribution of a solution base of the reduced equation (2.3.23), corresponding to Theorem 2.2.1(a), we have the following result, see [115]:

THEOREM 2.3.15. *In the higher order case* $n \geqslant 3$, *suppose that all coefficients* a_j, $j = 0, \ldots, n-2$, *of* (2.3.23) *are of order* $\rho(a_j) < \frac{1}{2(n-1)}$. *Then the linear differential*

equation (2.3.23) *cannot have linearly independent solution* $f_1, \ldots, f_n$ *each satisfying* $\lambda(f_j) < \infty$.

For additional results of similar type, see [111].

2.4. *Homogeneous linear differential equations with meromorphic coefficients*

If a linear differential equation (2.3.22) has meromorphic coefficients $a_j(z)$, $j = 1, \ldots, n-1$, of which at least one is non-entire, then the existence of a meromorphic solution, resp. of a meromorphic solution base, in the whole complex plane becomes the first problem to be treated. Indeed, algebraic and logarithmic branch points may appear for such linear differential equations more as a rule than as an exception. As a simple classical example, consider a linear differential equations of type

$$f'' + \frac{h(z)}{(z-z_0)^2} f = 0$$

with $h(z)$ analytic in an open disc $B(z_0, R)$. Supposing that the indicial equation

$$\rho(\rho - 1) + h(z_0) = 0$$

has its roots ρ_1, ρ_2 such that $\rho_1 - \rho_2$ is a non-zero integer, then the equation admits two linearly independent solutions

$$f_1(z) = (z-z_0)^{\rho_1} \sum_{j=0}^{\infty} a_j (z-z_0)^j, \quad a_0 \neq 0,$$

$$f_2(z) = k f_1(z) \log(z - z_0) + (z-z_0)^{\rho_2} \sum_{j=0}^{\infty} b_j (z-z_0)^j,$$

where $k = 0$ or $k = 1$, in some slit disc $D := B(z_0, r) \setminus \{z_0 + t; 0 \leqslant t < r\}$, $r \leqslant R$. On the other hand, any meromorphic function f may appear as a global solution by just defining $A(z) := -f''(z)/f(z)$. Therefore, restricting ourselves to the most simple case of (2.1.1) with a meromorphic coefficient $A(z)$, a natural problem is to ask, when Eq. (2.1.1) admits two linearly independent meromorphic solutions in the whole complex plane, i.e. to characterizing when all solutions of (2.1.1) are globally meromorphic. To this end, the following result may be found, see [107], pp. 120–124:

THEOREM 2.4.1. *A linear differential equation* (2.1.1) *with* $A(z)$ *meromorphic in the complex plane admits two meromorphic solutions in the complex plane if and only if at all poles* z_0 *of* $A(z)$, *the Laurent expansion of* $A(z)$ *at* z_0 *is of the form*

$$A(z) = \frac{b_0}{(z-z_0)^2} + \frac{b_1}{z-z_0} + b_2 + \cdots,$$

where

$$4b_0 = 1 - m^2$$

for an odd integer $m \geqslant 3$*, and where the determinant*

$$D(z_0) = \begin{vmatrix} 1-m & 0 & \cdots & 0 & b_1 \\ b_1 & 4-2m & & & b_2 \\ b_2 & b_1 & \ddots & (0) & b_3 \\ \vdots & \vdots & & & \vdots \\ b_{m-2} & b_{m-3} & \cdots & (m-1)^2-(m-1)m & b_{m-1} \\ b_{m-1} & b_{m-2} & \cdots & b_1 & b_m \end{vmatrix} \qquad (2.4.25)$$

vanishes.

In such a situation, when all solutions are meromorphic in the plane, a number of results are known to describe the distribution of zeros and poles of solutions, see [17]. It remains mostly open, however, to express these results in terms of the coefficient $A(z)$ only. To give a simple example, let f_1, f_2 be a meromorphic solution base. Then it is easy to see, from $(f_2/f_1)' = 1/f_1^2$, that all solutions have the same order of growth. On the other hand, contrary to the case of $A(z)$ entire, it may happen that all solutions are of finite order. A simple example may be provided by $f'' - (2/\cos^2 z)f = 0$ with $f_1(z) = \tan z$, $f_2(z) = 1 + z\tan z$. However, it remains open to characterize, in terms of $A(z)$ only, when the solutions are of finite order. Similarly, it is not known, in terms of $A(z)$ again, for which $A(z)$, there exist two linearly independent solutions f_1, f_2, each being zero-free in the whole complex plane.

Concerning the case when all solutions of (2.1.1) with a meromorphic coefficient $A(z)$ are of finite order, we may look at the Lamé equation

$$f'' - \big(q(q+1)\wp(z) + B\big)f = 0, \qquad (2.4.26)$$

where $\wp(z)$ is the Weierstraß $\wp$-function, q a natural number and B a complex constant. In this case of (2.4.26), all solutions are meromorphic and if non-trivial, they are of order $\rho(f) = 2$. Moreover, we have $\lambda(f - a) = 2$ for all $a \in \mathbb{C}$, see [157], pp. 594–595. More generally, the same conclusions hold for all non-trivial solutions of (2.1.1), provided the coefficient $A(z)$ is a doubly periodic meromorphic function such that its Laurent expansion $A(z) = (z - z_0)^2 \sum_{j=0}^{\infty} b_j (z - z_0)^j$ satisfies around every pole z_0 of $A(z)$ the following conditions:

(1) $b_0 = -q(z_0)(q(z_0) + 1)$ for an integer $q(z_0)$,

$$(2)\ D(a) = \begin{vmatrix} \mu_1 & 0 & \cdots & & 0 & b_1 \\ b_1 & \mu_2 & \ddots & (0) & \vdots & b_2 \\ b_2 & b_1 & \ddots & \ddots & \vdots & b_3 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ b_{2q(a)-1} & b_{2q(a)-2} & \cdots & b_1 & \mu_{2q(a)} & b_{2q(a)} \\ b_{2q(a)} & b_{2q(a)-1} & \cdots & b_2 & b_1 & b_{2q(a)+1} \end{vmatrix} = 0,$$

where $\mu_j = j^2 - (2q(z_0)+1)j$, $1 \leqslant j \leqslant 2q(z_0)$. This more general case may be regarded as a generalized Lamé equation.

In the case, when the coefficient $A(z)$ in (2.1.1) is periodic of period ω, but not doubly periodic, and all solutions are supposed to be meromorphic, we may apply, under mild conditions, the Floquet theory to conclude that there exists a solution base, the members of which are of the type $f_j(z) = R_j(z)\exp(\beta_j z)$, where β_j is a complex constant, and $R_j(z)$ is rational in $\exp(2\pi i z/\omega)$, see [187]. For a number of results in the periodic case, see [160].

For a few examples of results describing the distribution of zeros and poles of meromorphic solutions in terms of $A(z)$ only, see [17]:

THEOREM 2.4.2. *Let $A(z)$ be a transcendental meromorphic function of order ρ such that all solutions of* (2.1.1) *are meromorphic in the complex plane. Then we have*

(a) *If $0 < \rho < \infty$, and if f_1, f_2 are linearly independent, then*

$$\max\bigl(\bar{\lambda}(f_1), \bar{\lambda}(f_2), \lambda(1/f_1)\bigr) \geqslant \rho.$$

(b) *If $\rho = 0$, then at least one of the following three sets is an infinite set of points: the zero-set of f_1, the zero-set of f_2, the pole-set of f_1.*

(c) *If $\rho > 0$ and if $\bar{\lambda}(A) < \rho$, then $\max(\bar{\lambda}(f), \bar{\lambda}(1/f)) \geqslant \rho$ for all non-trivial solutions f.*

(d) *If $\bar{\lambda}(f_1) < \infty$ and $\bar{\lambda}(f_2) < \infty$ for two linearly independent solutions f_1, f_2, then $\max(\bar{\lambda}(f), \bar{\lambda}(1/f)) = \infty$, unless all solutions are of finite order.*

Observe that these assertions represent a meromorphic counterpart of Theorem 2.2.1.

Of course, more refined information about the zeros of solutions of (2.3.22) follow if one is restricting the subset of $\mathbb{C}$ to be considered. An interesting paper of this type is due to Y. Il'yashenko and S. Yakovenko [89]. They consider a finite interval in $K \subset \mathbb{R}$, and its sufficiently small neighborhood U such that the coefficients $a_j(z)$ of (2.3.22) are analytic in U. Provided $|a_j(z)| \leqslant A$ in U for all $j = 0, \ldots, n-1$, then all non-trivial solutions that are real on K have at most $\beta(A+n)$ distinct zeros on K, where the constant β depends on the geometry of the pair (K, U) only. A more general result, removing the assumption that a solutions is real on K, was due to D. Novikov and S. Yakovenko in [138]:

THEOREM 2.4.3. *Let $K \subset U$ be a compact set in an open set $U \subset \mathbb{C}$ and suppose that K, U both have simply connected interiors and piecewise smooth boundaries. More-*

over, suppose that K equals to the closure of its interior. Then all non-trivial solutions f of (2.3.22) *with coefficients analytic in a neighborhood of $\bar{U}$ have at most $\beta(A + n\log n)$ distinct zeros in K, where now $A := \max_{j=0,\dots,n-1} \max_{z\in U} |a_j(z)|$.*

For a number of related generalizations, see [138].

2.5. *Non-homogeneous linear differential equations*

In this section, we describe key results concerning the growth and oscillation of solutions of non-homogeneous linear differential equations

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_0(z)f = H(z) \tag{2.5.27}$$

with entire coefficients. It is well-known that all solutions of (2.5.27) are entire functions, see [107], Proposition 8.1. A typical phenomenon here is that the growth of solutions is essentially determined by the coefficients in the homogeneous part of (2.5.27), provided the finiteness degree of growth of $H(z)$ satisfies that $i(H) \leqslant 1 + \max_{j=0,\dots,n-1} i(a_j)$. As for the situation, when the coefficients $a_j(z)$, $j = 0,\dots,n-1$ are polynomials, while $H(z)$ is of finite order of growth, see two papers [55,57] by Gundersen, Steinbart and Wang. In what follows, we partially apply the same notations (2.1.12)–(2.1.15) as used with respect to the corresponding homogeneous equation (2.1.10) above. While looking at the results below, it also makes sense to compare with the corresponding results related to the homogeneous case (2.1.10) with polynomial coefficients. Recall that all solutions of the corresponding homogeneous linear differential equation are of growth

$$\rho(f) \leqslant \lambda := 1 + \max_{0\leqslant j\leqslant n-1} \frac{d_j}{n-j}$$

at most, while there always exist solutions of growth

$$\rho(f) = 1 + \max_{0\leqslant j\leqslant n-1} \frac{d_j}{n-j} = \alpha_1.$$

In the present non-homogeneous situation, elementary order considerations show that we always have $\rho(f) \geqslant \rho(H)$ for all solutions of (2.5.27). More precisely, we obtain

THEOREM 2.5.1. *All solutions f of* (2.5.27) *satisfy either $\rho(f) = \rho(H)$, or $\rho(f) = \alpha_j$ for some $1 \leqslant j \leqslant p$, provided $\alpha_j > \rho(H)$, and there always exists a solution f such that $\rho(f) = \max(\rho(H), 1 + \alpha_1)$. Moreover, if $\rho(H)$ is an integer, and if $\rho(f) > \rho(H)$, then $\rho(f) \geqslant \rho(H) + 1/n$.*

The last inequality here is sharp as shown by Example 4.1 in [57], and the assumption that $\rho(H)$ is an integer cannot be deleted, see Example 4.2 in [57].

Perhaps more interesting than the growth only is to see the influence of the non-homogeneity part H to the zero-distribution of f:

THEOREM 2.5.2. *For all solutions f of* (2.5.27), *we have*

$$\rho(f) - \lambda(f) \leqslant \rho(H) - \lambda(H).$$

Therefore, we always have $\lambda(f) \geqslant \lambda(H)$ and $\lambda(f) = \rho(f)$ as soon as $\lambda(H) = \rho(H)$.

THEOREM 2.5.3. *If $\lambda(H)$ is an integer, and if f is a solution of* (2.5.27) *such that $\lambda(f) > \lambda(H)$, then*

$$\lambda(f) \geqslant \lambda(H) + 1/n. \tag{2.5.28}$$

Again, observe that the inequality (2.5.28) is sharp, and the assumption that $\lambda(H)$ is an integer cannot be deleted, see examples in [57].

THEOREM 2.5.4. *If $\lambda(H) < \rho(H)$, and if f is a solution of* (2.5.27) *such that $\lambda(f) < \rho(f)$, then*

$$\rho(f) - \lambda(f) \geqslant \min\bigl(1/n, \rho(H) - \lambda(H)\bigr). \tag{2.5.29}$$

In particular, if $\rho(H) - \lambda(H) \leqslant 1/n$, and $\lambda(f) < \rho(f)$, then

$$\rho(f) - \lambda(f) = \rho(H) - \lambda(H). \tag{2.5.30}$$

Examples in [57] again show the sharpness of Theorem 2.5.4.

To obtain a counterpart of Theorem 2.1.7 in the non-homogeneous case (2.5.27), where $\lambda(H) < \rho(H) < \infty$, we need to write $H(z) = h(z)e^{Q(z)}$, where $Q(z) = \frac{1}{\beta} b z^{\beta} + \cdots$ is a polynomial of degree β and h is an entire function such that $\lambda(h) = \rho(h) < \beta$. Let next A_j be the leading coefficient of $a_j(z) = A_j z^{d_j}$ in (2.5.27), and set $a_n(z) \equiv 1$. Setting then

$$\tau := \max_{0 \leqslant j \leqslant n} \bigl\{d_j + j(\beta - 1)\bigr\},$$

we may define the constants $A_j^\star$ for $j = 0, \ldots, n$ as

$$A_j^\star := \begin{cases} A_k & \text{if } d_j + j(\beta - 1) = \tau, \\ 0 & \text{otherwise} \end{cases} \tag{2.5.31}$$

and a polynomial $H^\star(t)$ as

$$H^\star(t) := \sum_{j=0}^{n} A_j^\star t^j.$$

Using these notations, we obtain the following conclusions, see [55]:

THEOREM 2.5.5. *If b is a zero of multiplicity $m \geqslant 0$ of the polynomial $H^\star(t)$. Then Eq.* (2.5.27) *admits at most $m+1$ linearly independent solutions f such that $\lambda(f) < \rho(f)$.*

In particular, each solution f *of* (2.5.27) *satisfies* $\lambda(f) < \rho(f)$ *if and only if* $H^\star(t) = (t-b)^n$.

Observe that this theorem also covers the case, when $H^\star(b) \neq 0$ showing that in this case their is at most one solution f such that $\lambda(f) < \rho(f)$.

THEOREM 2.5.6. *If the non-homogeneity part* H *in Eq.* (2.5.27) *is of infinite order, then all solutions* f *of* (2.5.27) *satisfy* $\lambda(f) = \rho(f) = \infty$ *with at most one exceptional solution* f_0.

Returning back to the finite vs. global oscillation properties treated above in the case of homogeneous linear differential equations (2.1.10) with polynomial coefficients, the corresponding behavior in the non-homogeneous case appears to be more complicated, see [7], Theorem 1.1:

THEOREM 2.5.7. *Suppose that all coefficients in Eq.* (2.5.27) *are polynomials with* $H(z) \not\equiv 0$. *Then there exist finitely many real numbers* s_j, $j = 1, \dots, m+1$, *with* $0 = s_0 < s_1 < \cdots < s_m < s_{m+1} = 2\pi$ *such that for each* k, $0 \leqslant k \leqslant m$, *one of the following properties hold*: (1) *for any* $\theta \in (s_k, s_{k+1})$ *and for any* $\varepsilon > 0$, *there is a solution of* (2.5.27) *having infinitely many zeros in* $|\arg z - \theta| < \varepsilon$ *or* (2) *for any* $\varepsilon > 0$, *any solution of* (2.5.27) *admits at most finitely many solutions in the sector* $s_k + \varepsilon \leqslant \arg z \leqslant s_{k+1} - \varepsilon$.

Therefore, in the non-homogeneous case, Eq. (2.5.27) has either the finite oscillation property, the global oscillation property, or a mixture of these two properties. An example of the last situation is

$$f''' + (z^2+3)f'' + (2z^2+z+3)f' + (z^2+z+2)f = H(z),$$

which has the global oscillation property in the left half plane, while four rays $\theta = -\pi/2, -\pi/6, \pi/6, \pi/2$ determine the finite oscillation property of this equation in the right half plane, see [7], pp. 290–291, for details.

We finally remark that there are very few results concerning the growth and zero distribution of solutions of (2.5.27), with coefficients being entire functions so that at least one of the coefficients in the homogeneous part is transcendental. We restrict ourselves here to giving just a couple of typical results, see [27] and [49]:

THEOREM 2.5.8. *Suppose that the coefficients in* (2.5.27) *are entire functions of finite order so that* $\rho(a_j) < \rho(a_0)$, $j = 1, \dots, n-1$. *Then all solutions* f *of* (2.5.27) *satisfy* $\bar{\lambda}(f) = \lambda(f) = \rho(f) = \infty$ *with at most one exceptional solution* f_0 *of finite order. If such an exceptional solution* f_0 *exists, then* $\rho(f_0) \leqslant \max\{\rho(a_0), \rho(H), \bar{\lambda}(f_0)\} < \infty$.

THEOREM 2.5.9. *Suppose that the coefficients in* (2.5.27) *are entire functions of finite order so that at least one of* $a_j(z)$, $j = 0, \dots, n-1$ *is transcendental. Then at least one of the solutions of* (2.5.27) *satisfies* $\bar{\lambda}(f) = \lambda(f) = \rho(f) = \infty$.

See also [24] for a result concerning the iterated order of solutions.

3. Linear differential equations in the unit disc

Meromorphic solutions of linear differential equations in the unit disc behave to some extent similarly as to the case of the complex plane, provided the solutions are fast growing in the sense of the Nevanlinna theory, i.e. that $T(r, f)/\log(1/(1-r))$ is unbounded. This case will be considered in Section 3.4. The complementary case of slowly growing solutions, which becomes trivial in the plane case, needs extensive consideration in the unit disc. In fact, analytic functions in the unit disc typically appear to be of slow growth when they belong to certain complex function spaces such as Hardy spaces, Bloch space, Q_p-spaces for some p etc. This gives rise to the general problem of investigating connections between the function spaces where the solutions, resp. the coefficients belong to. This topic has attracted systematic studies only very recently. Quite likely, new observations and improvements to the presentation in this chapter are to be expected in the near future. Another key aspect in investigating linear differential equations in the unit disc is the distribution of zeros of solutions. Historically, this topic preceded growth considerations, and combining these classical results on zero distribution with the recent growth studies is an open task of research today.

3.1. *Classical results on zeros of solutions*

Certain results about the zeros of solutions of linear differential equations in a disc of the complex plane are classical, although perhaps not so well-known, see [76]. To give an example of such classical theorems, consider the linear differential equation

$$f'' + A(z)f = 0, \tag{3.1.1}$$

where $A(z)$ is analytic:

THEOREM 3.1.1. (See [76], p. 580.) *Suppose $A(z)$ is analytic in a disc $|z - z_0| < R$ and bounded there by $|A(z)| \leqslant M$. Let f be a solution of $f'' + A(z)f = 0$ in this disc. Then we have*:

(i) *If f satisfies the initial conditions $f(z_0) = 0$, $f'(z_0) = 1$, then f has no zeros in the punctured disc $0 < |z - z_0| < \min(R, \pi M^{-1/2})$.*

(ii) *If f satisfies the initial conditions $f(z_0) = 1$, $f'(z_0) = 0$, then f has no zeros in the disc $|z - z_0| < \min(R, \frac{1}{2}\pi M^{-1/2})$.*

This type of classical results, connected with the closely related problem of the univalence of solutions of (3.1.1) prompted quite a lot of activity in 1950's. Unfortunately, this activity gradually faded out, being perhaps again reviving. The basic notions to be considered in this connection are related to the oscillatory nature of solutions. For the sake of normalization, we shall consider the situation in the unit disc $\mathbb{D}$ only. Of course, by a simple rescaling, any finite disc in the complex plane could be treated as well.

The differential equation

$$f'' + A(z)f = 0, \tag{3.1.2}$$

where $A(z)$ is analytic in $\mathbb{D}$, is said to be *oscillatory*, if at least one of its non-trivial solutions has infinitely many zeros in the unit disc $\mathbb{D}$. Otherwise, Eq. (3.1.2) is called *non-oscillatory*. Two more notions applied in this connection are *disconjugate*, meaning that every non-trivial solution of (3.1.2) has at most one zero in $\mathbb{D}$, and *Blaschke-oscillatory* meaning that the zero sequence (z_j) of every non-trivial solution of (3.1.2) is a Blaschke sequence, i.e. that $\sum_{j=1}^{\infty}(1-|z_j|)$ converges.

The research activity in early 1950's was very much due to the short but important paper [132] by Nehari, see also [134,135]. In particular, he proved the following theorem, which connects the notion of univalence to solutions of (3.1.2) in the unit disc:

THEOREM 3.1.2. *Given two linearly independent solutions f_1, f_2 of Eq.* (3.1.2), *their quotient g takes a complex value α n times in $\mathbb{D}$ if and only if there exists a non-trivial solutions of* (3.1.2) *that has n zeros in $\mathbb{D}$. In particular, g is univalent if and only if Eq.* (3.1.2) *is disconjugate.*

Actually, the proof of this theorem is quite elementary: If $g = f_1/f_2$ takes a certain value $\alpha \in \mathbb{C}$ at some point z, then $f_1 - \alpha f_2$ vanishes at z.

Making use of Theorem 3.1.2, Nehari [132] proved the following non-oscillation theorem:

THEOREM 3.1.3. *If*

$$\bigl|A(z)\bigr| \leqslant \frac{1}{(1-|z|^2)^2}$$

in the unit disc $\mathbb{D}$, then Eq. (3.1.2) *is disconjugate.*

The same conclusion holds, if $|A(z)| \leqslant \frac{2}{1-|z|^2}$, see Pokornyi [142]. Observe that Theorem 3.1.3 is in some sense close to the best possible, as shown by Schwarz [152]:

THEOREM 3.1.4. *If the inequality in Theorem* 3.1.3 *holds close to the boundary of $\mathbb{D}$, i.e. in $r_0 \leqslant |z| < 1$ for some $r_0 > 0$, then equation* (3.1.2) *is non-oscillatory. Moreover, for each $\gamma > 0$, there exists $A(z)$ such that*

$$|A(z)| \leqslant \frac{1+4\gamma^2}{(1-|z|^2)^2}$$

and that Eq. (3.1.2) *becomes oscillatory.*

For the proof of the first assertion, see [152], pp. 160–162. On the other hand, taking

$$A(z) = \frac{1+4\gamma^2}{(1-z^2)^2},$$

the corresponding Eq. (3.1.2) has an analytic solution

$$f(z) = (z^2 - 1)^{1/2} \sin\left(\gamma \log \frac{1+z}{1-z} - C\right)$$

in the unit disc, which has infinitely many zeros in $\mathbb{D}$, provided C is real (and provided that the branches of square root and logarithm have been chosen suitably).

A substantial number of oscillation results for (3.1.2) in the unit disc come out by integration of $|A(z)|$. Denoting by $\mathrm{d}\sigma$ the standard area element in the complex plane, the following conclusions may be found in [124]:

THEOREM 3.1.5. *Equation* (3.1.2) *is disconjugate, resp. non-oscillatory, if* $\int_{\mathbb{D}} |A(z)|\,\mathrm{d}\sigma \leqslant \pi$, *resp.* $< \infty$.

PROOF. The first assertion is an immediate consequence of Theorem 3.1.3, combined with the standard estimate

$$\pi\bigl(1 - |\zeta|^2\bigr)^2 \bigl|A(\zeta)\bigr| \leqslant \int_{\mathbb{D}} \bigl|A(z)\bigr|\,\mathrm{d}\sigma, \quad |\zeta| < 1$$

for analytic functions in the unit disc.

The second assertion essentially follows from the invariance of area integrals of analytic functions under Möbius transformations of the independent variable, see [124], pp. 982–986. □

A corresponding pair of oscillation results follows by computing the boundary integral of $|A(z)|$ instead of the area integral. Observing that $I(\rho, A) := \int_0^{2\pi} |A(\rho \mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta$ is non-decreasing, we may define

$$I(1, A) := \lim_{\rho \to 1} I(\rho, A).$$

A counterpart to Theorem 3.1.5 is now

THEOREM 3.1.6. *Equation* (3.1.2) *is disconjugate, resp. non-oscillatory, if* $I(1, A) \leqslant 4\pi$, *resp.* $I(1, A) < \infty$.

PROOF. The first assertion follows by combining the disconjugacy criterion due to Pokornyi, see the remark after Theorem 3.1.3, with the inequality

$$2\pi\bigl(1 - |\zeta|^2\bigr)\bigl|A(\zeta)\bigr| \leqslant I(1, A)$$

due to Nehari, see [133], p. 127. On the other hand, the second assertion is an immediate consequence of the corresponding non-oscillation criterion in Theorem 3.1.5. □

For additional, related results, see [34].

An immediate problem open for further investigations is the sharpness of the bounds in disconjugacy criteria π and 4π in Theorems 3.1.5 and 3.1.6, respectively. The example

$f'' + \frac{\pi^2}{4} f = 0$ due to London [124] shows that the bounds are definitely not sharp. Indeed, we have in this case $\int_{\mathbb{D}} |A(z)|\,\mathrm{d}\sigma = \frac{\pi^3}{4} \approx 2.467\pi$ and $I(1, A) = \frac{\pi^3}{2} \approx 4.935\pi$. On the other hand, the boundary integral value is not too far from the best possible bound as shown by the following example due to Heittokangas [67]: Considering

$$f'' + \frac{10\zeta^2 - z^2}{4\zeta^4} f = 0, \quad |\zeta| < 1,$$

it is not difficult to see that this equation has a solution with two zeros at $\pm\zeta$, and so the equation is not disconjugate. On the other hand, by numerical computation, we see that the area integral $\int_{\mathbb{D}} |A(z)|\,\mathrm{d}\sigma(z) \approx 5\pi/(2\zeta^2)$, while $I(1, A) \approx 5.013\pi$ as $|\zeta|$ is conveniently chosen close to one. Therefore, it is natural to conjecture that the possible constant might be $5\pi/2$ and 5π, respectively.

It is trivial to observe that disconjugate and non-oscillatory equations (3.1.2) are Blaschke-oscillatory. However, characterizing Blaschke-oscillatory equations (3.1.2) remains defectively investigated by now. The key results into this direction read as follows, see Pommerenke [143] and Heittokangas [65,66]:

THEOREM 3.1.7. *If $A(z)$ is analytic in $\mathbb{D}$ and either $\int_{\mathbb{D}} |A(z)|^{1/2}\,\mathrm{d}\sigma(z) < \infty$ or $\int_{\mathbb{D}} |A(z)|(1 - |z|)\,\mathrm{d}\sigma(z) < \infty$, then Eq.* (3.1.2) *is Blaschke-oscillatory.*

THEOREM 3.1.8. *If $A(z)$ is analytic in $\mathbb{D}$ and Eq.* (3.1.2) *is Blaschke-oscillatory, then $\int_{\mathbb{D}} |A(z)|^{\alpha}\,\mathrm{d}\sigma(z) < \infty$ for each $\alpha \in (0, 1/2)$.*

The first of these results is an easy consequence of [143], Theorem 5, while the second one can be proved by using the unit disc version of the second main theorem of Nevanlinna theory, see [66], p. 126. Observe that the two integral conditions in Theorem 3.1.7 are independent of each other, as one may see by [70], Section 4. Also observe that there exist Blaschke-oscillatory equations of the form (3.1.2) for which neither one of the conditions in Theorem 3.1.7 is not satisfied, as one may easily see by looking at the equation

$$f'' - \frac{4}{(1 - z)^4} f = 0,$$

see [66], pp. 121–122. Concerning the next result, see [65], pp. 55–56:

THEOREM 3.1.9. *Suppose the coefficient $A(z)$ in* (3.1.2) *belongs to the Hardy space H^p. If $1/4 \leqslant p \leqslant \infty$, then* (3.1.2) *is Blaschke-oscillatory. Moreover, if $1 \leqslant p \leqslant \infty$, then* (3.1.2) *is non-oscillatory.*

This is sharp in the following sense: Eq. (3.1.2) with $A(z) := 4/(1-z)^4$ is not Blaschke-oscillatory, while $A \in H^p$ for all $p < 1/4$, but $A \notin H^{1/4}$, see [65], p. 56.

A natural question about zeros of solutions of (3.1.2) is their geometric distribution inside of the unit disc. This problem has been addressed by Schwarz [152], Beesack [21] and Herold [75] by considering the hyperbolic non-Euclidean distance between zeros, while

Schwarz [152] and Beesack [20] also considered Euclidean distance between zeros. To describe these results, recall first that the hyperbolic non-Euclidean distance between two points z_1, z_2 in the unit disc will be defined as

$$[z_1, z_2] := \int_\gamma \frac{\mathrm{d}\zeta}{1-|\zeta|^2},$$

where the path of integration γ is the circle arc joining z_1 and z_2, orthogonal to the unit circle. Schwarz now proved the following theorem:

THEOREM 3.1.10. *Let f be a non-trivial solution of* (3.1.2).

(a) *If*

$$\left|A(z)\right| \leqslant \frac{1+a^2}{(1-|z|^2)^2}$$

for some $a > 0$ and if z_1, z_2 are two zeros of f in $\mathbb{D}$, then

$$[z_1, z_2] > \log \frac{\sqrt{1+a^2}+1}{\sqrt{1+a^2}-1}.$$

(b) *On the other hand, if for any two zeros z_1, z_2 in the unit disc $\mathbb{D}$,*

$$[z_1, z_2] \geqslant \log \frac{\sqrt{1+a^2}+1}{\sqrt{1+a^2}-1}, \quad a > 0,$$

then

$$\left|A(z)\right| < \frac{3(1+a^2)}{(1-|z|^2)^2}.$$

Herold [75] improved the first part of the preceding theorem by proving

THEOREM 3.1.11. *Let f be a non-trivial solution of* (3.1.2) *and suppose that*

$$\left|A(z)\right| \leqslant \frac{1+a^2}{(1-|z|^2)^2}.$$

Then for any two zeros z_1, z_2 of f in the unit disc $\mathbb{D}$,

$$[z_1, z_2] \geqslant \frac{\pi}{a}.$$

Theorem 3.1.11 is in fact best possible, as shown by the following example which goes back to Schwarz, see [152], p. 171, and [75], p. 638:

The function in the unit disc $\mathbb{D}$ defined as

$$f(z) := \sqrt{1-z^2}\,\sin\left(\frac{\alpha}{2}\log\frac{1+z}{1-z}\right),$$

where $\log 1 = 0$, is a solution of

$$f'' + \frac{1+\alpha^2}{(1-z^2)^2} f = 0.$$

The zeros z_n of f in the unit disc are determined by

$$\log\frac{1+z_n}{1-z_n} = n\frac{2\pi}{\alpha}, \quad n \in \mathbb{Z}.$$

By an elementary computation,

$$[z_n, z_{n+1}] = \int_{z_n}^{z_{n+1}} \frac{\mathrm{d}t}{1-t^2} = \Big/_{z_n}^{z_{n+1}} \frac{1}{2}\log\frac{1+t}{1-t} = \frac{\pi}{\alpha}.$$

3.2. *Complex function spaces in the unit disc*

Describing the growth and zero distribution of solutions of linear differential equations in the unit disc essentially relies on characterizing the coefficients of the equation as well as of its solutions in terms of complex functions spaces in the unit disc. Indeed, we may consider meromorphic functions f in the unit disc in two classes: Defining

$$D(f) := \limsup_{r\to 1^-} \frac{T(r,f)}{\log(1/(1-r))},$$

we say that f is of *finite degree*, if $D(f) < \infty$, while if $D(f) = \infty$, we say that f is of *infinite degree*. Of course, we may also say that f is of degree β, if $D(f) = \beta$. Observe that the finiteness of degree of f could also be defined for analytic functions as

$$D_M(f) := \limsup_{r\to 1^-} \frac{\log M(r,f)}{\log(1/(1-r))}$$

instead of $D(f)$. Clearly, $D(f) \leqslant D_M(f)$. However, $D_M(f)$ is not equivalent to $D(f)$ in defining the finiteness of degree, as shown by $f(z) := \mathrm{e}^{1/(1-z)}$. In this case, $D_M(f) = \infty$ and $D(f) < \infty$ by simple computation.

It is important to observe that the unit disc version of Nevanlinna theory remains a powerful tool while considering fast growing solutions of complex differential equations in the unit disc, meaning that the solution under consideration is of infinite degree and is growing essentially faster than the coefficients of the equation. On the other hand, solutions of slow growth need different tools, as in this situation the growth of a solution is comparable with

the growth of coefficients. Due to recent studies, the theory of complex functions spaces takes the role of a key device to attack the situation.

For the convenience of the reader, we shortly recall basic definitions of the complex analytic spaces needed in what follows.

Perhaps the most classical linear space of analytic functions in the unit disc is the *Hardy space* H^p of analytic functions f satisfying the inequality

$$\sup_{0\leqslant r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^p \,d\theta\right)^{1/p}<\infty$$

for $0<p<\infty$, while for $p=\infty$, the corresponding Hardy space H^∞ is just the space of analytic functions bounded in the unit disc. For basic properties of Hardy spaces, see [36]. To define a natural extension of Hardy spaces H^p, $0<p<\infty$, we consider the p-characteristic of an analytic function f defined as

$$m_p(r,f):=\left(\frac{1}{2\pi}\int_0^{2\pi}\left(\log^+\left|f(re^{i\theta})\right|\right)^p \,d\theta\right)^{1/p} \tag{3.2.3}$$

for $0\leqslant r<1$. Now, the *generalized Nevanlinna class* N^p consists of all analytic functions f in the unit disc for which $\sup_{0\leqslant r<1} m_p(r,f)<\infty$. Observe that N^p is actually a linear space, and for each p, $\bigcup_0^\infty H^p\subset N^p$. The classical Nevanlinna class then is nothing but $N:=N^1$.

We next define the *weighted Hardy spaces* H_q^p, resp. H_q^∞ for $0\leqslant q<\infty$ as analytic functions f in the unit disc satisfying

$$\sup_{0\leqslant r<1}(1-r^2)^q\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^p \,d\theta\right)^{1/p}<\infty,$$

resp.

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^q\left|f(z)\right|<\infty.$$

Clearly, $H^p=H_0^p$ and $H^\infty=H_0^\infty$. Finally, we say that an analytic function $f\in H_{q,0}^\infty$, if

$$\lim_{|z|\to 1^-}\left(1-|z|^2\right)^q\left|f(z)\right|=0,$$

and correspondingly for $f\in H_{q,0}^p$. A closely related analytic function space is the *Korenblum space*

$$\mathcal{H}:=\bigcup_{q\in[0,\infty)}H_q^\infty.$$

In some sense, Bergman, resp. weighted Bergman, spaces may be understood as the area integral counterparts of Hardy spaces. Given $0<p<\infty$ and $-1<\alpha<\infty$, the *weighted*

Bergman space A^p_α is defined as the family of analytic functions f in the unit disc $\mathbb{D}$ for which the area integral over the unit disc satisfies

$$\left(\int_{\mathbb{D}} \left|f(z)\right|^p \left(1-|z|^2\right)^\alpha \, \mathrm{d}\sigma\right)^{1/p} < \infty.$$

For $\alpha = 0$, we obtain the classical *Bergman space* A^p. Observe that $A^p \subset H^\infty_{2/p}$ and $H^\infty_p \subset A^{\varepsilon/p}$ for each $\varepsilon \in (0,1)$, see [64], p. 11.

Applying the derivative of an analytic function f in the unit disc, a collection of further complex function spaces follow. The classical starting point in this direction is the space $\mathcal{B}$ of *Bloch functions* defined as those functions satisfying

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \left|f'(z)\right| < \infty.$$

Slightly more generally, we may consider the *α-Bloch space* $\mathcal{B}^\alpha$ defined by

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|f'(z)\right| < \infty$$

for $0 < \alpha < \infty$. A connection to weighted Hardy spaces appears by the equality $\mathcal{B}^\alpha = H^\infty_{\alpha-1}$ for $1 < \alpha < \infty$, see [196], Proposition 7.

There are now two standard types of corresponding area integral counterparts of Bloch spaces. The more classical of them is the notion of *Dirichlet spaces*, resp. *weighted Dirichlet spaces*. For $-1 < q < \infty$, an analytic function f in the unit disc is said to be in the weighted Dirichlet space $\mathcal{D}_q$, provided

$$\int_{\mathbb{D}} \left|f'(z)\right|^2 \left(1-|z|^2\right)^q \, \mathrm{d}\sigma < \infty.$$

Of course, $\mathcal{D} := D_0$ is the classical Dirichlet space.

We also need related notion of Dirichlet type spaces $\mathcal{D}^p$ consisting of analytic functions f in the unit disc such that

$$\int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^{p-1} \, \mathrm{d}\sigma < \infty.$$

Of course, this means that $f' \in A^p_{p-1}$.

More recent is the notion of Q_p-spaces, introduced by Aulaskari and Lappan [1] in 1994. These spaces are defined for $0 < p < \infty$ as analytic functions f in the unit disc that satisfy

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^2 \left(g(z,a)\right)^p \, \mathrm{d}\sigma(z) < \infty,$$

where $g(z,a) = \log((1-\bar{a}z)/(z-a))$ is the usual Green function of the unit disc with a logarithmic singularity at $a \in \mathbb{D}$. As shown by Aulaskari and Lappan in [1], Q_p equals to

the Bloch space $\mathcal{B}$, provided $p > 1$. Moreover, $Q_1 = BMOA$, where $BMOA$ is the space of analytic functions of *bounded mean oscillation*. It is also important to keep in mind the strict nesting property $Q_p \subset Q_q \subset BMOA \subset \mathcal{B}$ whenever $0 < p < q < 1$.

Finally, replacing the derivative with the spherical derivative, we obtain the definition of α*-normal functions* instead of α*-Bloch functions*: A meromorphic function f in the unit disc is said to belong to the class of α*-normal functions* $\mathcal{N}^\alpha$, provided

$$\sup_{z\in\mathbb{D}}\bigl(1-|z|^2\bigr)^\alpha f^\sharp(z) < \infty.$$

Of course, $\mathcal{N}^1$ is nothing but the classical class $\mathcal{N}$ of *normal meromorphic functions* in the unit disc. Finally, we say that f is strongly normal, if

$$\lim_{|z|\to 1^-}\bigl(1-|z|^2\bigr) f^\sharp(z) = 0.$$

3.3. *Growth estimates for homogeneous linear differential equations*

In this section, we recall some recent results, which have appeared to be extremely powerful while considering the growth of solutions of homogeneous linear differential equations

$$L(f) := f^{(n)} + a_{n-1}(z) f^{(n-1)} + \cdots + a_0(z) f = 0 \tag{3.3.4}$$

in the unit disc, with analytic coefficients. The first two results [68] below make use of the classical Gronwall lemma to obtain a growth estimate for solutions of (3.3.4). Surprisingly, these estimates are, qualitatively, best possible.

THEOREM 3.3.1. *Let f be a solution of* (3.3.4) *in a disc D_R of radius R, centered at the origin.*

(a) *If $0 < R \leqslant 1$, then there exist a constant $C_1 = C_1(k) > 0$, depending on the initial conditions at the origin, and a constant $C_2 = C_2(k) > 0$, such that*

$$\bigl|f(re^{i\theta})\bigr| \leqslant C_1 \exp\left(C_2 \sum_{j=0}^{n-1}\sum_{k=0}^{j}\int_0^r \bigl|a_j^{(k)}(se^{i\theta})\bigr|(R-s)^{n-j+k-1}\,ds\right) \tag{3.3.5}$$

for all $\theta \in [0, 2\pi)$ and $r \in [0, R)$.

(b) *If $1 < R \leqslant \infty$, then there exist a constant $C_1 = C_1(k) > 0$, depending on the initial conditions at the origin, and a constant $C_2 = C_2(k) > 0$, such that*

$$\bigl|f(re^{i\theta})\bigr| \leqslant C_1 r^{n-1} \exp\left(C_2 \sum_{j=0}^{n-1}\sum_{k=0}^{j}\int_0^r \bigl|a_j^{(k)}(se^{i\theta})\bigr| s^{n-j+k-1}\,ds\right) \tag{3.3.6}$$

for all $\theta \in [0, 2\pi)$ and $r \in (1, R)$.

Combining the p-characteristic of an analytic function with the preceding theorem it is a straightforward conclusion to obtain

COROLLARY 3.3.2. *Let f be a solution of* (3.3.4) *in a disc D_R of radius R, centered at the origin, and let $1 \leqslant p < \infty$ be given.*

(a) *If $0 < R \leqslant 1$, then there exist a constant $C_1 = C_1(k) > 0$, depending on the initial conditions at the origin, and a constant $C_2 = C_2(k) > 0$, both depending on p, such that*

$$m_p(r, f)^p \leqslant C_1 + C_2 \sum_{j=0}^{n-1} \sum_{k=0}^{j} \int_0^{2\pi} \int_0^r \left| a_j^{(k)}(s\mathrm{e}^{\mathrm{i}\theta}) \right| (R-s)^{p(n-j+k-1)} \,\mathrm{d}s \,\mathrm{d}\theta \tag{3.3.7}$$

for all $r \in [0, R)$.

(b) *If $1 < R \leqslant \infty$, then there exist a constant $C_1 = C_1(k) > 0$, depending on f, and a constant $C_2 = C_2(k) > 0$, both depending on p, such that*

$$m_p(r, f)^p \leqslant C_1 (\log r^{n-1})^p + C_2 \sum_{j=0}^{n-1} \sum_{k=0}^{j} \int_0^{2\pi} \int_0^r \left| a_j^{(k)}(s\mathrm{e}^{\mathrm{i}\theta}) \right| s^{p(n-j+k-1)} \,\mathrm{d}s \,\mathrm{d}\theta \tag{3.3.8}$$

for all $r \in (1, R)$.

REMARK. Of course, the right-hand side integrals in (3.3.7) and (3.3.8) may be replaced by area integrals over the disc of radius r, if needed. The same remark also applies to (3.3.10).

Closely related estimates follow by applying a slight variant, see [68], of a comparison theorem due to Herold ([73], Satz 1):

THEOREM 3.3.3. *Let f be a solution of* (3.3.4) *in a disc D_R of radius R, where $0 < R \leqslant \infty$, centered at the origin, let b be the number of non-zero elements in the coefficient sequence $a_0, \dots, a_{n-1}$. Given $\theta \in [0, 2\pi)$ and $\varepsilon > 0$, if $z_\theta = \sigma \mathrm{e}^{\mathrm{i}\theta} \in D_R$ is such that $a_j(z_\theta) \neq 0$ for some of the coefficients, then we have for all r such that $\sigma < r < R$,*

$$\left| f(r\mathrm{e}^{\mathrm{i}\theta}) \right| \leqslant C \exp\left(b \int_\sigma^r \max_{0 \leqslant j \leqslant n-1} \left| a_j(t\mathrm{e}^{\mathrm{i}\theta}) \right|^{1/(n-j)} \mathrm{d}t \right), \tag{3.3.9}$$

where $C > 0$ is a constant depending on b and $a_j(z_\theta)$, $f^{(j)}(z_\theta)$, $j = 0, \dots, n-1$. Actually,

$$C \leqslant (1+\varepsilon) \max_{j=0,\dots,n-1} \left(\frac{|f^{(j)}(z_\theta)|}{b^j \max_{k=0,\dots,n-1} |a_k(z_\theta)|^{j(n-k)}} \right).$$

Similarly as in the Gronwall estimate case, the following corollary is an easy consequence:

COROLLARY 3.3.4. *Let f be a solution of* (3.3.4) *in a disc D_R of radius R, where $0 < R \leqslant \infty$, centered at the origin, and let $1 \leqslant p < \infty$. Then, for all $0 \leqslant r < R$, we have*

$$m_p(r, f)^p \leqslant C\left(1 + \sum_{j=0}^{n-1} \int_0^{2\pi}\int_0^r \left|a_j(s\mathrm{e}^{\mathrm{i}\theta})\right|^{p/(n-j)} \mathrm{d}s\,\mathrm{d}\theta\right), \tag{3.3.10}$$

where $C = C(n) > 0$ is a constant depending on p, and on the initial values of f at a point z_θ, where at least one of the coefficients a_j takes a non-zero value.

We also recall certain pointwise estimates for the generalized meromorphic functions in the unit disc, corresponding to similar estimates due to Gundersen in the plane. For a more detailed exposition, see [33]. The first estimate here, see [69], Lemma B, is a variant of the original version in [33]:

THEOREM 3.3.5. *Given two integers k, j such that $k > j \geqslant 0$, $\varepsilon > 0$ and $\beta \in (0, 1)$, if f is meromorphic in the unit disc $\mathbb{D}$ and $f^{(j)}$ does not vanish identically, then for all z such that $r = |z| \notin E$, where E is of finite logarithmic measure, i.e. $\int_E (1-r)^{-1}\,\mathrm{d}r < \infty$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant \left(\left(\frac{1}{1-r}\right)^{2+\varepsilon} \max\left(\log\frac{1}{1-r}, T\big(s(r), f\big)\right)\right)^{n-j}, \tag{3.3.11}$$

where $s(r) := 1 - \beta(1-r)$. If f is of finite order $\rho(f)$ in the unit disc, then

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant \left(\frac{1}{1-r}\right)^{(n-j)(\rho(f)+2+\varepsilon)}, \quad r \notin E, \tag{3.3.12}$$

while if f is of finite iterated order $\rho_k(f)$ for some $k \geqslant 2$, then

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant \exp_{k-1}\left(\left(\frac{1}{1-r}\right)^{\rho(f)+\varepsilon}\right), \quad r \notin E. \tag{3.3.13}$$

Finally, we mention that the notion of disconjugacy treated relative to Eq. (3.1.1) above may be extended to the higher order case (3.3.4), see [116]: Equation (3.3.4) is said to be desconjugate in the unit disc $\mathbb{D}$, if all non-trivial solutions have at most $n-1$ zeros in $\mathbb{D}$. For disconjugacy results in this setting, see [58,101,116] and [117].

3.4. *Equations with fast growing solutions*

The key notion in these considerations, essentially started by Heittokangas in his thesis [64] in 2000, is the order, resp. iterated order, of an analytic, resp. meromorphic, function

f in the unit disc $\mathbb{D}$. Defining for analytic functions

$$\rho_j(f) := \limsup_{r\to 1^-} \frac{\log_j^+ T(r,f)}{-\log(1-r)}$$

and

$$\rho_{M,j}(f) := \limsup_{r\to 1^-} \frac{\log_{j+1}^+ M(r,f)}{-\log(1-r)},$$

it is immediate to observe that $\rho(f) \leqslant \rho_M(f) \leqslant \rho(f)+1$ for the usual order, while $\rho_j(f) = \rho_{M,j}(f)$ for $j \geqslant 2$. Therefore, unless otherwise specified, we use $\rho_j(f)$ to describe the order in what follows in this section. A starting point for order considerations of solutions of homogeneous linear differential equations (3.3.4) in the unit disc with analytic coefficients is the following counterpart of the Frei theorem, Theorem 2.3.1, in the complex plane:

THEOREM 3.4.1. *Let a_j be the last coefficient in the sequence $a_0,\dots,a_{n-1}$ not belonging to the Korenblum space $\mathcal{H}$. Then Eq.* (3.3.4) *possesses at most j linearly independent solutions of finite order of growth in $\mathbb{D}$.*

This theorem, due to Heittokangas, see [64], Theorem 6.3, has been recently generalized by using the notion of iterated order, see [25] and [69]. To this end, we introduce the unit disc counterpart of the *finiteness degree* $i(f)$ of f, and the related notions, see Section 2.3. So, $i(f) := 0$ for meromorphic functions of finite degree and $i(f) := \min\{s \in \mathbb{N};\ \rho_s(f) < \infty\}$ for meromorphic functions of infinite degree. Of course, this means that $i(f) = \infty$, if $\rho_s(f) = \infty$ for all $s \in \mathbb{N}$. Of course, we may use $\rho_M(f)$ as well, equivalently, to determine $i(f)$ for analytic functions in the unit disc. Exactly as in Section 2.3, we then define in this unit disc setting $\delta(L), \gamma_j(L), p(L)$ and $\kappa(L)$. Also, we may use $\gamma_{M,j}(L)$ and $\kappa_M(L)$, if needed, observing that $\gamma_\delta(L) = \gamma_{M,\delta}(L)$, if $\delta > 1$, and $\gamma_1(L) \leqslant \gamma_{M,1}(L) \leqslant \gamma_1(L)+1$, if $\delta = 1$. Similarly, $\kappa(L) = \kappa_M(L)$, if $p(L) > 1$ and $\kappa(L) \leqslant \kappa_M(L) \leqslant \kappa(L)+1$, if $p = 1$.

THEOREM 3.4.2. (See [25], Theorem 2.1.) *For Eq.* (3.3.4), *the following conditions always hold*:

(i) $\delta(L) \leqslant 1 + p(L)$,
(ii) *if* $1 < p < \infty$, *then* $\delta(L) = 1 + p(L)$ *and* $\kappa_M(L) = \gamma_{M,p+1}(L) = \gamma_{p+1} = \kappa(L)$,
(iii) *if* $p = 0$ *and* $m := \max_j D_M(a_j)$, *then* $\gamma_1(L) \leqslant \gamma_{M,1}(L) \leqslant 1 + m$.

There are a number of recent results, see [25,69] and [104], that are improvements of the preceding theorem:

THEOREM 3.4.3. (See [69], Theorem 1.1, [25], Theorem 2.2.) *Given $k \in \mathbb{N}$ and $\alpha \geqslant 0$, all solutions of* (3.3.4) *in $\mathbb{D}$ satisfy $\rho_{M,k+1}(f) \leqslant \alpha$ if and only if $\rho_{M,k}(a_j) \leqslant \alpha$ for all $j = 0,\dots,n-1$. Moreover, if q, $0 \leqslant q \leqslant n-1$ is the largest index such that $\rho_{M,k}(a_q) =$*

$\max_{0\leqslant j\leqslant n-1}\rho_{M,k}(a_j)$, *then each solution base of* (3.3.4) *possesses at least* $n-q$ *linearly independent solutions* f *such that* $\rho_{M,k+1}(f)=\rho_{M,k}(a_q)$.

This result is sharp in the sense that examples exist of a solution base containing exactly $n-q$ solutions of maximal growth, see [69], Example 1.3. On the other hand, examples of (3.3.4) exist such that all non-trivial solutions are of maximal growth, see [69], Example 1.4.

THEOREM 3.4.4. (See [25], Theorem 2.3, [69], Theorem 1.2.) *Suppose* $0<p<\infty$ *and* $i(a_0)=p$. *If* $\max_{1\leqslant j\leqslant n-1} i(a_j)<p$ *or if* $\max_{1\leqslant j\leqslant n-1}\rho_{M,p}(a_j)<\rho_{M,p}(a_0)$, *then for all solutions* f *of* (3.3.4), $i(f)=p+1$ *and* $\rho_{M,p+1}(f)=\rho_{M,p}(a_0)$.

The preceding theorems may be slightly refined by introducing the notion of *iterated type*: Given $k\in\mathbb{N}$, and an analytic function f in the unit disc $\mathbb{D}$ such that $0<\rho_{M,k}(f)<\infty$, we define the iterated k-type as

$$\tau_{M,k}(f):=\limsup_{r\to1^-}(1-r)^{\rho_{M,k}(f)}\log_k^+ M(r,f).$$

THEOREM 3.4.5. (See [69], Theorem 1.5.) *Given* $k\in\mathbb{N}$, *suppose that*

$$\max_{1\leqslant j\leqslant n-1}\rho_{M,k}(a_j)\leqslant\rho_{M,k}(a_0)$$

and that

$$\sum_{\rho_{M,k}(a_j)=\rho_{M,k}(a_0)}\tau_{M,k}(a_j)<\tau_{M,k}(a_0),$$

then all non-trivial solutions f *of* (3.3.4) *satisfy* $\rho_{M,k+1}(f)=\rho_{M,k}(a_0)$.

REMARK. Concerning Theorems 3.4.4 and 3.4.5, it remains open, whether a similar conclusion holds, if a dominating coefficient a_j exists, but $j\neq0$.

In addition to the growth considerations of fast growing solutions of (3.3.4), published results about the oscillation of such solutions in the unit disc are practically non-existent. The only exception we know is a forth-coming paper [26] by Cao and Yi, devoted to considering zero distribution of solutions of $f''+A(z)f=0$, where the coefficient $A(z)$ is of infinite degree $D(f)=\infty$. In this case it is well known that $\rho_2(f)=\rho_M(A)$, and oscillation results parallel to what are known from the complex plane case immediately follow by applying the unit disc version of the Nevanlinna theory, see [26], Theorem 2.2 through Theorem 2.4. See also [28].

3.5. *Equations with slowly growing solutions*

As pointed out above, Nevanlinna theory becomes ineffective as soon as a solution f of a differential equation, say of (3.3.4), is slowly growing in the rough sense that the growth

of a solution is not essentially faster than the growth of coefficients. Indeed, it may happen that the growth of solution is slower than that of the coefficients, as shown by the equation $f^{(n)} = (1-z)^{-\alpha} f = 0$, the solutions of which are bounded in the unit disc, provided $\alpha < n$. In the case of slowly growing solutions, complex function spaces appear to be the correct tool. The key questions regarding the growth may be formulated as follows, see [70], Section 1:

(1) Providing the coefficients a_j, $j = 0, \dots, n-1$ in (3.3.4) belong to a certain analytic function space (possibly depending on j), determine the function space(s) where solutions of (3.3.4) belong to.
(2) Providing that solutions f of (3.3.4) belong to a certain analytic function space, determine the function space(s) where the coefficients a_j belong to.

In what follows, the first problem here is called the *direct growth problem*, while the second is termed as the *inverse growth problem*. Concerning these two problems, the key results, see [70], may be expressed as follows:

THEOREM 3.5.1. *Let f be a non-trivial solution of* (3.3.4) *with analytic coefficients a_j, $j = 0, \dots, n-1$, in the unit disc $\mathbb{D}$. Then we conclude that*

(1) *$f \in N^p$, provided $1 \leqslant p < \infty$ and $a_j \in A^{p/(n-j)}$ for all $j = 0, \dots, n-1$, or $a_j \in A^p_{p(n-j-1)}$ for all $j = 0, \dots, n-1$,*
(2) *$f \in N$, provided $-1 < \alpha < 0$ and $a_j \in H^{1/(n-j)}_{(\alpha+1)(n-j)}$ for all $j = 0, \dots, n-1$,*
(3) *f is of finite degree $D(f)$, provided $a_j \in H^{1/(n-j)}_{n-j}$ for all $j = 0, \dots, n-1$,*
(4) *f is of finite order $\rho(f) \leqslant \alpha$, provided $0 < \alpha < \infty$ and $a_j \in A^{1/(n-j)}_{\alpha}$ for all $j = 0, \dots, n-1$, or $a_j \in A^1_{n-j-1+\alpha}$ for all $j = 0, \dots, n-1$,*
(5) *f is of finite order $\rho(f) \leqslant \alpha$, provided $\alpha > 0$ and $a_j \in H^{1/(n-j)}_{(\alpha+1)(n-j)}$ for all $j = 0, \dots, n-1$.*

Similarly, for the inverse problem, we obtain

THEOREM 3.5.2. *Let f be a non-trivial solution of* (3.3.4) *with analytic coefficients a_j, $j = 0, \dots, n-1$, in the unit disc $\mathbb{D}$. Then we conclude that*

(1) *The coefficients $a_j \in \bigcap_{0<p<1/(n-j)} A^p$ for all $j = 0, \dots, n-1$, provided all non-trivial solutions $f \in N$,*
(2) *the coefficients $a_j \in \bigcap_{0<p<1/(n-j)} H^p_{1/p}$ for all $j = 0, \dots, n-1$, provided all non-trivial solutions f are of finite degree $D(f)$,*
(3) *the coefficients $a_j \in \bigcap_{0<p<1/(n-j)} A^p_\alpha$ for all $j = 0, \dots, n-1$, provided all non-trivial solutions f are of finite order $\rho(f) \leqslant \alpha$,*
(4) *the coefficients $a_j \in \bigcap_{0<p<1/(n-j)} H^p_{(\alpha+1)/p}$ for all $j = 0, \dots, n-1$, provided $0 \leqslant \alpha < \infty$ and all non-trivial solutions f are of finite order $\rho(f) \leqslant \alpha$.*

For a quick, not quite complete, overview of the preceding two theorems, see the table in [70], Section 1. See also [70], Section 4, for a number of inclusions between the function spaces applied here.

Refining the reasoning used to prove Theorems 3.5.1(4) and 3.5.2(3), one may indeed even prove the following characterization, see [105]:

THEOREM 3.5.3. *All solutions f of the linear differential equations* (3.3.4) *satisfy* $\rho(f) \leqslant \alpha$ *if and only if the coefficients* $a_j \in \bigcap_{0<p<1/(n-j)} A^p_\alpha$ *for all* $j = 0, \dots, n-1$.

A slight improvement has been proved in [105] as follows:

THEOREM 3.5.4. *Denoting by* $\mathbb{A}^p_\alpha$ *the space of analytic function f in the unit disc $\mathbb{D}$ such that* $\alpha = \inf\{t \geqslant 0;\ f \in \bigcap_{0<s<p} A^s_t\}$, *we conclude that all non-trivial solutions f of* (3.3.4) *satisfy*

$$\min\left\{\alpha_0, \min_{j=1,\dots,n-1}\left\{\alpha_j + \frac{k(\alpha_0-\alpha_j)}{j}\right\}\right\} \leqslant \rho(f) \leqslant \max_{j=0,\dots,n-1} \alpha_j,$$

provided the coefficients $a_j \in \mathbb{A}^{1/(n-j)}_{\alpha_j}$ *for all* $j = 0, \dots, n-1$.

Moreover, making use of the standard order reduction procedure, and using the same notations as in the preceding theorem, we conclude

THEOREM 3.5.5. *If* $q \in \{0, \dots, n-1\}$ *is the smallest index such that* $\alpha_q = \max_{j=0,\dots,n-1} \alpha_j$, *then each solution base of* (3.3.4) *possesses at least* $n-q$ *linearly independent solutions f of order* $\rho(f) = \alpha_q$.

Replies to the direct problem under the requirement that all solutions of (3.3.4) belong to, say H^∞_p, are not included in the preceding theorems. This type of recent investigations may be found in [71]. Main results in this important paper can be collected as follows:

THEOREM 3.5.6. *Given* $0 < \delta < 1$, *for every* $p > 0$ *there exists a constant* $\alpha = \alpha(p,n) > 0$ *such that if the coefficients* a_j *of* (3.3.4) *satisfy*

$$\sup_{|z|\geqslant\delta} \left|a_j(z)\right|\left(1-|z|^2\right)^{n-j} \leqslant \alpha \tag{3.5.14}$$

for $j = 0, \dots, n-1$, *then all solutions f of* (3.3.4) *are in the weighted Hardy space* H^∞_p. *In particular, the conclusion remains true, if the supremum in* (3.5.14) *is taken over the whole unit disc* $\mathbb{D}$.

Proving the next theorem, see also [71], one has to make use of the Carleson measure, and the related notion of a Carleson box

$$S(I) := \left\{z \in \mathbb{D};\ \frac{z}{|z|} \in I, 1-|I| \leqslant |z|\right\}$$

attached to a subarc of the unit disc boundary $\partial\mathbb{D}$ such that the arc length $|I| \leqslant 1$.

THEOREM 3.5.7. *Given* $0 < \delta < 1$, *all solutions* f *of* (3.3.4) *belong to the space* $\mathcal{D}^p \cap H_p^\infty$, *provided one of the following conditions hold*:

(1) *For every* $0 < p \leqslant 2$, *there exists a constant* $\alpha = \alpha(p,n)$ *such that the coefficients* a_j *satisfy the condition*

$$\sup_{\zeta \geqslant \delta} \int_{\mathbb{D}} \left|a_j(z)\right|^p \left(1 - |z|^2\right)^{p(n-j)-1} \frac{1 - |\zeta|^2}{|1 - \bar{\zeta} z|^2} \, \mathrm{d}\sigma_z \leqslant \alpha$$

for all $j = 0, \ldots, n-1$.

(2) *For every* $0 < p \leqslant 2$, *there exists a constant* $\beta = \beta(p,n)$ *such that the coefficients* a_j *satisfy the condition*

$$\sup_{0<|I|\leqslant\delta} \int_{S(I)} \left|a_j^k(z)\right|^p \left(1 - |z|^2\right)^{p(n+k-j)-1} \, \mathrm{d}\sigma_z \leqslant \beta$$

for all $j = 0, \ldots, n-1$ *and all* $k = 0, \ldots, n-1$.

For additional results of similar type as the two theorems above, see [144]. We also refer to [43] for results concerning the growth of $A(z)$, provided f is an α-normal solution of $f^{(k)} + A(z)f = 0$.

4. Non-linear differential equations in a complex domain

4.1. *Introductory remarks*

We first recall three key results, repeatedly used in studying non-linear algebraic differential equations in the complex plane, namely those called as the Clunie, Mohon'ko and Valiron–Mohon'ko lemmas. In this section, we mostly refer to [107], where complete proofs and references to the original works can be found.

We start with the Clunie lemma, see [107], Lemma 2.4.2:

THEOREM 4.1.1. *Let* f *be a meromorphic solution of*

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f), Q(z, f)$ *are polynomials in* f *and its derivatives with meromorphic coefficients, say* $a_\lambda, \lambda \in I$, *such that* $m(r, a_\lambda) = S(r, f)$ *for all* $\lambda \in I$. *If the total degree of* $Q(z, f)$ *in* f *and its derivatives is* $\leqslant n$, *then*

$$m\big(r, P(z, f)\big) = S(r, f).$$

REMARK. Observe that there are numerous variants of this theorem in the literature, see e.g. [107], Section 2.4. Perhaps the most general version may be found in a forth-coming paper [192] by Yang and Ye. This version reads as follows:

THEOREM 4.1.2. *Let f be a meromorphic solution of*

$$S(z, f)P(z, f) = Q(z, f),$$

where $S(z, f)$ is a polynomials in f, and $P(z, f)$, $Q(z, f)$ are differential polynomials in f, all three with small coefficients $a(z)$ in the sense of $m(r, a) = S(r, f)$. If the degree of $S(z, f)$ is greater or equal to the total degree of $Q(z, f)$ in f and its derivatives, then $m(r, P(z, f)) = S(r, f)$.

We next proceed to the Mohon'ko lemma, see [107], Proposition 9.2.3:

THEOREM 4.1.3. *Let f be a meromorphic solution of an algebraic differential equation*

$$P(z, f, f', \ldots, f^{(n)}) = 0, \tag{4.1.1}$$

where P is a polynomial in $f, f', \ldots, f^{(n)}$ with meromorphic coefficients small with respect to f. If a complex constant c does not satisfy Eq. (4.1.1), *then*

$$m\left(r, \frac{1}{f - c}\right) = S(r, f).$$

In this case as well, there are several variants in the literature. We recall here the following version, due in this generality to Ishizaki and Yanagihara [95]:

THEOREM 4.1.4. *Let f be a meromorphic solution of an algebraic differential equation* (4.1.1), *with meromorphic coefficients small with respect to f. Moreover, let $Q(z, f)$ be an irreducible polynomial in f with meromorphic coefficients small with respect to f. If the function elements determined by equation $Q(z, f) = 0$ do not satisfy Eq.* (4.1.1), *then we have for $Q = Q(z, f(z))$ that $m(r, 1/Q) = S(r, f)$.*

REMARK. The Ishizaki–Yanagihara variant of the Mohon'ko lemma has not been used very often in the theory of complex differential equations. However, we believe the importance of this version has been underestimated.

To close this section, we recall the Valiron–Mohon'ko theorem, see [107], Theorem 2.2.5:

THEOREM 4.1.5. *Let f be a meromorphic function, and let $R(z, f) = P(z, f)/Q(z, f)$ be an irreducible rational function in f with meromorphic coefficients small with respect to f. Then the characteristic function of $R(z, f(z))$ satisfies*

$$T\big(r, R(z, f)\big) = dT(r, f) + S(r, f),$$

where $d := \max(\deg_f P(z, f), \deg_f Q(z, f))$.

4.2. *Malmquist type theorems*

Malmquist type theorems essentially contain necessary existence conditions for meromorphic solutions of algebraic differential equations that are fast growing with respect to the coefficients of the equation. The original version, due to Malmquist [125] in 1913, reads as follows:

THEOREM 4.2.1. *If a differential equation of the form*

$$f' = R(z, f),$$

where $R(z, f)$ is rational in both arguments and irreducible in f, admits a transcendental meromorphic solution, then the equation reduces to a Riccati differential equation

$$f' = a_0(z) + a_1(z)f + a_2(z)f^2$$

with rational coefficients.

Proving this theorem with the aid of the Nevanlinna theory is almost trivial, just making use of the Valiron–Mohon'ko theorem and comparing the characteristic functions of both sides of the equation. Yosida was the first one to offer this type of proof in 1933. Introducing the notion of a *small function* $\alpha(z)$ with respect to another function f by the requirement $T(r, \alpha) = S(r, f)$, where $S(r, f)$ stands for a quantity $o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure, a slight improvement easily follows by the same reasoning:

THEOREM 4.2.2. *If a differential equation of the form*

$$(f')^n = R(z, f), \tag{4.2.2}$$

where $R(z, f)$ is rational and irreducible in f with meromorphic coefficients, admits a meromorphic solution f such that all coefficients of $R(z, f)$ are small with respect to f, then (4.2.2) *reduces into*

$$(f')^n = \sum_{j=0}^{2n} \alpha_j(z) f^j, \tag{4.2.3}$$

where at least one of the coefficients $\alpha_j(z)$ does not vanish identically.

In fact, the reduced equation (4.2.3) may still be reduced further. This reduction was first made by Steinmetz [169] in the basic case of rational coefficients, then partially extended to the case of meromorphic coefficients by von Rieth [185], and finally completed in [62]. In the general case (with non-rational coefficients), the proof in [62] makes use of a slight extension of the Hayman–Miles theorem estimating $T(r, f)$ in terms of $T(r, f')$ in a set of positive lower logarithmic density, see [60] and [62], Lemma D:

THEOREM 4.2.3. *If a differential equation of the form* (4.2.2), *where* $R(z, f)$ *is rational and irreducible in* f *with meromorphic coefficients, admits a meromorphic solution* f *such that all coefficients of* $R(z, f)$ *are small with respect to* f, *then* (4.2.2) *reduces by a suitable non-singular Möbius transformation* $f = (\alpha y + \beta)/(\gamma y + \delta)$ *into one of the following equation types* (*or an integer power of one of these*):

$$y' = a(z) + b(z)y + c(z)y^2, \tag{4.2.4}$$

$$(y')^2 = a(z)\big(y - b(z)\big)^2(y - \tau_1)(y - \tau_2), \tag{4.2.5}$$

$$(y')^2 = a(z)(y - \tau_1)(y - \tau_2)(y - \tau_3)(y - \tau_4), \tag{4.2.6}$$

$$(y')^3 = a(z)(y - \tau_1)^2(y - \tau_2)^2(y - \tau_3)^2, \tag{4.2.7}$$

$$(y')^4 = a(z)(y - \tau_1)^2(y - \tau_2)^3(y - \tau_3)^3, \tag{4.2.8}$$

$$(y')^6 = a(z)(y - \tau_1)^3(y - \tau_2)^4(y - \tau_3)^5, \tag{4.2.9}$$

where $\tau_1, \tau_2, \tau_3, \tau_4$ *are complex constants, and the coefficients* $a(z), b(z), c(z)$ *are small meromorphic functions. Moreover,* $a(z)$ *does not vanish identically in* (4.2.5)–(4.2.9).

As pointed out by Steinmetz in [169], examples of all six types in Theorem 4.2.3 with a meromorphic solution such that the coefficients are small may easily be constructed.

Somewhat similar considerations concerning the following two differential equations

$$(f')^n = q(z)P(f)P_0(f')(f - z)^m \tag{4.2.10}$$

and

$$(f')^n = q(z)\mathrm{e}^{P_1(z)}P(f)(f - z)^m, \tag{4.2.11}$$

where q is a rational function, P, P_0, P_1 are polynomials, $P_0 \not\equiv 0$ and m, n are natural numbers, have been made by Ishizaki and Wang in [94]. We remark that these differential equations have a certain interest in the field of complex dynamics, as shown by Bergweiler, Terglane and Wang. As in the case of Malmquist–Yosida equations, existence of transcendental meromorphic solutions implies that the equations reduce to certain special forms:

THEOREM 4.2.4. *If Eq.* (4.2.10) *admits a transcendental meromorphic solution, then* (4.2.10) *reduces to one of the following special forms*:

$$f' = q(z)(f - z),$$

$$f' = q(z)(f - z)^2,$$

$$f' = q(z)(f - \tau_1)(f - z),$$

$$(f')^2 = q(z)(f - \tau_1)(f - z)^2,$$

$$(f')^2 = q(z)(f - \tau_1)(f - \tau_2)(f - z)^2,$$

where τ_1, τ_2 *are distinct complex constants.*

THEOREM 4.2.5. *If Eq.* (4.2.11) *admits a transcendental meromorphic solution, then* (4.2.11) *reduces to one of the following special forms*:

$$
\begin{aligned}
&f' = q(z)e^{P_1(z)}(f - z),\\
&f' = q(z)e^{P_1(z)}(f - z)^2,\\
&f' = q(z)e^{P_1(z)}(f - \tau_1)(f - z),\\
&(f')^2 = q(z)e^{P_1(z)}(f - \tau_1)(f - z)^2,\\
&(f')^2 = q(z)e^{P_1(z)}(f - \tau_1)(f - \tau_2)(f - z)^2,
\end{aligned}
$$

where τ_1, τ_2 *are distinct complex constants.*

Concerning first order differential equations in general, first recall a classical theorem due to L. Fuchs:

THEOREM 4.2.6. *If the solutions of a differential equation*

$$P(z, f, f') := \sum_{j=0}^{n} A_j(z, f)(f')^j = 0 \tag{4.2.12}$$

where $P(z, f, f')$ *is an irreducible polynomial in* f, f' *with analytic coefficients, is free of movable singularities other than poles, then*

$$\deg_f A_j(z, f) \leqslant 2(n - j).$$

In particular, $A_0(z, f)$ *is independent of* f.

We remark in this connection that the notion of *movable singularities* has had wide use in the theory of complex differential equations, although it is not easy to find a rigorous definition in the literature. We recall the following one due to Bieberbach in [23]: Considering a complex differential equation (D), let S_D be the set of those points z_0 where at least one of the solutions of (D) has a singularity other than a pole. Then (D) is *free of movable singularities*, if the interior of S_D is empty. Isolated points in S_D are called *fixed singularities* of (D).

Somewhat surprisingly, the assertion as in Theorem 4.2.6 results while extending Theorem 4.2.2 to equations of type (4.2.12). This extension is called to be of the second Malmquist theorem type, see [126] for the original paper and [38], Theorem 6. To find a rigorous proof of this result, see [38], Sections 2–4. In a form corresponding to the previous versions of the (first) Malmquist theorem, we may write

THEOREM 4.2.7. *If a differential equation of the form* (4.2.12), *where the left-hand side* $P(z, f, f')$ *is an irreducible polynomial in* f, f' *with meromorphic coefficients, admits a meromorphic solution* f *such that all coefficients are small with respect to* f, *then* $\deg_f A_j(z, f) \leqslant 2(n - j)$. *In particular,* $A_0(z, f)$ *is independent of* f.

As pointed out by A. Eremenko in [38], the preceding way of considering a solution f of a differential equation such that all coefficients are small with respect to f, may easily be expressed in a slightly more general form. To this end, let $\varphi : [0, \infty) \to (0, \infty)$ satisfy $\log r = \mathrm{O}(\varphi(r))$ as $r \to \infty$, and denote by $\mathcal{M}_\varphi$ the family of all meromorphic functions that satisfy $T(r, f) = \mathrm{O}(\varphi(r))$ as $r \to \infty$ outside of a possible exceptional set E, depending on f, of finite linear measure. As one may easily see, $\mathcal{M}_\varphi$ is an algebraically closed field. Supposing now that all coefficients of $P(z, f, f')$ are in $\mathcal{M}_\varphi$, we say that a solution f of Eq. (4.2.12) is *admissible*, provided $f \notin \mathcal{M}_\varphi$. We may now reformulate Theorem 4.2.7 as follows, see [38], Theorem 6, and [86], Section 4:

THEOREM 4.2.8. *If a differential equation of the form* (4.2.12), *where the left-hand side $P(z, f, f')$ now is an irreducible polynomial in f, f' with meromorphic coefficients in $\mathcal{M}_\varphi$, admits a meromorphic solution $f \notin \mathcal{M}_\varphi$, then*

$$\deg_f A_j(z, f) \leqslant 2(n - j).$$

In particular, $A_0(z, f)$ is independent of f.

It remains now to consider such issues as the growth, value distribution, factorization etc. of the special types of differential equations described above with fast growing meromorphic solutions. As an example of such results we recall the following theorem due to S. Bank and R. Kaufman in [11]:

THEOREM 4.2.9. *Let f be a transcendental meromorphic solutions of the Malmquist–Yosida equation* (4.2.2) *with polynomial coefficients. Then the order $\rho(f)$ is either zero, or a positive multiple of* $1/2$, *or a positive multiple of* $1/3$. *Conversely, any such number appears as the order of growth of a transcendental meromorphic solution in the plane of such Eq.* (4.2.2).

Observe that the assertion of Theorem 4.2.9 remains valid for all first-order algebraic differential equations with polynomial coefficients, as shown by Eremenko in [40].

Concerning Malmquist type theorems in the second order case, it would be natural to ask for a complete list of differential equations of type

$$f'' = R(z, f, f'), \tag{4.2.13}$$

where $R(z, f, f')$ is rational in f, f' with meromorphic coefficients, such that all the coefficients (4.2.13) are small with respect to some solution f that does not satisfy first order differential equations of type (4.2.12), save for a possible Möbius transformation type change of variable in f. The requirement not to satisfy a first order algebraic differential equations is quite obvious, since almost arbitrarily complicated second order differential equations may be derived from (4.2.12) by differentiation. It has been conjectured that the existence of a fast growing solution implies that (4.2.13) reduces to

$$f'' = L(z, f)(f')^2 + M(z, f)f' + N(z, f), \tag{4.2.14}$$

where $L(z, f), M(z, f), N(z, f)$ are rational in f with small meromorphic coefficients, save for a Möbius transformation in f. In this full generality, the problem remains open, and we are aware of just one result that could be called as a second order Malmquist type theorem:

THEOREM 4.2.10. *If Eq.* (4.2.13), *where $R(z, f, f')$ is rational in all of its arguments, possesses a meromorphic solution f of infinite order, then* (4.2.13) *reduces into* (4.2.14).

Observe that the proof in [120] is quite different to the usual Malmquist theorem reasoning based on Nevanlinna theory; this time the well known Zalcman lemma and some pointwise derivative estimates due to Barsegian will be applied. Observe as well that such a solution f cannot satisfy a first-order algebraic differential equation by the classical Gol'dberg theorem [50], see Theorem 4.7.1.

Except for the preceding Theorem 4.2.10, all results below related to the second order Malmquist type topic are restricted to considering equation (4.2.14) under the assumption that all coefficients are small with respect to a meromorphic solution f. We recall two results of this type:

THEOREM 4.2.11. (See [172], Theorem 2.) *Writing* (4.2.14) *in the form*

$$P_3(z, f)f'' = P_2(z, f)(f')^2 + P_1(z, f)f' + P_0(z, f),$$

where P_0, P_1, P_2, P_3 are relatively prime polynomials with rational coefficients, suppose that f is a transcendental meromorphic solution. Then

$$\deg_f P_3 \leqslant 4, \quad \deg_f (fP_2 - 2P_3) \leqslant 3, \quad \deg_f P_1 \leqslant 4, \quad \deg_f P_0 \leqslant 6.$$

These bounds are best possible as shown by [172], Example 1.

THEOREM 4.2.12. (See [121], Theorem 1.) *Suppose that $L(z, f), M(z, f), N(z, f)$ in Eq.* (4.2.14) *are rational in both arguments, and suppose that f is a transcendental meromorphic solution of* (4.2.14). *Moreover, we assume that*

$$L(z, f) = \frac{a_n(z)f^n + \cdots + a_s(z)f^s}{b_m(z)f^m + \cdots + b_r(z)f^r},$$

where either $m - n < 1$ or $r - s > 1$. Then $\rho(f) < \infty$.

Concerning this last result, the simple example $f(z) = \exp(e^z)$ solving $ff'' - (f')^2 - ff' = 0$ shows that the assumptions $m - n < 1$, $r - s > 1$ cannot be deleted simultaneously.

In the autonomous case, Liao and Ye [122] were able to extract the possible types with fast growing solutions as follows:

THEOREM 4.2.13. *If the differential equation* (4.2.14) *with constant coefficients possesses a transcendental meromorphic solution f of order $\rho(f) > 2$, then the equation reduces into one of the following nine special cases*:

$$f'' = \sum_{j=1}^{4} \frac{1}{f-a_j}\left(\frac{1}{2}(f')^2 + C_j\right) + b_0 f' + c_2 f^2 + c_1 f + c_0, \tag{4.2.15}$$

$$\begin{aligned} f'' = {} & \frac{1}{f-a_1}\big((f')^2 + B_1 f' + C_1\big) + \frac{1}{f-a_2}\left(\frac{1}{2}(f')^2 + C_2\right) \\ & + \frac{1}{f-a_3}\left(\frac{1}{2}(f')^2 + C_3\right) + b_0 f' + c_2 f^2 + c_1 f + c_0, \end{aligned} \tag{4.2.16}$$

$$\begin{aligned} f'' = {} & \frac{1}{f-a_1}\left(\frac{1}{2}(f')^2 + C_1\right) + \frac{1}{f-a_2}\left(\frac{1}{2}(f')^2 + C_2\right) \\ & + \frac{1}{f-a_3}\left(\frac{1}{2}(f')^2 + C_3\right) + (b_1 f + b_0) f' \\ & + c_3 f^3 + c_2 f^2 + c_1 f + c_0, \end{aligned} \tag{4.2.17}$$

$$\begin{aligned} f'' = {} & \frac{1}{f-a_1}\left(\frac{2}{3}(f')^2 + B_1 f' + C_1\right) + \frac{1}{f-a_2}\left(\frac{2}{3}(f')^2 + B_2 f' + C_2\right) \\ & + \frac{1}{f-a_3}\left(\frac{2}{3}(f')^2 + B_3 f' + C_3\right) + b_0 f' + c_2 f^2 + c_1 f + c_0, \end{aligned} \tag{4.2.18}$$

$$\begin{aligned} f'' = {} & \frac{1}{f-a_1}\left(\frac{1}{2}(f')^2 + C_1\right) + \frac{1}{f-a_2}\left(\frac{3}{4}(f')^2 + B_2 f' + C_2\right) \\ & + \frac{1}{f-a_3}\left(\frac{3}{4}(f')^2 + B_3 f' + C_3\right) + b_0 f' + c_2 f^2 + c_1 f + c_0, \end{aligned} \tag{4.2.19}$$

$$\begin{aligned} f'' = {} & \frac{1}{f-a_1}\left(\frac{1}{2}(f')^2 + C_1\right) + \frac{1}{f-a_2}\left(\frac{2}{3}(f')^2 + B_2 f' + C_2\right) \\ & + \frac{1}{f-a_3}\left(\frac{5}{6}(f')^2 + B_3 f' + C_3\right) + b_0 f' + c_2 f^2 + c_1 f + c_0, \end{aligned} \tag{4.2.20}$$

$$\begin{aligned} f'' = {} & \frac{1}{f-a_1}\big(A_1 (f')^2 + B_1 f' + C_1\big) + \frac{1}{f-a_2}\big(A_2 (f')^2 + B_2 f' + C_2\big) \\ & + (b_1 f + b_0) f' + c_3 f^3 + c_2 f^2 + c_1 f + c_0, \end{aligned} \tag{4.2.21}$$

$$\begin{aligned} f'' = {} & \frac{1}{f-a_1}\big(A_1 (f')^2 + B_1 f' + C_1\big) + (b_1 f + b_0) f' \\ & + c_3 f^3 + c_2 f^2 + c_1 f + c_0, \end{aligned} \tag{4.2.22}$$

$$f'' = (b_1 f + b_0) f' + c_3 f^3 + c_2 f^2 + c_1 f + c_0, \tag{4.2.23}$$

where a_j *for* $j = 1, \ldots, 4$, B_j, C_j *for* $j = 1, \ldots, 4$, b_1, b_0 *and* c_j $j = 1, \ldots, 3$ *are complex constants, and where either* $A_1 = A_2 = 1/2$ *or* $A_1 = (k_1 - 1)/k_1$, $A_2 = (k_2 - 1)/k_2$ *and* $(k_1 - 1)/k_1 + (k_2 - 1)/k_2 = (k + 1)/k$ *for some integers* $k_1 \geqslant 2, k_2 \geqslant 2$ *and* $k \geqslant 3$.

The proof of Theorem 4.2.13 again relies, among other tools, to the Zalcman lemma. As shown by concrete examples, see [122], all nine cases in Theorem 4.2.13 may actually appear. The non-autonomous case of (4.2.14) remains open. Although it is quite likely that a result similar to Theorem 4.2.13 would remain true for fast growing solutions, some modification of the proof in [122] is definitely needed.

In addition to Theorems 4.2.11 and 4.2.13, special cases of (4.2.14) have been treated in a number of papers: The case when $L(z, f) = M(z, f) \equiv 0$ has been due to Wittich in [188], the case $M(z, f) = N(z, f) \equiv 0$ by von Rieth [185] and the case $L(z, f) \equiv 0$ by Steinmetz [171]. Ishizaki gives in [92] a fairly detailed treatment of the case $L(z, f) \equiv 0$ as well as of the general case, provided $L(z, f) \equiv 1$. See also [91].

Finally, we mention that a number of Malmquist type theorems have been proved for complex differential equations of higher order. As a typical example of such theorems, we recall

THEOREM 4.2.14. (See [46], Satz 5, [169], p. 21.) *If a differential equation of the form*

$$P(z, f, f', \ldots, f^{(n)}) := \sum_{\lambda \in I} c_\lambda(z) f^{\lambda_0} (f')^{\lambda_1} \cdots (f^{(n)})^{\lambda_n} = R(z, f) \tag{4.2.24}$$

where $P(z, f, f', \ldots, f^{(n)})$ *is an irreducible polynomial in* $f, f', \ldots, f^{(n)}$ *with meromorphic coefficients, and* $R(z, f)$ *is rational and irreducible in* f *with meromorphic coefficients, admits a meromorphic solution* f *such that all coefficients of* P *and* R *are small with respect to* f, *then* $R(z, f)$ *is a polynomial in* f *and*

$$\deg_f R(z, f) \leqslant \Delta := \max_{\lambda \in I} (\lambda_0 + \lambda_1 + \cdots + \lambda_n).$$

4.3. *Riccati differential equation*

As the most simple non-linear differential equation, the Riccati differential equation

$$f' = a(z) + b(z) f + c(z) f^2 \tag{4.3.25}$$

deserves a separate treatment, not the least due to its important role in optimal control (usually in the frame of matrix-valued complex differential equation). First recall (Corollary 1.2.7) that all solutions of (4.3.25) are meromorphic functions whenever the coefficients $a(z), b(z), c(z)$ are entire. The situation becomes more complicated, if the coeffi-

cients (or at least one of them) is non-entire meromorphic. Since (4.3.25) may be transformed to

$$f' = A(z) + f^2 \tag{4.3.26}$$

by a simple transformation, which preserves the meromorphic nature of the coefficients, we mostly refer to (4.3.26) in what follows.

THEOREM 4.3.1. *If the Riccati differential equation* (4.3.26) *admits three distinct meromorphic solutions in the complex plane, then all solutions of* (4.3.26) *are meromorphic, and they constitute a one-parameter family* $\mathfrak{F} := (f_C)_{C\in\mathbb{C}}$, *with* $f_0 \notin \mathfrak{F}$ *for one of the three original solutions. Two distinct meromorphic solutions are sufficient to imply the same conclusion, if for one of them, say* f_0, $2f_0$ *has integer residues at all of its poles.*

Concerning the number of distinct meromorphic solutions of (4.3.26) deduced from the properties of the meromorphic coefficient $A(z)$, the following conclusions can be made, see [9] and [107], Section 9.1:

THEOREM 4.3.2.

(1) *If* $A(z)$ *is entire, or if all poles* ζ *of* $A(z)$ *are double such that the coefficients of their Laurent expansions*

$$A(z) = \sum_{j=-2}^{\infty} b_{j+2}(z-\zeta)^j$$

satisfy that (i) $4b_0 = 1 - m^2$ *for a natural number* $m \geqslant 2$ *and that* (ii) *the determinant condition* (2.4.25) *holds, then all solutions of* (4.3.26) *are meromorphic.*

(2) *If* $A(z)$ *has a simple pole, then* (4.3.26) *has at most one meromorphic solution.*

(3) *If* $A(z)$ *has a double pole such that the coefficients of the Laurent expansion at this pole does not satisfy the conditions in* (1), *then* (4.3.26) *has at most two meromorphic solutions. If, however,* $4b_0 = 1$, *then at most one meromorphic solution exists.*

(4) *If* $A(z)$ *has a multiple pole of odd multiplicity, then* (4.3.26) *has no meromorphic solutions. If* $A(z)$ *has multiple poles only, all with even multiplicity, then two meromorphic solutions may exist.*

As an example of Theorem 4.3.2(1), see [108], we may consider equation $f'' - \frac{2}{\cos^2 z} f = 0$ which has two linearly independent meromorphic solutions $f_1(z) = \tan z$ and $f_2(z) = 1 + z\tan z$. By a simple computation, we see that the conditions given in Theorem 4.3.2(1) are satisfied. For a collection of examples covering the remaining cases, see [9].

It is well known that all meromorphic solutions of (4.3.25) are of finite order whenever $a(z), b(z), c(z)$ are rational functions. As for the case of non-rational coefficients, very few papers only may be found. We may refer to [9] for the normalized equation (4.3.26) with $A(z)$ entire, and to [166] for (4.3.26) with $A(z)$ doubly periodic meromorphic. Concerning

basic observations for the value distribution of meromorphic solutions of (4.3.25), resp. of (4.3.26), see [107] and [9].

What also remains more or less open, is to consider Riccati differential equation in the unit disc. We mention here a result due to Yamashita [191]:

THEOREM 4.3.3. *Assume that in Eq.* (4.3.25) *the coefficients belong to the following weighted Hardy spaces*: $a(z) \in H_2^\infty, b(z) \in H_1^\infty, c(z) \in H_0^\infty$. *If* $\inf_{z\in\mathbb{D}} |c(z)| > 0$, *then each holomorphic solution* f *of* (4.3.25) *may be expressed in the form* $f = Cy''/y'$, *where* $C > 0$ *is a constant and* y *is a non-constant function analytic and univalent in* $\mathbb{D}$.

Quite likely, some more results could be easily obtained by recalling the close connection between (4.3.26) and the linear differential equation $u'' + A(z)u = 0$, and making use of the results described in Section 3.

4.4. *Schwarzian differential equations*

A Schwarzian differential equation means a complex differential equation of the form

$$\big(S(f)(z)\big)^p = R(z, f) = \frac{P(z, f)}{Q(z, f)}, \tag{4.4.27}$$

where p is a natural number, $R(z, f)$ is an irreducible rational function in f with meromorphic coefficients, and where

$$S(f)(z) := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \tag{4.4.28}$$

is the standard Schwarzian derivative of a meromorphic function f. We also denote by d the degree of $R(z, f)$ with respect to f.

Observe first that Schwarzian differential equations are closely related with Riccati differential equations. In fact, denote by $\mathcal{R}$ the class of meromorphic functions f that satisfy a Riccati differential equation (4.3.25), $f' = a(z) + b(z)f + c(z)f^2$, such that the coefficients $a(z), b(z), c(z)$ are small with respect to f. Moreover, denote by $\mathcal{R}_k$ the subclass of functions f in $\mathcal{R}$ that have k Picard exceptional values. Then we have the following result, see [92], Theorem 3.2.13:

THEOREM 4.4.1. *If* $f \in \mathcal{R}_k$ *with* $k = 1, 2$, *then* f *also satisfies a Schwarzian differential equation* (4.4.27) *with* $p = 1$ *and* $d = 4 - 2k$. *If* $f \in \mathcal{R}_0$, *then* f *also satisfies a Schwarzian differential equation* (4.4.27) $p = 1$ *and* $d = 4$ *or* $d = 2$ *depending on whether* $D(z) = b(z)^2 - 4a(z)c(z) \equiv 0$, *or not.*

Not surprisingly, the Schwarzian differential equation (4.4.27) has similarities with the Malmquist–Yosida differential equation (4.2.2). To this end, compare the following theorem from [90] with Theorem 4.2.3:

THEOREM 4.4.2. *Suppose that the autonomous Schwarzian differential equation*

$$S(f)(z) = R(f) \tag{4.4.29}$$

admits a transcendental meromorphic solution f. Then for a non-singular Möbius transformation $u := (\alpha f + \beta)/(\gamma f + \delta)$ with complex coefficients, Eq. (4.4.29) *reduces into one of the following forms*:

$$S(u)(z) = c\frac{(u-\sigma_1)(u-\sigma_2)(u-\sigma_3)(u-\sigma_4)}{(u-\tau_1)(u-\tau_2)(u-\tau_3)(u-\tau_4)},$$
$$S(u)^3(z) = c\frac{(u-\sigma_1)^3(u-\sigma_2)^3}{(u-\tau_1)^3(u-\tau_2)^2(u-\tau_3)},$$
$$S(u)^3(z) = c\frac{(u-\sigma_1)^3(u-\sigma_2)^3}{(u-\tau_1)^2(u-\tau_2)^2(u-\tau_3)^2},$$
$$S(u)^2(z) = c\frac{(u-\sigma_1)^2(u-\sigma_2)^2}{(u-\tau_1)^2(u-\tau_2)(u-\tau_3)},$$
$$S(u)(z) = c\frac{(u-\sigma_1)(u-\sigma_2)}{(u-\tau_1)(u-\tau_2)},$$
$$S(u)(z) = c,$$

where $c \in \mathbb{C}$, and $\tau_1, \tau_2, \tau_3, \tau_4, \sigma_1, \sigma_2, \sigma_3, \sigma_4$ are complex constants so that $\tau_1, \tau_2, \tau_3, \tau_4$ are mutually distinct.

As for the corresponding non-autonomous case, Ishizaki proved in [93] the following corresponding result:

THEOREM 4.4.3. *Suppose that the non-autonomous Schwarzian differential equation* (4.4.27) *with $p = 1$ admits a transcendental meromorphic solution f such that all meromorphic coefficients of $R(z, f)$ are small with respect to f. Then either the solution f satisfies a Riccati differential equation with small meromorphic coefficients, or f satisfies a first order algebraic differential equation*

$$(f')^2 + B(z, f)f' + A(z, f) = 0 \tag{4.4.30}$$

where $A(z, f), B(z, f)$ are polynomials in f with small meromorphic coefficients, or (4.4.27) *reduces to one of the following two forms*:

$$S(u)(z) = \frac{P(z, u)}{(f + b(z))^2},$$
$$S(u)(z) = c(z),$$

where $b(z), c(z)$ are small meromorphic functions.

REMARK. In the case of (4.4.30), we may apply a transformation $f = \tau + 1/w$, $\tau \in \mathbb{C}$, to obtain a first order differential equation of the same form (4.4.30) such that $\deg_w B(z, w) \leqslant 1$, $\deg_w A(z, w) \leqslant 3$.

For corresponding results concerning a more complicated Schwarzian type differential equation, see [87].

The growth of solutions of Schwarzian differential equations has been treated by Liao and Ye recently, see [123]:

THEOREM 4.4.4. *Suppose f is a transcendental meromorphic solution of a Schwarzian differential equation* (4.4.27) *with polynomial coefficients. Then the following conclusions hold*:

(1) *If f has a Picard exceptional value, then $\rho(f)$ is finite.*
(2) *If f has two completely ramified values, then $\rho(f) \leqslant 2$.*
(3) *If the Schwarzian equation under consideration is autonomous, then $\rho(f) \leqslant 2$.*
(4) *If the right-hand side of* (4.4.27) *is independent of f, hence a rational function $R(z) = P(z)/Q(z)$, then f is of finite order and*

$$\rho(f) = (m - n + 2p)/(2p) > 0,$$

where $m = \deg P$, $n = \deg Q$. In particular, if $R(z)$ reduces to a constant, then $\rho(f) = 1$.

Finally, we recall the following result concerning the deficiencies, see [90]:

THEOREM 4.4.5. *Suppose f is a transcendental meromorphic solution of a Schwarzian differential equation* (4.4.27) *with polynomial coefficients, and let $\alpha_1, \dots, \alpha_s$ be distinct complex numbers. Then*

$$d + 2p \sum_{j=1}^{s} \delta(\alpha_j, f) \leqslant 4p,$$

where $d = \deg_f R(z, f)$.

4.5. *Painlevé differential equations*

The following problem has been originally proposed by Picard: Given $R(z, f, f')$ rational in its arguments, what are the second order ordinary differential equations in the complex plane of the form $f'' = R(z, f, f')$ having the property that the singularities other than poles of any solution of this equation depend on the equation in question only and not on the constants of integration? Painlevé (1895) and Gambier (1910) proved that there are fifty canonical equations of this form having the property proposed by Picard. This property is known as the *Painlevé property*, and the differential equations possessing this property

are called equations of Painlevé type. Among the fifty equations obtained by Painlevé and Gambier, the following six equations, known as the *Painlevé equations*, are the most interesting ones:

$$f'' = 6f^2 + z, \tag{4.5.31}$$

$$f'' = 2f^3 + zf + \alpha, \tag{4.5.32}$$

$$f'' = \frac{(f')^2}{f} - \frac{1}{z}f' + \frac{1}{z}(\alpha f^2 + \beta) + \gamma f^3 + \frac{\delta}{f}, \tag{4.5.33}$$

$$f'' = \frac{(f')^2}{2f} + \frac{3}{2}f^3 + 4zf^2 + 2(z^2 - \alpha)f + \frac{\beta}{f}, \tag{4.5.34}$$

$$\begin{aligned} f'' = {} & \frac{3f-1}{2f(f-1)}(f')^2 - \frac{1}{z}f' + \frac{1}{z^2}(f-1)^2\left(\alpha f + \frac{\beta}{f}\right) \\ & + \frac{\gamma f}{z} + \frac{\delta f(f+1)}{f-1}, \end{aligned} \tag{4.5.35}$$

$$\begin{aligned} f'' = {} & \frac{1}{2}\left(\frac{1}{f} + \frac{1}{f-1} + \frac{1}{f-z}\right)(f')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{f-z}\right)f' \\ & + \frac{f(f-1)(f-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{f^2} + \frac{\gamma(z-1)}{(f-1)^2} + \frac{\delta z(z-1)}{(f-z)^2}\right), \end{aligned} \tag{4.5.36}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary complex constants. In fact, solutions of eleven other equations of the Painlevé type may be expressed in terms of solutions of the preceding six Painlevé equations, while the thirty-three remaining Painlevé type equations are solvable in terms of solutions of linear differential equations, or in terms of elliptic functions.

The Painlevé equations form a vast field nowadays, in particular due to the importance of their solutions as a kind of non-linear special functions. The field has remarkably extended during the last four decades or so, very much falling in the borderline area of mathematical physics. As for some of the several key references of this field, we may refer to [42,96,137] and [139], as examples of a number of excellent references. In this connection, we restrict ourselves to considering solutions of Painlevé differential equations from the point of view of complex analysis, with main emphasis in the value distribution of solutions of Painlevé differential equations. In particular, we omit results related to asymptotic representations, integrability, Riemann–Hilbert problems, isomonodromic deformations and inverse scattering theory. Moreover, we do not go in detail into Bäcklund transformations.

The Painlevé property proposes that solutions of the six Painlevé equations are close to being meromorphic functions in the complex plane. It is easy to find non-meromorphic solutions to the third (4.5.33), fifth (4.5.35) and sixth (4.5.36) Painlevé equations. However, for (4.5.33) and (4.5.35) the only possible non-polar singularities are at $z = 0$ and $z = \infty$, and hence the transformation $z = \exp t$ might result in an equation with meromorphic solutions only, while for (4.5.36) with three possible fixed singularities at $z = 0, 1, \infty$, such

a transformation does not work. Indeed, applying for (4.5.33) a slightly different transformation, we obtain the modified forms of (4.5.33) and (4.5.35):

$$f'' = \frac{(f')^2}{f} + \alpha f^2 + \gamma f^3 + \beta e^z + \frac{\delta e^{2z}}{f} \tag{4.5.37}$$

and

$$\begin{aligned} f'' = {} & \left(\frac{1}{2f} + \frac{1}{f-1}\right)(f')^2 + (f-1)^2\left(\alpha f + \frac{\beta}{f}\right) + \gamma e^z f \\ & + \frac{\delta e^{2z} f(f+1)}{f-1}. \end{aligned} \tag{4.5.38}$$

Making use of these modified forms, we may express the meromorphic nature of the solutions of Painlevé equations as follows:

THEOREM 4.5.1. *All local solutions of first* (4.5.31), *second* (4.5.32), *modified third* (4.5.37), *fourth* (4.5.34) *and modified fifth* (4.5.38) *Painlevé equations can be analytically continued to single-valued meromorphic functions in the complex plane.*

THEOREM 4.5.2. *Given $z_0 \in \mathbb{C} \setminus \{0, 1\}$ and a local solution f of the sixth Painlevé equation* (4.5.36) *in a neighborhood of z_0, f can be analytically continued without restriction to a possible multi-valued meromorphic function in $\mathbb{C} \setminus \{0, 1\}$.*

The original idea of proving these two theorems goes back to Painlevé in the early years of 1900's. The first paper containing a rigorous function theoretic proof for these two theorems, as far as we know, is due to Hukuhara [88]. Unfortunately, this paper was never published, and so it remained quite unknown. Recently, several papers appeared to offer such proofs, see [80–83,140,161] and [173].

After the meromorphic nature of the Painlevé functions has been settled, natural questions about their growth and value distribution appears. Concerning the growth, we have the following results:

THEOREM 4.5.3. *All solutions f of the first Painlevé equation* (4.5.31) *are of order* $\rho(f) = 5/2$.

The inequality $\rho(f) \geqslant 5/2$ goes back to Mues and Redheffer [130]. The inverse inequality is due to Steinmetz [174] and Shimomura [163], see also [162]. The proofs are quite different: Steinmetz makes use of a clever rescaling of the equation, while Shimomura carries through a careful analysis of solutions f in the neighborhoods of its poles, making use of the fact that f is the derivative of a meromorphic function.

THEOREM 4.5.4. *The solutions f of the second Painlevé equation* (4.5.32), *resp. the fourth Painlevé equation* (4.5.34), *are either rational functions, or transcendental meromorphic of order $\rho(f)$ satisfying* $3/2 \leqslant \rho(f) \leqslant 3$, *resp.* $2 \leqslant \rho(f) \leqslant 4$.

This theorem was proved almost simultaneously in three papers with different methods, see [84,162] and [175].

Observe that the rational solutions for Painlevé differential equations may be completely characterized and constructed. Indeed, (4.5.31) admits no rational solutions, while for (4.5.32), there exists exactly one rational solution if and only if the parameter $\alpha \in \mathbb{Z}$. This solution vanishes identically, when $\alpha = 0$, from which the rational may be computed, at least in principle, for $\alpha \neq 0$, by the standard Bäcklund transformations, see [51], p. 100. Concerning the fourth Painlevé equation (4.5.34), rational solutions exist if and only if the parameters α, β satisfy that $\alpha = n_1$ and either $\beta = -\frac{2}{9}(6n_2 - 3n_1 + \varepsilon)^2$, or $\beta = -2(1 + 2n_3 - n_1)^2$, where n_1, n_2, n_3 are integers, and $\varepsilon = \pm 1$, see [131] and [195]. As for solutions f of order $\rho(f) = 3/2$ for (4.5.32), a family of such solutions, called Airy solutions, are characterized by the fact that they simultaneously satisfy a first order algebraic differential equation, see [51], Theorem 21.1. Concerning solutions f of order $\rho(f) = 2$ for (4.5.34), several such families of solutions are known, see [51], Chapter 6. Concerning the rational solutions of (4.5.33), (4.5.35) and (4.5.36), the case of (4.5.35) has been solved by Kitaev, Law and McLeod, see [103], while for (4.5.33) and (4.5.36), we refer to [195] by Yuan and Li. Of course, rational solutions of (4.5.33) and (4.5.35) result in special solutions f of (4.5.37) and (4.5.38) of growth $T(r, f) = O(r^K)$ for some $K > 0$.

Proceeding to the growth of modified third and fifth Painlevé equations, Shimomura proved in [165] that these are of finite hyper-order:

THEOREM 4.5.5. *All solutions f of* (4.5.37), *resp. of* (4.5.38), *satisfy* $T(r, f) = O(\exp(\Lambda r))$ *for some strictly positive real number Λ independent of f.*

The best possible value of Λ remains unknown. However, a result in [19], Theorem 2, indicates that $\Lambda \leqslant 2$ might be true. Moreover, a similar bound might be true for solutions f of the sixth Painlevé equation as well in the sense that around an essential singularity of f, the local hyper-order of f might be $\leqslant 2$.

We now proceed to considering the value distribution of Painlevé functions. The key quantities to be considered in this connection are their defect and ramification indices, including constant and slowly moving targets. We remark that most results of this type presently known are, quite likely, not best possible. Therefore, we mention a number of open questions in this connection. Most of the results referred to below are due to Shimomura, [155,158,156,159].

THEOREM 4.5.6. *Let f be a solution of* (4.5.31), *and let α be a meromorphic function small with respect to f. Then the defect of f, resp. the ramification index of f, with respect to α satisfies the inequality $\delta(\alpha, f) \leqslant 1/2$, resp. $\vartheta(\alpha, f) \leqslant 5/12$. If $\alpha(z) \equiv a \in \mathbb{C}$, then $\vartheta(a, f) \leqslant 1/6$. Finally, $\delta(\infty, f) = 0$.*

REMARK. Theorem 4.5.6 is originally due to Schubart and Wittich [151]. In the proof due to Shimomura [158], the key ingredients are the classical Clunie and Mohon'ko lemmas, see Theorems 4.1.1 and 4.1.3. The proof in [158] immediately implies that $\delta(\alpha, f) = 0$, if $\alpha(z)$ does not satisfy the same Painlevé equation (4.5.31). As the existence of such a pair

$\alpha(z), f(z)$ seems quite unlikely, we conjecture that actually $\delta(\alpha, f) = 0$. Also, it is not known whether the upper bound $1/6$ for the ramification index is best possible.

THEOREM 4.5.7. *Let f be a solution of* (4.5.32), *and let α be a meromorphic function small with respect to f. Then the defect of f with respect to α satisfies the inequality $\delta(\alpha, f) \leqslant 1/2$.*

REMARK. As for the second Painlevé equation (4.5.32), results related to the ramification indices $\vartheta(\alpha, f)$ of their solutions seem to be missing in the literature.

Concerning the modified third Painlevé differential equation (4.5.37), we first restrict our considerations to its fast growing solutions f meaning that $S(r) \equiv r$ is small with respect to f, see [155]. Secondly, we assume that the parameters satisfy the conditions $(\alpha, \gamma) \neq (0, 0)$ and $(\beta, \delta) \neq (0, 0)$.

THEOREM 4.5.8. *Under the conditions just described, $\delta(a, f) = 0$ for each extended complex number a. If the parameter $\delta \neq 0$, then $\vartheta(a, f) \leqslant 1/4$ for each complex number $a \neq 0$, while if $\delta = 0$ and $\beta \neq 0$, then $\vartheta(a, f) \leqslant 1/6$.*

REMARK. If $(\alpha, \gamma) = (0, 0)$ or $(\beta, \delta) = (0, 0)$, then (4.5.37) reduces to a Riccati differential equation with constant coefficients as shown in [81], pp. 326–327. In this case, defects and ramification indices may easily be computed.

Defect indices for solutions of the fourth Painlevé equation (4.5.34) become more complicated. We first consider the case of constant targets:

THEOREM 4.5.9. (See [170].) *Let f be a transcendental solution of* (4.5.34), *and let $a \in \mathbb{C} \cup \{\infty\}$ be a constant target value. We then have*:

(1) *If $a = \infty$, then $\delta(a, f) = 0$.*
(2) *If $(\beta, a) \in \mathbb{C} \times \mathbb{C} \setminus (0, 0)$, then $\delta(a, f) = 0$.*
(3) *If $(\beta, a) = (0, 0)$, and f satisfies one of the two Riccati differential equations $f' = \mp(f^2 + 2zf)$, then $\delta(a, f) = 1$.*
(4) *If $(\beta, a) = (0, 0)$, and f does not satisfy the two Riccati differential equations $f' = \mp(f^2 + 2zf)$, then $\delta(a, f) \leqslant 1/2$.*

PROOF. The case (1) is an immediate consequence of the Clunie lemma. As for the case (2), we refer to [170]. For the remaining cases (3) and (4), one needs to solve the Riccati differential equations $f' = \mp(f^2 + 2zf)$. Since $f = 0$ is a solution trivially, and for each $c \in \mathbb{C}$,

$$f_c^{\pm}(z) = \exp(\mp z^2)\left(c \pm \int_0^z \exp(\mp t^2)\,\mathrm{d}t\right)^{-1}$$

also solves the corresponding Riccati equation, we conclude by [107], Proposition 9.1.1, that the complete solution family is given by $\{0\} \cup \{f_c^{\pm}; c \in \mathbb{C}\}$. Since the functions $f_c^{\pm}$ are

entire, the conclusion of (3) immediately follows. The remaining case (4) is again referred to [170]. □

Observe that the solutions $f_c^{\pm}$ above are of order $\rho(f_c^{\pm}) = 2$.

Proceeding to the case of slowly moving targets, first recall the expression

$$H_\nu(z) = (-1)^\nu \exp(z^2)\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^\nu \exp(-z^2)$$

for the Hermite polynomials with $\nu \in \mathbb{N} \cup \{0\}$. For these values of ν, the Hermite–Weber differential equation $u'' + (-\frac{t^2}{4} + \nu + \frac{1}{2})u = 0$ admits two linearly independent solutions $D_\nu(t)$, $D_{-\nu-1}(\mathrm{i}t)$, for which we have

$$D_\nu(\sqrt{2}\,z) = 2^{-\nu/2} \exp(-z^2/2) H_\nu(z).$$

Let us first consider the special parameter values $\alpha = -(n + \frac{1}{2} \pm \frac{1}{2})$, $\beta = -2(n + \frac{1}{2} \mp \frac{1}{2})^2$ in (4.5.34), where $n \in \mathbb{Z}$ can be given arbitrarily. Then equation (4.5.34) possesses a one-parameter family $f_{n,c}^{\mp}$, $c \in \mathbb{C} \cup \{\infty\}$, of solutions, expressible in the form

$$f_{n,c}^{\mp}(z) = -z \mp \frac{\eta'_{n,c}(z)}{\eta_{n,c}(z)},$$

where

$$\eta_{n,c}(z) = c_1 D_n(\sqrt{2}\,z) + c_2 D_{-n-1}(\sqrt{2}\,\mathrm{i}z), \quad c = c_1/c_2.$$

These solutions are of order $\rho(f_{n,c}^{\mp}) = 2$, except for $g_n^{\mp} := f_{n,\infty}^{\mp}$, provided $n \in \mathbb{N} \cup \{0\}$ and for $g_n^{\mp} := f_{n,0}^{\mp}$, provided $n \in -\mathbb{N}$. These exceptional cases are rational solutions. For more details, as well as for proofs of the next theorem, see [159]. Again, the proofs strongly rely on using the Clunie and Mohon'ko lemmas.

THEOREM 4.5.10. *Let f be an arbitrary transcendental solution of* (4.5.34), *and let g be a meromorphic function small with respect to f. Then we have*:

(1) *If $(f, g) = (f_{n,c}^{\mp}, g_n^{\mp})$ is one of the special pairs determined above, then $\delta(g, f) = 1$.*
(2) *Otherwise, $\delta(g, f) \leqslant 1/2$, if $\beta \neq 0$, and $\delta(g, f) \leqslant 3/4$, if $\beta = 0$.*

REMARK. It is not known, whether the inequalities in Theorem 4.5.10(2) are best possible. Moreover, possible results concerning the ramification indices remain open.

The value distribution of (4.5.38) has been treated by Shimomura in [156]. First observe that if the parameters γ, δ vanish, then (4.5.38) easily reduces to a Riccati differential equation, see [82], pp. 134–135. Similarly as in the case of (4.5.37), we have to restrict ourselves to fast growing solutions f of (4.5.38), again meaning that $S(r) \equiv r$ is small with respect to f. We first have to extract two special families of solutions: (1) Provided

$\alpha = 0$ and $-4\beta\delta + (\gamma \pm (-2\delta)^{1/2})^2 = 0$, then (4.5.38) possesses a one-parameter family of solutions

$$f^{\pm}_{\gamma,\delta,c}(z) := \exp\bigl(\kappa_{\pm} z \mp (-2\delta)^{1/2} e^{z}\bigr)\left(c - \kappa_{\pm} \int_0^z \exp\bigl(-\kappa_{\pm} s \pm (-2\delta)^{1/2} e^{s}\bigr)\, ds\right),$$

where $\kappa_{\pm} = 1 \pm \gamma(-2\delta)^{-1/2}$ and $c \in \mathbb{C}$, denoted by $\mathcal{V}$ below. (2) Provided $\beta = 0$ and $-4\alpha\delta + (-\gamma \pm (-2\delta)^{1/2})^2 = 0$, then (4.5.38) possesses a one-parameter family of solutions

$$g_{\gamma,\delta,c}(z) := 1/f_{-\gamma,\delta,c}(z),$$

denoted by $\mathcal{W}$ below.

THEOREM 4.5.11. *Let f be a fast growing solution of* (4.5.38), *where its parameters satisfy* $(\gamma, \delta) \neq (0, 0)$, *and let a be an extended complex number. Then* $\delta(a, f) = 0$ *except of the following special cases*:

(1) *If $f \in \mathcal{V}$, then $\delta(\infty, f) = 1$ and if $f \in \mathcal{W}$, then $\delta(0, f) = 1$.*
(2) *If $\alpha = 0$ and $f \notin \mathcal{V}$, then $\delta(\infty, f) \leqslant 1/2$ and if $\beta = 0$ and $f \notin \mathcal{W}$, then $\delta(0, f) \leqslant 1/2$.*
(3) *If $\alpha + \beta = 0$, $\gamma = 0$ and $\delta \neq 0$, then $\delta(-1, f) \leqslant 1/2$.*

THEOREM 4.5.12. *Let f be a fast growing solution of* (4.5.38), *where its parameters satisfy* $(\gamma, \delta) \neq (0, 0)$, *and let $a \in \mathbb{C} \setminus \{0, 1\}$. Then we have the following inequalities*:

(1) *If $\delta \neq 0$, then $\vartheta(a, f) \leqslant 1/4$.*
(2) *If $\delta = 0$ and $\gamma \neq 0$, then $\vartheta(a, f) \leqslant 1/6$.*
(3) *If $\alpha + \beta = 0$, $\gamma = 0$ and $\delta \neq 0$, then $\vartheta(-1, f) = 0$.*

Moreover,

(4) *If $\alpha \neq 0$, resp. $\beta \neq 0$, resp. $\delta \neq 0$, then $\vartheta(\infty, f) = 0$, resp. $\vartheta(0, f) = 0$, $\vartheta(1, f) = 0$. Finally, if $\delta = 0$, then $\vartheta(1, f) = 1/2$.*

REMARK. As one may immediately see, ramification indices remain open in several special cases. It is also not known, whether the inequalities in the preceding two theorems are best possible.

Concerning finally (4.5.36), the most general of the Painlevé equations, it would be natural to study the defects and the ramification indices of its solutions locally in a neighborhood of their essential singularities. However, this has not been done, so far we know.

Finally, it is worth to be mentioned that for meromorphic Painlevé functions, the asymptotic inequality in the second main theorem of Nevanlinna theory becomes an asymptotic equality, see [51], Section 11.

We then proceed to considering, shortly, higher order Painlevé equations, i.e. certain hierarchies of equations, each starting from a given Painlevé equation. Equations in these hierarchies all possess the Painlevé property. However, it remains open to what extent the solutions of these differential equations indeed are meromorphic functions. We conjecture that the solutions to equations in the hierarchies built on the first, second and fourth

Painlevé equations are meromorphic, but there is no proof for this by now. Also, considering the growth and value distribution of meromorphic solutions of higher order Painlevé equations is still very much an open area of research. The first and the second Painlevé hierarchies only have been somewhat treated presently, see below.

To define the *first Painlevé hierarchy*, let $D = \mathrm{d}/\mathrm{d}z$ be the usual differential operator, and D^{-1} stand for its inverse. Defining $d^1(f) := -4f$, we inductively determine

$$d^{n+1}(f) := D^{-1}\big((D^3 - 8fD - 4f')d^n(f)\big).$$

Then we may consider the equations in the first Painlevé hierarchy defined as

$$d^{n+1}(f) + 4z = 0. \tag{4.5.39}$$

It is immediate to observe that we get the first Painlevé equation (4.5.31) from the case $n = 1$, while for $n = 2$ we obtain

$$f^{(4)} = 20ff'' + 10(f')^2 - 40f^3 + z,$$

and so on. Value distribution of meromorphic solutions of equation in the first Painlevé hierarchy (4.5.39) has been treated by Shimomura [164], He [61] and Li [118]. Some of the main results in these papers are as follows:

THEOREM 4.5.13. *Given $n \in \mathbb{N}$, let f be a meromorphic solution of* (4.5.39). *Then the following assertions follow*:

(1) *The solution f is transcendental meromorphic with double poles only and of order $\rho(f) \geqslant \frac{2n+3}{n+1}$.*
(2) *For each extended complex number a, $\delta(a, f) = 0$.*
(3) *The second main theorem for f takes the form of an asymptotic equality*

$$m(r, f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) + N_1(r, f) + N\left(r, \frac{1}{f'}\right) = 2T(r, f) + S(r, f).$$

REMARK. It is a natural conjecture that $\rho(f) = \frac{2n+3}{n+1}$. Unfortunately, this remains open.

As for the *second Painlevé hierarchy*, we first define the operator

$$S[f] := 4f^2 + 4f'D^{-1}f - D^2$$

and then construct the algebraic differential equation

$$D^{-1}S^{m-1}[f]f' + zf + \alpha = 0, \tag{4.5.40}$$

where $\alpha \in \mathbb{C}$ is a parameter, for $m \in \mathbb{N} \setminus \{1\}$. This differential equation is of order $2m - 2$ and for $m = 2$ we obtain the second Painlevé equation (4.5.32), for $m = 3$ we get

$$f^{(4)} = 10f^2f'' + 10f(f')^2 - 6f^5 - zf - \alpha,$$

and so on. Now, the growth and value distribution of meromorphic solutions of equations in the second Painlevé hierarchy (4.5.40) has been investigated in two papers by Li and He [119,118]. Some of their results read as follows:

THEOREM 4.5.14. *Given $m \in \mathbb{N} \setminus \{1\}$, let f be a transcendental meromorphic solution of* (4.5.40). *Then the following assertions hold*:

(1) *All poles of f are simple and $\delta(\infty, f) = 0$.*
(2) *If $\alpha \neq 0$, then $\delta(a, f) = 0$ for each $a \in \mathbb{C}$.*
(3) *If $\alpha = 0$, then $\delta(a, f) = 0$ for each $a \in \mathbb{C} \setminus \{0\}$ and $\delta(0, f) \leqslant 1/2$.*

REMARKS.

(1) There exist no results concerning the growth of meromorphic solutions of (4.5.40), although it is natural again to conjecture that these functions are of finite order.
(2) It would be interesting to investigate the residues of the poles of f for the parameter values $\alpha = \varepsilon/2$, $\varepsilon^2 = 1$. This has been done in the case of $m = 3$, see [119], Theorem 2.4. Compare also with the corresponding results for (4.5.32), i.e. in the case of $m = 2$, see [51], Section 21.
(3) In the second Painlevé hierarchy as well, transcendental meromorphic solutions again have an asymptotic equality in the second main theorem in the form

$$m\left(r, \frac{1}{f}\right) + \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) + N_1(r, f) + N\left(r, \frac{1}{f'}\right) = 2T(r, f) + S(r, f),$$

where each $a_j \neq 0$, see [118], Theorem 3.1.

4.6. *Briot–Bouquet differential equations*

By Briot–Bouquet differential equations we understand autonomous differential equations in the complex plane of the form $P(f, f^{(k)}) = 0$, where P is a polynomial in both variables, having constant coefficients, of course. Investigations concerning Briot–Bouquet differential equations go back to a classical paper by Briot and Bouquet in 1856, where they claim that all meromorphic solutions of $P(f, f') = 0$ belong to one of the following classes of functions: (1) rational functions, (2) rational functions of e^{az} for some complex constant $a \neq 0$, (3) elliptic functions. Denoting this collection of meromorphic functions by $\mathfrak{W}$, one can show [79] that any meromorphic function $f \in \mathfrak{W}$ satisfies a Briot–Bouquet equation, see also [78]. In the reverse direction, this remains an open

CONJECTURE. *Any meromorphic solution in the complex plane of a Briot–Bouquet differential equation is in the collection $\mathfrak{W}$ of meromorphic functions.*

Concerning the existence of meromorphic functions for Briot–Bouquet differential equations, the following fact is classical: Denote by $S = S(P)$ the compact Riemann surface determined by the algebraic curve $P(u, v) = 0$. A necessary condition for the existence

of a non-constant meromorphic solution f to $P(f, f^{(k)}) = 0$ is that the genus $g(S)$ of the Riemann surface S is either $g(S) = 0$ or $g(S) = 1$. The following partial result to the conjecture above are known:

THEOREM 4.6.1. *Let f be a meromorphic solution of a Briot–Bouquet differential equation $P(f, f^{(k)}) = 0$ of the first or second order, i.e. if $k = 1$ or $k = 2$. Then $f \in \mathfrak{W}$.*

REMARK. There are several presentations of this result in the classical literature of the 19th century. However, these proofs hardly can be accepted of being rigorous according to the present standards. For a modern proof, see [12], Theorems 1 and 5.

THEOREM 4.6.2. (See [37].) *Suppose f is a non-constant meromorphic solution of a Briot–Bouquet differential equation $P(f, f^{(k)}) = 0$, and let g be the genus of the Riemann surface determined by the algebraic curve $P(u, v) = 0$. Then the following assertions hold*:

(1) *If $g = 1$, then f is an elliptic function.*
(2) *If $g = 0$, k is an odd natural number and f is entire, then $f \in \mathfrak{W}$.*

REMARK. Observe that there is an extensive literature in the unit disc about the differential equation

$$f + \frac{zf'}{\beta f + \gamma} = h(z),$$

where β, γ are complex constants and $h(z)$ is an analytic function. This differential equation is said to be a Briot–Bouquet differential equation as well. We omit these investigations in this connection, as their main connection is to the subordination theory, and not so much to the theory of complex differential equations.

4.7. *Algebraic differential equations*

Algebraic differential equations

$$\Omega(z, f, f', \dots, f^{(n)}) = \sum_{\lambda \in I} a_\lambda(z) f^{\lambda_0} (f')^{\lambda_1} \cdots (f^{(n)})^{\lambda_n} = 0, \tag{4.7.41}$$

where Ω is a polynomial in the variables $f, f', \dots, f^{(n)}$ with meromorphic coefficients, and I is a finite index set, form an extensive field of research. However, open problems abound, and the overall impression is still somehow scattered. It seems to us that obtaining some coherence in the whole area needs to combine tools from various fields such as complex analysis and differential field theory. To get a quick overview of a number of open research problems in the field of algebraic differential equations, we refer to the problem collections [148–150] composed by Rubel.

While considering Eq. (4.7.41) below, we need some degree and weight notations as follows: The *degree* $|\lambda|$ of a single term in (4.7.41) will now be defined as

$$|\lambda| := \lambda_0 + \cdots + \lambda_n,$$

and its *weight* $\|\lambda\|$ as

$$\|\lambda\| := \lambda_0 + 2\lambda_1 + \cdots + (n+1)\lambda_n.$$

Of course, the degree and the weight of the whole Eq. (4.7.41) will then be defined as

$$|\Omega| := \max_{\lambda\in I} |\lambda|, \qquad \|\Omega\| := \max_{\lambda\in I} \|\lambda\|.$$

One of the key open problems in the field of algebraic differential equations is the following

CONJECTURE. *All meromorphic solutions of algebraic differential equations* (4.7.41) *with polynomial coefficients satisfy the growth inequality*

$$T(r, f) \leqslant a \exp_{n-1}(br^c),$$

where $\exp_1(z) \equiv e^z$, $\exp_{j+1}(z) := \exp(\exp_j(z))$, *and where* a, b, c *are positive real constants*.

This conjecture has been verified for the first order algebraic differential equations as shown by the classical result due to Gol'dberg [50]:

THEOREM 4.7.1. *All meromorphic solutions* f *of first order algebraic differential equations* $\Omega(z, f, f') = 0$ *with polynomial coefficients are of finite order of growth*.

REMARKS.

(1) As mentioned above, see the remark after Theorem 4.2.9, the order of transcendental meromorphic solutions is either zero, or a multiple of $1/2$ or of $1/3$, see [40].

(2) If f is an entire solution of $\Omega(z, f, f') = 0$ with polynomial coefficients, then $\rho(f) > 1/2$. This is an improvement, due to Strelitz [177], of a classical result by Pólya who proved that $\rho(f) > 0$. However, a meromorphic solution may well be of order zero, see an example in [107], pp. 234–236, due to Bank and Kaufman. We remark here that the claim $\rho(f) > 0$ remains true for entire solutions f of second order algebraic differential equations $\Omega(z, f, f', f'') = 0$ as shown by Zimogljad [197], while for algebraic differential equations of order $\geqslant 3$, examples of entire solutions of zero order have been constructed by Valiron.

Proceeding to first order algebraic differential equations with meromorphic coefficients, observe first that it is not possible to estimate the growth of meromorphic solutions in terms of the growth of the coefficients only, uniformly for all such equations, not even in the linear case. This follows of a somewhat surprising arbitrary growth theorem due to Bank in [2]:

THEOREM 4.7.2. *Given any strictly positive real function Φ on the positive real axis, there exists a meromorphic function h such that h'/h is of finite order and $T(r_n, h) > \Phi(r_n)$ for a sequence tending to $+\infty$.*

Some of the results describing quantities needed for such a uniform estimate are presented below. Such a uniform estimate in terms of the coefficients only may be found for entire solutions, see [107], Theorem 11.10:

THEOREM 4.7.3. *Let $\Omega(z, f, f') = 0$ be a first order algebraic differential equation with entire coefficients of finite order $\leqslant \rho < \infty$. Then all entire solutions f satisfy*

$$T(r, f) = \mathrm{O}(\exp r^{\rho+\varepsilon})$$

for any $\varepsilon > 0$. Hence, the hyper-order of f satisfies $\rho_2(f) \leqslant \rho$.

As for second order algebraic differential equations $\Omega(z, f, f', f'') = 0$, we first remark that the growth conjecture above has been verified for entire solutions in the homogeneous case by Steinmetz, see [176]:

THEOREM 4.7.4. *Suppose an algebraic differential equation*

$$\Omega(f, f', f'') = 0$$

is homogeneous in f, f', f'' with polynomial coefficients. Then each meromorphic solution of such an equation takes the form $f(z) = \frac{g_1(z)}{g_2(z)} \exp(g_3(z))$, where g_1, g_2, g_3 are entire functions of finite order.

However, the non-homogeneous case remains open in general, even for entire solutions.

Concerning algebraic differential equations of arbitrary order, there seems to be very few systematic investigations about this topic. As for differential equations of a suitable form, reasoning of Malmquist or Clunie type may sometimes help. Recalling Theorem 4.2.14, if we are considering algebraic differential equations of the form

$$\Omega(z, f, f', \ldots, f^{(n)}) = R(z, f) \tag{4.7.42}$$

with meromorphic coefficients, all of them small with respect to f, and if $\deg_f R(z, f) > \|\Omega\|$, then all possible meromorphic solutions have their growth comparable, in some sense, with the growth of the coefficients. Indeed, denoting by $\Phi(r)$ the maximum of $\log r$ and the maximum of the characteristic functions of the meromorphic coefficients of Eq. (4.7.42), then we have $T(r, f) = \mathrm{O}(\Phi(r))$, at least on a set of r-values of infinite linear measure. However, it is quite likely that such an estimate actually holds nearly everywhere. However, this has been proved in the polynomial case of $R(z, f)$ only, see [178], as far as we know.

Three natural questions concerning meromorphic solutions of algebraic differential equations (4.7.41) appear: (1) to obtain growth estimates for solutions, (2) to work out

criteria under which meromorphic solutions are of finite order of growth and (3) to work criteria under which meromorphic solutions exist and if so, produce the number of meromorphic solutions, resp. the dimension of this family. The last question remains untouched here; all studies we know in this direction are restricted to first order algebraic differential equations of a special form such as $f' = P(z, f)$ with polynomial coefficients.

Concerning the growth estimates, Theorem 4.7.2 above implies that no uniform growth estimates exist in terms of the coefficients only. It appears that information about the poles and distinct zeros are needed to obtain a uniform estimate. To this end, we divide Eq. (4.7.41) in its homogeneous parts as

$$\Omega(z, f, f', \dots, f^{(n)}) = \sum_{q=0}^{|\Omega|} \Omega_q(z, f, f', \dots, f^{(n)}) = 0,$$

denoting by $A_q(z)$ the sum of all coefficients in the homogeneous part Ω_q. As before, we define

$$\Phi(r) := \max_{\lambda \in I}\bigl(\log r, T(r, a_\lambda)\bigr)$$

over all coefficients of P.

THEOREM 4.7.5. (See [13].) *Let f be a meromorphic solution of* (4.7.41). *If f does not satisfy some of the homogeneous equations $\Omega_q(z, f, f', \dots, f^{(n)}) = 0$ of* (4.7.41), *then for any $\sigma > 1$, we have, for some $K > 0$,*

$$T(r, f) \leqslant KG(\sigma r)$$

for all r sufficiently large, where

$$G(r) = \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + \Phi(r).$$

On the other hand, if f satisfies all of the homogeneous equations $\Omega_q = 0$ of (4.7.41), *while for some q, $0 \leqslant q \leqslant |\Omega|$, we have $A_q(z) \not\equiv 0$, then*

$$T(r, f) = \mathrm{O}\bigl(rN(\sigma r, f) + r^2 \exp\bigl(\mathrm{d}G(\sigma r)\log\bigl(rG(\sigma r)\bigr)\bigr)\bigr).$$

REMARKS.

(1) In principle, this enables us to determine all quantities needed to obtain a uniform growth estimate for meromorphic solutions. Indeed, if A_q vanishes in the last part of the preceding theorem, we may continue by looking at

$$\sum_{q=0}^{|\Omega|-1} \Omega_q(z, f, f', \dots, f^{(n)}) = 0.$$

Inductively, if needed, we may proceed this way down to a first order algebraic differential equation, which always can be treated by Theorem 4.7.1.

(2) In the special case of linear differential equations, the original result has been substantially improved recently by Chiang and Hayman, see [32], p. 456. In particular, if the meromorphic solution f has a few poles only in the sense that $\delta(\infty, f) > 0$, then a uniform growth estimate follows in terms of the coefficients only.

Concerning criteria to deduce existence of finite order meromorphic solutions, only a few results are known for non-linear algebraic differential equations. We recall here the following interesting result due to Hayman, see [59]:

THEOREM 4.7.6. *Consider an algebraic differential equation* (4.7.41) *with polynomial coefficients, and denote by* Λ *the collection of terms that have maximal weight among all terms of maximal degree* $|\Omega|$. *Suppose that*

$$\sum_{\lambda\in\Lambda} a_\lambda(z) \not\equiv 0.$$

Then all entire solutions f *of* (4.7.41) *are of finite order*

$$\rho(f) \leqslant \max\bigl(2|\Omega|, 1 + |\Omega|\bigr).$$

Observe that the assertion fails for meromorphic solutions as shown by equation

$$f^2 f'' - 2f(f')^2 - f^2 f' + f'' - f' = 0,$$

where $\sum_{\lambda\in\Lambda} a_\lambda(z) = -1$, having a meromorphic solution $f(z) = \tan \mathrm{e}^z$ of infinite order.

A simple example not covered by Theorem 4.7.6 is equation

$$ff'' - (f')^2 = a_2(z) f'' + a_1(z) f' + a_0(z) f + b(z) \tag{4.7.43}$$

with rational coefficients. Hayman conjectured [59] that all entire solutions are of finite order. This has been proved in the case of constant coefficients by Chiang and Halburd [31], and considerations in [19] indicate that the conjecture might be true.

The growth of meromorphic solutions of algebraic differential equations in the unit disc still remains very much an open field of research, and some of the results obtained are too complicated to be repeated here, see e.g. [167]. Concerning first order algebraic differential equations, a unit disc counterpart to Theorem 4.7.1 has been proved by Bank in [3]:

THEOREM 4.7.7. *Consider a first order algebraic differential equation*

$$\Omega(z, f, f') = \sum_{\lambda\in I} a_\lambda(z) f^{\lambda_0} (f')^{\lambda_1} = 0$$

with analytic coefficients of finite order in the unit disc. Denote $d := \max_{\lambda\in I} \rho_M(a_\lambda)$. *Then all analytic solutions* f *of* $\Omega(z, f, f') = 0$ *in the unit disc are of finite order. More precisely,*

$$M(r, f) \leqslant \exp\bigl(\exp\bigl((1-r)^{-(d+4)}\bigr)\bigr)$$

for all $r < 1$ *close enough to* 1.

A natural problem would be to explore connections between the function spaces the solutions, resp. the coefficients of an algebraic differential equation belong to, in the same spirit as was presented for linear differential equations in Chapter 3. There are a couple of results only of this type. We quote here

THEOREM 4.7.8. *Given* $1 \leqslant \alpha < \infty$, *let* f *be a meromorphic solution of*

$$(f')^n = \sum_{j=1}^{m} P_j(z, f) D_j[f],$$

where

$$P_j(z, f) \sum_{k=0}^{m_j} a_{kj}(z) f^k, \qquad D_j[f] = (f')^{j_1} \cdots (f^{(s)})^{j_s}$$

with analytic coefficients a_{kj} *in the unit disc so that*

$$n > \nu := \max_{1\leqslant j\leqslant m} (j_1 + \cdots + s j_s).$$

If

$$\sup_{z\in\mathbb{D}} \bigl(1-|z|\bigr)^{\alpha(n-\nu)} \max_{1\leqslant j\leqslant m} \sum_{k=0}^{m_j} \bigl|a_{kj}(z)\bigr| < \infty,$$

then $f \in \mathcal{N}^\alpha$, *i.e.* f *is* α*-normal.*

Respectively, if

$$\lim_{|z|\to 1^-} \bigl(1-|z|\bigr)^{\alpha(n-\nu)} \max_{1\leqslant j\leqslant m} \sum_{k=0}^{m_j} \bigl|a_{kj}(z)\bigr| = 0,$$

then f *is strongly normal.*

4.8. *Algebraic differential equations and differential fields*

The theory of differential fields is an extensive topic which apparently has natural connections to investigations in the area of algebraic differential equations. However, these two

areas have been developed rather separately of each other, and their intersection clearly is much more narrow than it could be.

To fix our notations, we restrict ourselves to considering differential fields $\mathcal{L}$ that are subfields of the field of meromorphic functions. Moreover, we steadily assume that $\mathbb{C} \subset \mathcal{L}$. Of course, results to be applied from the theory of differential fields primarily are those dealing with fields of characteristic zero. In what follows, we are considering meromorphic functions (and their subfields) in the complex plane, although some of the results apply to the unit disc case as well. Meromorphic functions that satisfy an algebraic differential equation with coefficients in $\mathcal{L}$ may be called *differentially algebraic over* $\mathcal{L}$, being denoted as $\mathcal{A}(\mathcal{L})$. It is easy to see that $\mathcal{A}(\mathcal{L})$ is a differential field. But $\mathcal{A}(\mathcal{L})$ is also closed in the sense that whenever $f \in \mathcal{A}(\mathcal{A}(\mathcal{L}))$, then $f \in \mathcal{A}(\mathcal{L})$. A meromorphic function $f \notin \mathcal{A}(\mathcal{L})$ is called *differentially transcendental over* $\mathcal{L}$. Being differential algebraic has the elementary nesting property: If $\mathcal{L} \subset \mathcal{L}_1 \subset \mathcal{L}_2$ are subfields of meromorphic functions, and if $\mathcal{L}_1 \subset \mathcal{A}(\mathcal{L})$ and $\mathcal{L}_2 \subset \mathcal{A}(\mathcal{L}_1)$, then $\mathcal{L}_2 \subset \mathcal{A}(\mathcal{L})$. Finally, we say that the differential field $\mathcal{A}(\mathbb{C})$ is the field of *differentially elementary* meromorphic functions.

Perhaps the most typical papers on the borderline of algebraic differential equations and differential fields are those proving a differentially transcendental nature of a given meromorphic function (or a certain family of meromorphic functions). This type of investigations go back to Hölder who proved (in 1887) that the gamma-function $\Gamma \notin \mathbb{C}(z)$, hence is not differentially elementary. The Hölder theorem has been improved by Bank and Kaufman in [10]. For the convenience, we denote by T_1 the translation $z \mapsto z + 1$.

THEOREM 4.8.1. *Suppose $\mathcal{L}$ is a subfield of meromorphic functions, satisfying the following three properties*: (i) $\mathbb{C}(z) \subset \mathcal{L}$, (ii) *if $f \in \mathcal{L}$, then $f \circ T_1 \in \mathcal{L}$*, (iii) *if $f \in \mathcal{L}$ and $f \circ T_1 - f \in \mathbb{C}(z)$, then $f \in \mathbb{C}(z)$. Then the gamma-function Γ is differentially transcendental over $\mathcal{L}$, hence over $\mathcal{A}(\mathcal{L})$ as well. In particular, Γ is differentially transcendental over the field of meromorphic functions f that satisfy $T(r, f) = o(r)$.*

For a simple proof of the following result, see [127]:

THEOREM 4.8.2. *The Riemann zeta-function ζ is not differentially elementary.*

A natural extension of the notion of algebraic independence may be applied to define *differential independence* of meromorphic functions. In particular, if f, g are two meromorphic functions, they are said to differentially independent (over $\mathbb{C}$), if either f is differentially transcendental over $\mathbb{C}(g)$, or g is differentially transcendental over $\mathbb{C}(f)$.

THEOREM 4.8.3. (*See [127].*) *The following pairs of meromorphic functions are differentially independent over* $\mathbb{C}$:

(1) $\Gamma(\sin(2\pi z))$ *and* $\Gamma(z)$;
(2) $\zeta(\sin(2\pi z))$ *and* $\Gamma(z)$.

REMARK. A natural conjecture would be that $\zeta(z)$ and $\Gamma(z)$ are differentially independent over $\mathbb{C}$, not the least due to Theorem 4.8.3(2). However, this remains an open problem.

Another group of results in the borderline area of algebraic differential equations and differential fields are related to the following classical lemma, see [107], Lemma 14.3.1, originally due to Siegel. We remark that this lemma is part of the bridge between complex differential equations and transcendental number theory, again a borderline area, which apparently needs more exploration.

THEOREM 4.8.4. *Consider a second-order linear differential equation*

$$f'' + a(z)f' + b(z)f = 0 \tag{4.8.44}$$

over a subfield $\mathcal{L}$ of meromorphic functions. Suppose (4.8.44) *admits a solution f_0 that is* (*algebraically*) *transcendental over $\mathcal{L}$, but satisfies a first-order algebraic differential equation over $\mathcal{L}$. Then there exists another non-trivial solution f of* (4.8.44) *such that f'/f is algebraic over $\mathcal{L}$.*

Example of a consequence of the Siegel lemma is concerned with the basic linear differential equation $f'' + A(z)f = 0$, where $A(z)$ is a non-constant polynomial of degree n. Denoting by $\mathcal{L}_{(n+2)/2}$ the differential field of meromorphic functions of order $< (n+2)/2$, we get

THEOREM 4.8.5. *If n is odd, then no non-trivial solution of $f'' + A(z)f = 0$ with a polynomial $A(z)$ of degree n can satisfy a first-order algebraic differential equations over $\mathcal{L}_{(n+2)/2}$. On the other hand, if n is even, then at least one of any two linearly independent solutions of $f'' + A(z)f = 0$ cannot satisfy a first-order algebraic differential equation over $\mathcal{L}_{(n+2)/2}$.*

For further results of similar type, see [4,5].

5. Algebroid solutions of complex differential equations

5.1. *Introduction to algebroid functions*

Algebroid functions in the complex plane are formed by collections of analytic function elements that satisfy an irreducible algebraic equation

$$A_n(z)f^n + A_{n-1}(z)f^{n-1} + \cdots + A_1(z)f + A_0(z) = 0 \tag{5.1.1}$$

with meromorphic coefficients. This means that the collection of algebroid functions in the complex plane form an extension field of the field of meromorphic functions. Moreover, an algebroid function f as defined above is n-valued, hence f may be understood as a one-valued meromorphic function on an n-sheeted Riemann surface. Typical examples of elementary algebroid functions are nth roots of meromorphic functions. In many aspects, algebroid functions behave similarly as to meromorphic functions. However, there are fundamental differences such as, for example, the phenomenon that the derivative f' of an algebroid function may have a pole at a point where the function f itself takes a

finite value. A simple example is $\sqrt{z}$ whose derivative $1/(2\sqrt{z})$ has a pole at the origin. Of course, the branch points of f also add an aspect which has no direct counterpart in the realm of meromorphic functions.

Nevanlinna theory of meromorphic functions has a natural extension to algebroid functions, soon developed in early 1930's, most notably by Valiron [184], Selberg [153] and Ullrich [183]. Since Nevanlinna theory is a powerful tool to investigating algebroid solutions of complex differential equations, we shortly describe some of its similarities and deviations as compared to the original meromorphic version.

Let f be a non-constant ν-valued algebroid function, that has at z_0 an a-point, resp. a pole, with local expansions about z_0 of the form

$$f(z) = a + b_\tau (z - z_0)^{\tau/\lambda} + \cdots, \quad \tau \geqslant 1, \tag{5.1.2}$$

resp.

$$f(z) = b_{-\tau}(z - z_0)^{-\tau/\lambda} + \cdots, \quad \tau \geqslant 1, \tag{5.1.3}$$

where $\lambda \leqslant \nu$. The *non-integrated counting function* $n(r, 1/(f-a))$ for the a-points, resp. $n(r, f)$ for the poles in the disc $|z| \leqslant r$, is the sum of the τ's in the preceding expansions, over all a-points, resp. over all poles, in $|z| \leqslant r$. The *integrated counting function* is now defined as in the usual meromorphic theory, taking however into account the fact that we are now operating on an ν-sheeted surface S_f over the disc $|z| \leqslant r$, i.e.,

$$N\left(r, \frac{1}{f-a}\right) := \frac{1}{\nu} \int_0^r \frac{n(t,a) - n(0,a)}{t}\, \mathrm{d}t + \frac{1}{\nu} n(0,a) \log r,$$

resp.

$$N(r, f) := \frac{1}{\nu} \int_0^r \frac{n(t,f) - n(0,f)}{t}\, \mathrm{d}t + \frac{1}{\nu} n(0,f) \log r.$$

Similarly, we define the *proximity function* as

$$m\left(r, \frac{1}{f-a}\right) := \frac{1}{\nu} \sum_{j=1}^{\nu} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f_j(r\mathrm{e}^{\mathrm{i}\theta}) - a|}\, \mathrm{d}\theta,$$

resp.

$$m(r, f) := \frac{1}{\nu} \sum_{j=1}^{\nu} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f_j(r\mathrm{e}^{\mathrm{i}\theta}) \right| \mathrm{d}\theta,$$

where f_j stands for the values of f on the jth sheet of the surface S_f. The *characteristic function* $T(r, f) := m(r, f) + N(r, f)$ is now defined as usual, and the first main theorem takes its usual form $T(r, 1/(f-a)) = T(r, f) + \mathrm{O}(1)$. Moreover, an algebroid function f is algebraic if and only if $T(r, f) = \mathrm{O}(\log r)$, and the lemma of the logarithmic derivative

takes its usual form $m(r, f^{(k)}/f) = S(r, f)$, $k \in \mathbb{N}$, in general and $m(r, f^{(k)}/f) = \mathrm{O}(\log r)$ for an algebroid function of finite order.

The key deviation from the usual meromorphic theory comes out of the *counting function of branch points* of f, which may be determined from the expansions (5.1.2), (5.1.3) as

$$n_3(r, f) := \sum_{|z|\leqslant r,\, a\in\widehat{\mathbb{C}}} (\lambda - 1)$$

first in the non-integrated form, and then the integrated counting function of branch points $N_3(r, f)$ follows exactly as above for $N(r, f)$, say. We also make use of the corresponding function $N_{3,\infty}(r, f)$ for branched poles above the disc $|z| \leqslant r$. The essential consequences may now be expressed as

PROPOSITION 5.1.1. *Let f be a ν-valued algebroid function. Then*

(1) $N_3(r, f) \leqslant 2(\nu - 1)T(r, f) + \mathrm{O}(1)$

and

(2) $T(r, f') \leqslant 2T(r, f) + N_3(r, f) + S(r, f) \leqslant 2\nu T(r, f) + S(r, f)$.

For more details and proofs concerning the basic Nevanlinna theory of algebroid functions, see [184,153,183] and [100].

We next give the algebroid versions of the Clunie, Mohon'ko and Valiron–Mohon'ko lemmas, see Section 4.1 for the corresponding meromorphic versions:

THEOREM 5.1.2. *Let f be an algebroid solution of*

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f)$, $Q(z, f)$ are polynomials in f and its derivatives with algebroid coefficients, say a_λ, $\lambda \in I$, such that $m(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. If the total degree of $Q(z, f)$ in f and its derivatives is $\leqslant n$, then

$$m\bigl(r, P(z, f)\bigr) = S(r, f).$$

REMARK. The standard proof of the meromorphic version applies here. One just needs to replace the notions from meromorphic value distribution theory with the corresponding algebroid ones. The key point is that the logarithmic derivative lemma applies in the same form as in the meromorphic case.

THEOREM 5.1.3. (See [128].) *Let f be a meromorphic solution of an algebraic differential equation*

$$P(z, f, f', \ldots, f^{(n)}) = 0, \tag{5.1.4}$$

where P is a polynomial in $f, f', \dots, f^{(n)}$ with meromorphic coefficients small with respect to f. If a complex constant c does not satisfy Eq. (5.1.4), *then*

$$m\left(r, \frac{1}{f-c}\right) = S(r, f).$$

THEOREM 5.1.4. (See [128,129].) *Let f be an algebroid function, and let $R(z, f) = P(z, f)/Q(z, f)$ be an irreducible rational function in f with algebroid coefficients small with respect to f. Then the characteristic function of $R(z, f(z))$ satisfies*

$$T\bigl(r, R(z, f)\bigr) = dT(r, f) + S(r, f),$$

where $d := \max(\deg_f P(z, f), \deg_f Q(z, f))$.

5.2. *Malmquist type theorems for algebroid functions*

We start by giving the algebroid counterpart of Theorem 4.2.2:

THEOREM 5.2.1. *If a differential equation of the form*

$$(f')^n = R(z, f), \tag{5.2.5}$$

where $R(z, f) = P(z, f)/Q(z, f)$ is rational and irreducible in f with meromorphic coefficients, admits a ν-valued algebroid solution f such that all coefficients of $R(z, f)$ are small with respect to f, then

$$q := \deg Q \leqslant 2n(\nu - 1), \qquad p := \deg P \leqslant q + 2n \leqslant 2n\nu.$$

REMARKS.

(1) This result was originally proved by Yosida [194]. He also proved that all algebroid solutions of (5.2.5) with small meromorphic coefficients are of finite order of growth, see [194], Theorem 7.

(2) As a deviation from the meromorphic case, observe that $R(z, f)$ does not necessarily reduce to a polynomial in f as shown by an example in [45]: The equation $f^2 + f \tan z + z = 0$ defines a two-valued algebroid function that satisfies

$$f' = \frac{z^2 + f + (1+2z)f^2 + f^4}{z - f^2}.$$

However, if the solution has not too much of the branch points, then $R(z, f)$ reduces to polynomial:

THEOREM 5.2.2. (See [100].) *Under the same assumptions as in Theorem* 5.2.1, *if*

$$\xi(f) := \limsup_{r\to\infty} \frac{N_3(r, f)}{T(r, f)} < \frac{1}{n},$$

then $R(z, f)$ reduces to a polynomial in f of degree $\leqslant 2n$.

REMARKS.

(1) The assumption that $\xi(f) < 1/n$ cannot be improved, as shown in [145], Beispiel 7.1: Consider $f(z) := (\mathrm{e}^z - 1)^{-1}$, and define algebroid functions w, v by the algebraic equation $w^2 - f(z) = 0$ and $v := (w-1)^{-1}$. Then v satisfies

$$v' = -\frac{(v+1)(2v^2+2v+1)}{2v},$$

while $\xi(v) = 1$.

(2) In the case of Theorem 5.2.2, the reduction may still be continued further in the algebroid case similarly as in the meromorphic case, see [100], Theorem 3.10 and [145], Satz 7.3.

(3) Concerning some introductory remarks of the value distribution of algebroid solutions of (5.2.5) with small meromorphic coefficients, see [145], Section 8.

(4) The Malmquist type theorems above for algebroid solutions correspond to Theorem 4.2.2 for meromorphic solutions. Concerning a counterpart of Theorem 4.2.7 for the algebroid case, see [39], [86], Satz 4.5 and [128], Theorem 5.

5.3. *Algebroid solutions of algebraic differential equations*

Investigations concerning algebroid solutions of algebraic differential equations

$$\Omega(z, f, f', \ldots, f^{(n)}) = 0, \tag{5.3.6}$$

where Ω is a polynomial in the variables $f, f', \ldots, f^{(n)}$ with meromorphic coefficients, are quite few, although this could also be an extensive field of research.

In considering an expression such as

$$\Omega(z, f, f', \ldots, f^{(n)}) = \sum_{\lambda \in I} a_\lambda(z) f^{\lambda_0} \cdots (f^{(n)})^{\lambda_n},$$

we apply the same notations concerning degree and weight as in Section 4.7, i.e.

$$|\Omega| := \max_{\lambda \in I} |\lambda|, \qquad \|\Omega\| := \max_{\lambda \in I} \|\lambda\|.$$

In addition, we need a couple of notations for certain modified versions of the weight:

$$|\Omega|_0 := \max_{\lambda \in I} \sum_{j=1}^{n} j\lambda_j, \qquad |\Omega|_1 := \max_{\lambda \in I} \sum_{j=1}^{n} (2j-1)\lambda_j.$$

Similarly as in the meromorphic case, see Theorem 4.2.14, Theorem 5.2.1 may be extended to algebraic differential equations of the form

$$\Omega(z, f, f', \dots, f^{(n)}) = R(z, f) = \frac{P(z, f)}{Q(z, f)}, \tag{5.3.7}$$

where $P(z, f)$, $Q(z, f)$ are polynomials in f with meromorphic coefficients, and $\Omega(z, f, f', \dots, f^{(n)})$ is a differential polynomial in f with meromorphic coefficients. Making use of the notations above for degree, weight etc. of Ω, and denoting $p := \deg P$, $q := \deg Q$, we get the following

THEOREM 5.3.1. (See [181].) *Suppose f is a ν-valued algebroid solution of Eq.* (5.3.7) *with small meromorphic coefficients. Then we have*

$$\max\bigl(p, q + \|\Omega\|\bigr) \leqslant \|\Omega\| + |\Omega|_1 \xi(f),$$
$$p \leqslant \min\bigl(q + |\Omega| + |\Omega|_0\bigl(1 - \Theta(\infty, f) + \xi_\infty(f)\bigr), \|\Omega\| + |\Omega|_1 \xi(f)\bigr),$$

where $\xi_\infty(f) := \limsup_{r\to\infty} \frac{N_{3,\infty}(r,f)}{T(r,f)}$.

REMARK.
(1) For several closely related variants of Theorem 5.3.1, see [48] and [47].
(2) As in the model case of Theorem 5.2.1, the reduced form may have a non-trivial denominator, contrary to the meromorphic case, see Theorem 4.2.14. However, if $\xi(f) = 0$, then the denominator becomes trivial, and we have

COROLLARY 5.3.2. *If $\xi(f) = 0$, then*

$$q = 0, \qquad p \leqslant \min\bigl(\|\Omega\|, |\Omega| + |\Omega|_0\bigl(1 - \Theta(\infty, f)\bigr)\bigr).$$

REMARK. Several of the items in the list of references below contain some additional results concerning algebroid solutions of algebraic differential equations. In particular, we refer to [39] and [86], Section 4. See also [179,180].

5.4. *Algebroid solutions of linear differential equations*

The only paper we know investigating algebroid solutions of linear differential equations of type

$$f'' + A(z)f = 0,$$

where $A(z)$ is meromorphic, is due to Katajamäki, see [100], Section 5. We quote here a couple of his results,

THEOREM 5.4.1. *If the quotient of two local solutions of the equation* $f'' + A(z)f = 0$ *admits in a disc, where* $A(z)$ *is holomorphic, a* ν*-valued,* $\nu \geqslant 2$*, algebroid continuation into the complex plane, then at each pole* z_0 *of* $A(z)$ *either* (1) *the Laurent expansion of* $A(z)$ *about* z_0 *has the form*

$$A(z) = \frac{1-m^2}{4(z-z_0)^2} + \cdots ,$$

where $m = p/q$ *is rational but not an integer,* p, q *have no non-trivial common factors, and* $q|\nu$*, or* (2) m *is an integer* $\geqslant 2$ *and the determinant condition* $D(z_0) = 0$ *in* (2.4.25) *holds.*

This theorem now has the following consequence:

THEOREM 5.4.2. *If equation* $f'' + A(z)f = 0$ *admits a* ν*-valued,* $\nu \geqslant 2$*, algebroid solution in the complex plane, then the equation has no non-trivial meromorphic solutions.*

THEOREM 5.4.3. *If equation* $f'' + A(z)f = 0$ *admits a* 2*-valued algebroid solution* f *in the complex plane, then* f *must be a square-root of a meromorphic function, i.e.* f^2 *is meromorphic.*

As for algebroid solutions of linear differential equations of higher order, see Tohge [182].

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CHAPTER 4

Transversal Periodic-to-Periodic Homoclinic Orbits

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Contents

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1. Introduction

The possible presence of transversal homoclinic points and the complicated dynamics associated with them was noticed by Poincaré [21] while studying the restricted three-body problem. Later Birkhoff [3] proved that every transversal homoclinic point of a two-dimensional diffeomorphism is accumulated by periodic points. Then Smale's [25] theorem showed that the existence of a transversal homoclinic orbit implied the presence of a horseshoe, and hence complicated dynamics.

Note that Smale's result was proved for diffeomorphisms. A transversal homoclinic point is a point where the stable and unstable manifolds of a hyperbolic fixed point intersect transversally. The corresponding notion for a flow is a point where the stable and unstable manifolds of a hyperbolic periodic orbit intersect transversally. We call the orbit of such a point a transversal periodic-to-periodic homoclinic orbit, the subject of the present chapter. Sil'nikov [22] proved a result analogous to Smale's for transversal periodic-to-periodic homoclinic orbits. A detailed history of homoclinic orbits is given by Sil'nikov [23].

A homoclinic orbit of a diffeomorphism is transversal if and only if the invariant set consisting of it and the associated hyperbolic fixed point is hyperbolic (confer [19]). This, in turn, happens if and only if the variational equation along the homoclinic orbit has an exponential dichotomy. Analogously a periodic-to-periodic homoclinic orbit of a flow is transversal if and only if the invariant set consisting of it and the associated hyperbolic periodic orbit is hyperbolic. It turns out that happens if and only if the variational equation along the homoclinic orbit has what we call a trichotomy. Trichotomy is similar to an exponential dichotomy but with the difference that there is an additional one-dimensional subspace of bounded solutions. Trichotomies play an important role in our development of the theory and so Section 2 is devoted to their study.

In Section 3 we first define the Poincaré map associated with a periodic orbit of an autonomous system of ordinary differential equations. Then we construct the local and global stable and unstable manifolds of a hyperbolic periodic orbit, thereby introducing the concept of asymptotic phase. These, of course, are well-known results but we give detailed proofs because, for later application, we need the results in a special form. Note that our smoothness assumptions may not be optimal.

In Section 4 we study properties of periodic-to-periodic homoclinic orbits. First we define a Poincaré map associated with such an orbit similar to that given by Hale and Lin in [11]. Then we give equivalent conditions for transversality of the periodic-to-periodic homoclinic orbits including its equivalence with the transversality of the corresponding homoclinic orbit of the Poincaré map. Note that the possibility of doing this was suggested to me in a conversation with Daniel Stoffer in Santander airport. Then Smale's theorem enables to deduce the existence of chaotic behaviour in the neighbourhood of a transversal periodic-to-periodic homoclinic orbit. Note that in [19] a shadowing theorem for hyperbolic sets of flows was used to give a complete symbolic description of the dynamics in the neighbourhood of a transversal periodic-to-periodic homoclinic orbit.

In Section 5 we consider an autonomous system depending on a parameter where we assume the unperturbed system has a hyperbolic periodic orbit with associated transversal homoclinic orbit. We show that the perturbed system has a similar structure.

Even though, as we found in Section 5, transversal periodic-to-periodic homoclinic orbits persist under perturbation, it does not seem possible to write down an explicit system

with an explicit transversal periodic-to-periodic homoclinic orbit. This is also the situation with homoclinic points of diffeomorphisms. In the latter case, Melnikov [15] finds diffeomorphisms with transversal homoclinic points as period maps of differential equations with time-periodic vector fields which are perturbations of autonomous systems with nontransversal homoclinic orbits. In Section 6 we prove an analogous theorem for periodic-to-periodic homoclinic orbits of flows. We begin with a system with a nontransversal homoclinic orbit; such systems can be easily written down explicitly. Then we show that a perturbed system has a transversal orbit nearby provided a certain Melnikov function has a simple zero. Note that the existence of transversal periodic-to-periodic homoclinic orbits in singularly perturbed systems has been shown by Szmolyan in [26]; see also [1]. Note also that in [7] Deng and Sakamoto show that periodic-to-periodic homoclinic orbits arise as the result of a Sil'nikov–Hopf bifurcation. However, these orbits may be nontransversal.

In Section 7, which is based on Coomes, Koçak and Palmer [4], it is shown how transversal periodic-to-periodic homoclinic orbits can be constructed via numerical shadowing. We begin with a numerically computed orbit which appears to be bi-asymptotic to an also numerically computed apparent periodic orbit. The main theorem gives computable criteria for verifying the existence of a true periodic orbit and an associated true transversal homoclinic orbit near the numerically computed orbits. Other authors, for example, Dieci and Rebaza [9] and Pampel [20], inspired by the work of Beyn [2], approximate a connecting orbit by the solution of a certain boundary value problem and they derive estimates for the error in the approximation. However, these works are carried out on the assumption that a true connecting orbit exists. Examples of transversal periodic-to-periodic homoclinic in the restricted three-body problem are studied in Kirchgraber and Stoffer [14].

2. Trichotomies

It turns out that a periodic-to-periodic homoclinic orbit is transversal if and only if its variational equation has a trichotomy in the sense defined below. So trichotomies are an important tool in our study and, in this section, we prove the results about trichotomies that we will need in the sequel.

DEFINITION 2.1. Let $A(t)$ be a continuous $n \times n$ matrix function. The linear system

$$\dot{x} = A(t)x \tag{1}$$

is said to have a *trichotomy* on an interval J if there exist supplementary projections P_0, P_+, P_- with rank $P_0 = 1$ and positive constants K, α such that

$$\begin{aligned} &\left|X(t)P_0X^{-1}(s)\right| \leqslant K \quad \text{for all } t, s,\\ &\left|X(t)P_+X^{-1}(s)\right| \leqslant K\mathrm{e}^{-\alpha(t-s)} \quad \text{for } t \geqslant s,\\ &\left|X(t)P_-X^{-1}(s)\right| \leqslant K\mathrm{e}^{-\alpha(s-t)} \quad \text{for } t \leqslant s \end{aligned} \tag{2}$$

where $X(t)$ is the fundamental matrix for (1) with $X(0) = I$.

Note that if $P_0 = 0$ we have an exponential dichotomy. Sometimes we refer to K as the "constant" and to α as the "exponent".

Trichotomies have similar properties to exponential dichotomies. We describe these properties in the remarks below.

REMARK 2.2. *Characterization of projections*. If $J = [t_0, \infty)$, then it is clear that

$$\mathcal{R}(P_+) = \{\xi\colon\ |X(t)\xi| \to 0 \text{ as } t \to \infty\},$$
$$\mathcal{R}(P_0 + P_+) = \Big\{\xi\colon\ \sup_{t\geqslant t_0} |X(t)\xi| < \infty\Big\}.$$

Now we show that $\mathcal{R}(P_0)$ can be any complementary subspace U_0 to $\mathcal{R}(P_+)$ in $\mathcal{R}(P_0 + P_+)$ and that $\mathcal{R}(P_-)$ can be any complementary subspace W_0 to $\mathcal{R}(P_0 + P_+)$ in $\mathbf{R}^n$. To see this, let Q_0, Q_+ and Q_- be the projections associated with the splitting

$$\mathbf{R}^n = U_0 \oplus \mathcal{R}(P_+) \oplus W_0.$$

Then for $t, s \geqslant t_0$, since $\mathcal{R}(P_+) = \mathcal{R}(Q_+)$ so that $P_+Q_+ = Q_+$ and $Q_+P_+ = P_+$,

$$\begin{aligned}
&\big|X(t)(Q_+ - P_+)X^{-1}(s)\big| \\
&\quad = \big|X(t)P_+(Q_+ - P_+)(I - P_+)X^{-1}(s)\big| \\
&\quad \leqslant \big|X(t)P_+X^{-1}(t_0)\big|\big|X(t_0)(Q_+ - P_+)X^{-1}(t_0)\big|\big|X(t_0)(P_0 + P_-)X^{-1}(s)\big| \\
&\quad \leqslant K\mathrm{e}^{-\alpha(t-t_0)}\big|X(t_0)(Q_+ - P_+)X^{-1}(t_0)\big|2K \\
&\quad = 2K^2\big|X(t_0)(Q_+ - P_+)X^{-1}(t_0)\big|\mathrm{e}^{-\alpha(t-t_0)} \\
&\quad = K_1\mathrm{e}^{-\alpha(t-t_0)}
\end{aligned}$$

and since $\mathcal{N}(P_-) = \mathcal{N}(Q_-)$ so that $P_-Q_- = P_-$ and $Q_-P_- = Q_-$,

$$\begin{aligned}
&\big|X(t)(Q_- - P_-)X^{-1}(s)\big| \\
&\quad = \big|X(t)(P_0 + P_+)(Q_- - P_-)P_-X^{-1}(s)\big| \\
&\quad \leqslant \big|X(t)(P_0 + P_+)X^{-1}(t_0)\big|\big|X(t_0)(Q_- - P_-)X^{-1}(t_0)\big|\big|X(t_0)P_-X^{-1}(s)\big| \\
&\quad \leqslant 2K\big|X(t_0)(Q_- - P_-)X^{-1}(t_0)\big|K\mathrm{e}^{-\alpha(s-t_0)} \\
&\quad = K_2\mathrm{e}^{-\alpha(s-t_0)}.
\end{aligned}$$

In particular this implies that if $t \geqslant t_0$,

$$\big|X(t)(Q_+ - P_+)X^{-1}(t)\big| \leqslant K_1, \qquad \big|X(t)(Q_- - P_-)X^{-1}(t)\big| \leqslant K_2.$$

Then going through the first argument again with t_0 replaced by s, with $s \geqslant t_0$, we find that for $t \geqslant s \geqslant t_0$

$$\left|X(t)(Q_+ - P_+)X^{-1}(s)\right| \leqslant K_3 e^{-\alpha(t-s)},$$

where $K_3 = 2K^2K_1$, and going through the second argument again with t_0 replaced by t, with $t \geqslant t_0$, we find that for $s \geqslant t \geqslant t_0$

$$\left|X(t)(Q_- - P_-)X^{-1}(s)\right| \leqslant K_4 e^{-\alpha(s-t)},$$

where $K_4 = 2K^2K_2$. Also we note that for $t, s \geqslant t_0$

$$\begin{aligned}\left|X(t)(Q_0 - P_0)X^{-1}(s)\right| &\leqslant \left|X(t)(Q_+ - P_+)X^{-1}(s)\right| \\ &\quad + \left|X(t)(Q_- - P_-)X^{-1}(s)\right| \\ &\leqslant K_1 + K_2.\end{aligned}$$

Then we see that for $t, s \geqslant t_0$

$$\begin{aligned}\left|X(t)Q_0X^{-1}(s)\right| &\leqslant \left|X(t)P_0X^{-1}(s)\right| + \left|X(t)(Q_0 - P_0)X^{-1}(s)\right| \\ &\leqslant K + K_1 + K_2,\end{aligned}$$

for $t \geqslant s \geqslant t_0$

$$\begin{aligned}\left|X(t)Q_+X^{-1}(s)\right| &\leqslant \left|X(t)P_+X^{-1}(s)\right| + \left|X(t)(Q_+ - P_+)X^{-1}(s)\right| \\ &\leqslant (K + K_3)e^{-\alpha(t-s)}\end{aligned}$$

and for $s \geqslant t \geqslant t_0$

$$\begin{aligned}\left|X(t)Q_-X^{-1}(s)\right| &\leqslant \left|X(t)P_-X^{-1}(s)\right| + \left|X(t)(Q_- - P_-)X^{-1}(s)\right| \\ &\leqslant (K + K_4)e^{-\alpha(s-t)}.\end{aligned}$$

If $J = (-\infty, t_0]$, then it is clear that

$$\begin{aligned}\mathcal{R}(P_-) &= \left\{\xi\colon \left|X(t)\xi\right| \to 0 \text{ as } t \to -\infty\right\}, \\ \mathcal{R}(P_0 + P_-) &= \left\{\xi\colon \sup_{t \leqslant t_0}\left|X(t)\xi\right| < \infty\right\},\end{aligned}$$

and we can similarly prove that $\mathcal{R}(P_0)$ can be any complementary subspace to $\mathcal{R}(P_-)$ in $\mathcal{R}(P_0 + P_-)$ and that $\mathcal{R}(P_+)$ can be any complementary subspace to $\mathcal{R}(P_0 + P_-)$ in $\mathbf{R}^n$.

If $J = \mathbf{R}$, it is clear that

$$\begin{aligned}\mathcal{R}(P_+) &= \left\{\xi\colon \left|X(t)\xi\right| \to 0 \text{ as } t \to \infty\right\}, \\ \mathcal{R}(P_-) &= \left\{\xi\colon \left|X(t)\xi\right| \to 0 \text{ as } t \to -\infty\right\},\end{aligned}$$

and

$$\mathcal{R}(P_0) = \Big\{\xi\colon \sup_{t\in\mathbf{R}} \big|X(t)\xi\big| < \infty\Big\}.$$

So, in this case, up to a scalar multiple, the linear system has a unique nontrivial bounded solution $x_0(t)$ where $x_0(0)$ spans the range of P_0 and, moreover, $\inf_{t\in\mathbf{R}} |x_0(t)| \geqslant K^{-1}|x_0(0)| > 0$.

REMARK 2.3. *If $t_0 < t_1$, trichotomy on $[t_1,\infty)$ implies trichotomy on $[t_0,\infty)$.* Let (1) have a trichotomy on $[t_1,\infty)$ as in Definition 2.1. For $\sigma = 0,+,-$, define

$$M_\sigma = \sup\Big\{\mathrm{e}^{\alpha(t-s)}\big|X(t)P_\sigma X^{-1}(s)\big|\colon\ t_0 \leqslant t, s \leqslant t_1\Big\}.$$

Then for $t_0 \leqslant t, s \leqslant t_1$ and $\sigma = 0,+,-$,

$$\big|X(t)P_\sigma X^{-1}(s)\big| \leqslant M_\sigma \mathrm{e}^{-\alpha(t-s)}.$$

Then, for example, if $t_0 \leqslant s \leqslant t_1 \leqslant t$,

$$\big|X(t)P_+X^{-1}(s)\big| \leqslant \big|X(t)P_+X^{-1}(t_1)\big|\big|X(t_1)P_+X^{-1}(s)\big| \leqslant M_+K\mathrm{e}^{-\alpha(t-s)}.$$

It follows that if $t_0 \leqslant s \leqslant t$,

$$\big|X(t)P_+X^{-1}(s)\big| \leqslant K_1\mathrm{e}^{-\alpha(t-s)},$$

where

$$K_1 = \max\{K, M_+K\}.$$

The inequalities for P_0 and P_- can be similarly derived.

Similarly, if $t_0 > t_1$, trichotomy on $(-\infty,t_1]$ implies trichotomy on $(-\infty,t_0]$.

REMARK 2.4. *Trichotomy on $\mathbf{R}$ from trichotomies on $\mathbf{R}_+$ and $\mathbf{R}_-$.* If (1) has a trichotomy on $\mathbf{R}$, then it certainly has trichotomies on $\mathbf{R}_+$ and $\mathbf{R}_-$ with projections of the same rank and, as we saw at the end of Remark 2.2, (1) has, up to a scalar multiple, a unique nontrivial bounded solution $x_0(t)$ which also satisfies $\inf_{t\in\mathbf{R}} |x_0(t)| > 0$.

The converse statement is also true. Suppose the projections on $[0,\infty)$ are P_0, P_+, P_- and the projections on $(-\infty,0]$ are Q_0, Q_+, Q_- with K and α as the constants in both cases. Now if $\xi \in \mathcal{R}(P_+) \cap \mathcal{R}(Q_0 + Q_-)$, then $X(t)\xi$ is bounded on the whole line and hence is a multiple of $x_0(t)$. However, $X(t)\xi \to 0$ as $t \to \infty$ and so $\xi = 0$. Hence $\mathcal{R}(P_+)$ is complementary to $\mathcal{R}(Q_0 + Q_-)$ and so, by Remark 2.2, the Q_σ can be chosen so that $\mathcal{R}(Q_+) = \mathcal{R}(P_+)$. Similarly, we can assume $\mathcal{R}(P_-) = \mathcal{R}(Q_-)$. Next since $\inf_{t\in\mathbf{R}} |x_0(t)| > 0$, $\operatorname{span}\{x_0(0)\}$ is complementary to $\mathcal{R}(P_+)$ in $\mathcal{R}(P_0 + P_+)$ and also complementary to

$\mathcal{R}(Q_-)$ in $\mathcal{R}(Q_0 + Q_-)$. Hence we can assume $\mathcal{R}(P_0) = \mathcal{R}(Q_0) = \operatorname{span}\{x_0(0)\}$. Thus we can arrange that $P_\sigma = Q_\sigma$ for $\sigma = 0, +, -$. Then we know that

$$\left|X(t)P_+X^{-1}(s)\right| \leqslant K\mathrm{e}^{-\alpha(t-s)}$$

for $s \leqslant t \leqslant 0$ and $0 \leqslant s \leqslant t$. If $s \leqslant 0 \leqslant t$, then

$$\begin{aligned}\left|X(t)P_+X^{-1}(s)\right| &\leqslant \left|X(t)P_+X^{-1}(0)\right|\left|X(0)P_+X^{-1}(s)\right| \\ &\leqslant K\mathrm{e}^{-\alpha t}K\mathrm{e}^{-\alpha(0-s)} \\ &= K^2\mathrm{e}^{-\alpha(t-s)}.\end{aligned}$$

Using two more similar arguments, we see that (1) has a trichotomy on $\mathbf{R}$ with constants K^2 and α.

REMARK 2.5. *The case when $A(t)$ is bounded.* If $A(t)$ is bounded, we can weaken the inequalities in the definition of trichotomy to having for all ξ

$$\begin{aligned}&\left|X(t)P_0\xi\right| \leqslant K\left|X(s)P_0\xi\right| \quad \text{for all } t, s, \\ &\left|X(t)P_+\xi\right| \leqslant K\mathrm{e}^{-\alpha(t-s)}\left|X(s)P_+\xi\right| \quad \text{for } t \geqslant s, \\ &\left|X(t)P_-\xi\right| \leqslant K\mathrm{e}^{-\alpha(s-t)}\left|X(s)P_-\xi\right| \quad \text{for } t \leqslant s\end{aligned}$$

since these inequalities imply that for $t \geqslant s$

$$\frac{|X(t)P_+\xi||X(s)P_0\xi|}{|X(s)P_+\xi||X(t)P_0\xi|} \leqslant K^2\mathrm{e}^{-\alpha(t-s)}$$

and

$$\frac{|X(t)P_0\xi||X(s)P_-\xi|}{|X(s)P_0\xi||X(t)P_-\xi|} \leqslant K^2\mathrm{e}^{-\alpha(t-s)}.$$

This means the solution subspaces corresponding to P_+, P_0 and P_- are exponentially separated and so by Lemma 1 in [16] the norms of the projections $X(t)P_\sigma X^{-1}(t)$ are bounded by a constant M (say) and so, for example, if $s \leqslant t$ then for all ξ

$$\left|X(t)P_+X^{-1}(s)\xi\right| \leqslant K\mathrm{e}^{-\alpha(t-s)}\left|X(s)P_+X^{-1}(s)\xi\right| \leqslant KM\mathrm{e}^{-\alpha(t-s)}|\xi|$$

so that

$$\left|X(t)P_+X^{-1}(s)\right| \leqslant KM\mathrm{e}^{-\alpha(t-s)}.$$

REMARK 2.6. *The adjoint equation*. By taking adjoints in the inequalities (2) and exchanging s and t, we find that

$$\begin{aligned}
&\left|Y(t)P_0^*Y^{-1}(s)\right| \leqslant K \quad \text{for all } t, s,\\
&\left|Y(t)P_-^*Y^{-1}(s)\right| \leqslant K\mathrm{e}^{-\alpha(t-s)} \quad \text{for } t \geqslant s,\\
&\left|Y(t)P_+^*X^{-1}(s)\right| \leqslant K\mathrm{e}^{-\alpha(s-t)} \quad \text{for } t \leqslant s,
\end{aligned}$$

where $Y(t) = X^{-1}(t)^*$. So the adjoint system has a trichotomy on J with projections P_0^*, P_-^*, P_+^*.

REMARK 2.7. *Roughness theorem*. For this, for convenience, we restrict to the case where J is $\mathbf{R}$, $\mathbf{R}_+$ or $\mathbf{R}_-$. We show the following result.

Suppose (1) *has a trichotomy as in Definition* 2.1*, letting $x_0(t)$ be the solution with $P_0x_0(0) = x_0(0)$ and $|x_0(0)| = 1$. Let $B(t)$ be a continuous matrix function with $|B(t)| \leqslant \delta$ for all t in J such that*

$$\dot{x} = \left[A(t) + B(t)\right]x \tag{3}$$

has a solution $y_0(t)$ with

$$0 < \Delta \leqslant \left|y_0(t)\right| \leqslant L < \infty$$

for all t in J and also

$$\left|y_0(0) - x_0(0)\right| < K^{-1}$$

when J is a half-line.

Let $\varepsilon > 0$ be given with $\varepsilon < \alpha/2$. Then if δ is sufficiently small, (3) *has a trichotomy on J with constant depending only on K, L, Δ* (*and $|y_0(0) - x_0(0)|$ when J is a half-line*) *and exponent $\alpha - \varepsilon$.*

Note first that the shifted equation

$$\dot{x} = \left[A(t) + \alpha/2\right]x$$

has an exponential dichotomy on J with projection P_+ and constants $2K$, $\alpha/2$. It follows from the roughness theorem for dichotomies in [5, Lecture 5] that, if δ is sufficiently small, the perturbed system

$$\dot{x} = \left[A(t) + B(t) + \alpha/2\right]x \tag{4}$$

has an exponential dichotomy on J with both exponents $\alpha/2 - \varepsilon$ and constant K_1 depending on K and projection Q_+ having the same rank as P_+. In particular, if $Y(t)$ is the fundamental matrix of (3) with $Y(0) = I$, then

$$\left|Y(t)Q_+Y^{-1}(s)\right| \leqslant K_1 e^{-(\alpha-\varepsilon)(t-s)} \tag{5}$$

for $t \geqslant s$.

Next the shifted equation

$$\dot{x} = \left[A(t) - \alpha/2\right]x$$

has an exponential dichotomy on J with projection $P_0 + P_+$ and constants $2K$, $\alpha/2$. It follows from the roughness theorem for dichotomies that, if δ is sufficiently small, the perturbed system

$$\dot{x} = \left[A(t) + B(t) - \alpha/2\right]x$$

has an exponential dichotomy on J with projection $I - Q_-$ having the same rank as $P_0 + P_+$ with exponent $\alpha/2 - \varepsilon$ and constant K_1. In particular,

$$\left|Y(t)Q_-Y^{-1}(s)\right| \leqslant K_1 e^{-(\alpha-\varepsilon)(s-t)} \tag{6}$$

for $t \leqslant s$.

Now if $J = \mathbf{R}$ or $\mathbf{R}_+$,

$$\mathcal{N}(Q_-) = \left\{\xi\colon \sup_{t\geqslant 0} e^{-\alpha t/2}\left|Y(t)\xi\right| < \infty\right\},$$

$$\mathcal{R}(Q_+) = \left\{\xi\colon \sup_{t\geqslant 0} e^{\alpha t/2}\left|Y(t)\xi\right| < \infty\right\}$$

so that $\mathcal{R}(Q_+) \subset \mathcal{N}(Q_-)$ and $y_0(0) \in \mathcal{N}(Q_-)$. It follows that $Q_-Q_+ = 0$ and $Q_-y_0(0) = 0$. Similarly, if $J = \mathbf{R}$ or $\mathbf{R}_-$, $Q_+Q_- = 0$ and $Q_+y_0(0) = 0$.

Then, when $J = \mathbf{R}$, if we define $Q_0 = I - Q_+ - Q_-$, we see that Q_0, Q_+, Q_- are supplementary projections such that the range of Q_0 is spanned by $y_0(0)$. Now for all t and s

$$\left|y_0(t)\right| \leqslant L\Delta^{-1}\left|y_0(s)\right|$$

and so for all ξ

$$\left|Y(t)Q_0\xi\right| \leqslant L\Delta^{-1}\left|Y(s)Q_0\xi\right|.$$

Hence

$$\left|Y(t)Q_0Y^{-1}(s)\xi\right| \leqslant L\Delta^{-1}\left|Y(s)Q_0Y^{-1}(s)\xi\right|.$$

However,

$$\left|Y(s)Q_0Y^{-1}(s)\right| = \left|Y(s)(I - Q_+ - Q_-)Y^{-1}(s)\right| \leqslant 2K_1 + 1$$

so that

$$\left|Y(t)Q_0Y^{-1}(s)\right| \leqslant L(2K_1+1)\Delta^{-1}$$

for all t and s. This together with (5) and (6) shows that (3) has a trichotomy if δ is sufficiently small with constant $\max\{K_1, L(2K_1+1)\Delta^{-1}\}$ and exponent $\alpha-\varepsilon$.

Now suppose $J=\mathbf{R}_+$. Coppel chooses Q_+ so that

$$\mathcal{N}(Q_+)=\mathcal{N}(P_+)=\operatorname{span}\bigl\{x_0(0)\bigr\}\oplus\mathcal{R}(P_-)$$

and $|Q_+-P_+|\leqslant N\delta$, where N depends only on K and α. Similarly, he chooses Q_- so that

$$\mathcal{R}(Q_-)=\mathcal{R}(P_-)$$

and $|Q_- - P_-|\leqslant N\delta$. Since $\mathcal{R}(Q_+)\subset\mathcal{N}(Q_-)$, $y_0(0)\in\mathcal{N}(Q_-)$ and clearly $y_0(0)\notin\mathcal{R}(Q_+)$ it follows by consideration of the dimensions that

$$\mathcal{N}(Q_-)=\operatorname{span}\bigl\{y_0(0)\bigr\}\oplus\mathcal{R}(Q_+).$$

Then

$$\mathbf{R}^n=\mathcal{N}(Q_-)\oplus\mathcal{R}(Q_-)=\operatorname{span}\bigl\{y_0(0)\bigr\}\oplus\mathcal{R}(Q_+)\oplus\mathcal{R}(Q_-). \tag{7}$$

Let $\tilde{Q}_+$ be the projection with range $\mathcal{R}(Q_+)$ and nullspace $\operatorname{span}\{y_0(0)\}\oplus\mathcal{R}(Q_-)$ and set $\tilde{Q}_0=I-\tilde{Q}_+-Q_-$. We see that $\tilde{Q}_0$, $\tilde{Q}_+$, Q_- are supplementary projections such that the range of $\tilde{Q}_0$ is spanned by $y_0(0)$.

By [5, p. 14], we know we still get a dichotomy for (4) with the altered projection $\tilde{Q}_+$ with exponent $\alpha/2-\varepsilon$ but different constant depending on K, α and $|\tilde{Q}_+-Q_+|$. However we need to ensure that the constant can be chosen as a fixed number depending only on the other constants provided δ is sufficiently small. To this end, we need to estimate $|\tilde{Q}_+-Q_+|$. Using $P_0+P_++P_-=I$, $\tilde{Q}_+Q_+=Q_+$, $\tilde{Q}_+Q_-=0$ and $Q_+P_-=0$, we find that

$$\begin{aligned}
|\tilde{Q}_+-Q_+| &= \bigl|(\tilde{Q}_+-Q_+)P_0+(\tilde{Q}_+-Q_+)(P_++P_-)\bigr| \\
&= \bigl|(\tilde{Q}_+-Q_+)P_0+(\tilde{Q}_+-Q_+)(P_++P_--Q_+-Q_-)\bigr| \\
&\leqslant \bigl|(\tilde{Q}_+-Q_+)P_0\bigr|+2N\delta|\tilde{Q}_+-Q_+|.
\end{aligned}$$

Now if $P_0x=\lambda x_0(0)$ so that $|\lambda|\leqslant K|x|$,

$$\begin{aligned}
&\bigl|(\tilde{Q}_+-Q_+)P_0x\bigr| \\
&\quad=|\lambda|\bigl|(\tilde{Q}_+-Q_+)x_0(0)\bigr| \\
&\quad\leqslant K|x|\bigl|(\tilde{Q}_+-Q_+)x_0(0)\bigr|
\end{aligned}$$

$$
\begin{aligned}
&= K|x|\big|\tilde{Q}_+\big(x_0(0) - y_0(0)\big)\big| \quad \text{since } Q_+x_0(0) = \tilde{Q}_+y_0(0) = 0 \\
&\leqslant K|x||\tilde{Q}_+|\big|x_0(0) - y_0(0)\big| \\
&\leqslant K|x|\big[|Q_+| + |\tilde{Q}_+ - Q_+|\big]\big|x_0(0) - y_0(0)\big|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&|\tilde{Q}_+ - Q_+| \\
&\quad\leqslant K\big[|Q_+| + |\tilde{Q}_+ - Q_+|\big]\big|x_0(0) - y_0(0)\big| + 2N\delta|\tilde{Q}_+ - Q_+| \\
&\quad\leqslant KK_1\big|x_0(0) - y_0(0)\big| + \big(K\big|x_0(0) - y_0(0)\big| + 2N\delta\big)|\tilde{Q}_+ - Q_+|.
\end{aligned}
$$

So, provided

$$
\begin{aligned}
&2N\delta < 1 - K\big|x_0(0) - y_0(0)\big|, \\
&|\tilde{Q}_+ - Q_+| \leqslant \big(1 - K\big|x_0(0) - y_0(0)\big| - 2N\delta\big)^{-1}KK_1\big|x_0(0) - y_0(0)\big|.
\end{aligned}
$$

Then it follows from [5, p. 14] that (4) has an exponential dichotomy with the altered projection $\tilde{Q}_+$, with the same exponent $\alpha/2 - \varepsilon$ and with constant depending only on K, K_1 and $|x_0(0) - y_0(0)|$, provided δ is sufficiently small. Then we finish the proof that (3) has a trichotomy on $\mathbf{R}_+$ as in the case $J = \mathbf{R}$.

The case $J = \mathbf{R}_-$ is similarly handled.

REMARK 2.8. *Smoothness of the projections*. We can show the smoothness of the projections in a trichotomy with respect to a parameter. Suppose (1) has a trichotomy as in the definition. Let $B(t,\mu)$ be a continuous matrix function where the continuity in μ is uniform in t and $B(t,0) = 0$. Suppose also that the system

$$
\dot{x} = \big[A(t) + B(t,\mu)\big]x \tag{8}
$$

has for μ small a solution $y_0(t,\mu)$ with $0 < \Delta \leqslant |y_0(t,\mu)| \leqslant L < \infty$ such that $y_0(0,\mu)$ is continuous. Then by Remark 2.7, if μ is sufficiently small, system (8) has a trichotomy on J. Now we suppose in addition that $B_\mu(t,\mu)$ exists and is bounded and continuous where the continuity in μ is uniform with respect to t and also that $y_0(0,\mu)$ is C^1 in μ.

Let the projections be $Q_0(\mu) = I - Q_+(\mu) - Q_-(\mu)$, $Q_+(\mu)$ and $Q_-(\mu)$ as found in Remark 2.7. Now it follows from theorems on exponential dichotomies (see Theorem 4 in [13], Proposition 2.3 in [18]) that $Q_+(\mu)$ and $Q_-(\mu)$ are C^1 in μ. This completes the proof when $J = \mathbf{R}$. Suppose now $J = \mathbf{R}_+$. Then we found in (7) that

$$
\mathbf{R}^n = \operatorname{span}\big\{y_0(0,\mu)\big\} \oplus \mathcal{R}\big(Q_+(\mu)\big) \oplus \mathcal{R}\big(Q_-(\mu)\big) \tag{9}
$$

and we let $\tilde{Q}_+(\mu)$ be the projection with range $\mathcal{R}(Q_+(\mu))$ and nullspace $\operatorname{span}\{y_0(0,\mu)\} \oplus \mathcal{R}(Q_-(\mu))$ and set $\tilde{Q}_0(\mu) = I - \tilde{Q}_+(\mu) - Q_-(\mu)$. Since the subspaces in the splitting (9) are C^1 in μ, it follows that the associated projections $\tilde{Q}_0(\mu)$, $\tilde{Q}_+(\mu)$ and $Q_-(\mu)$ are C^1 in μ also.

Next we develop a necessary and sufficient condition for a trichotomy in terms of exponential dichotomy of an associated difference equation.

PROPOSITION 2.9. *Suppose $A(t)$ is a continuous matrix function with $|A(t)| \leqslant M$ for all t in $\mathbf{R}$. Let $x_0(t)$ be a nonzero bounded solution of (1) with $0 < \Delta \leqslant |x_0(t)| \leqslant L < \infty$ for all t.*

Suppose $\{s_k\}_{k \in Z}$ is a sequence of times such that for all k, $0 < s_{\min} \leqslant s_{k+1} - s_k \leqslant s_{\max} < \infty$ and let $\{Q_k\}_{k \in Z}$ be a sequence of projections of rank $n-1$ *with $|Q_k|, |I - Q_k| \leqslant N$ such that the nullspace of Q_k is spanned by $x_0(s_k)$.*

Then (1) *has a trichotomy on* $\mathbf{R}$ *if and only if the difference equation*

$$u_{k+1} = Q_{k+1} X(s_{k+1}) X^{-1}(s_k) u_k, \quad u_k \in \mathcal{R}(Q_k),\ k \in Z, \tag{10}$$

has an exponential dichotomy, where $X(t)$ is the fundamental matrix for (1) *with $X(0) = I$.*

PROOF. We write $A_k = X(s_{k+1}) X^{-1}(s_k)$ and note that

$$Q_{k+1} A_k Q_k = Q_{k+1} A_k \tag{11}$$

since $A_k x_0(s_k) = x_0(s_{k+1})$.

Suppose first that (1) has a trichotomy and let $P_0(t)$, $P_+(t)$ and $P_-(t)$ be the associated projections, where $P_\sigma(t) = X(t) P_\sigma X^{-1}(t)$. Then, by Remark 2.2, the range of $P_0(t)$ is spanned by $x_0(t)$. Define the sequence

$$R_k = Q_k P_k^+,$$

where $P_k^+ = P_+(s_k)$. Note that $R_k P_k^+ = R_k$ and since $P_k^+ x_0(s_k) = 0$ also $R_k Q_k = R_k$. Then

$$R_k^2 = R_k Q_k P_k^+ = R_k P_k^+ = R_k$$

so that R_k is a projection. Moreover, R_k is invariant with respect to (10) since

$$\begin{aligned} Q_{k+1} A_k R_k &= Q_{k+1} A_k Q_k P_k^+ \\ &= Q_{k+1} A_k P_k^+ \\ &= Q_{k+1} P_{k+1}^+ A_k \\ &= R_{k+1} A_k \\ &= R_{k+1} Q_{k+1} A_k. \end{aligned}$$

Now the transition matrix for (10) is given by

$$\Phi(k, m) = \begin{cases} Q_k A_{k-1} \cdots A_m & \text{if } k > m, \\ I & \text{if } k = m, \\ Q_k A_k^{-1} \cdots A_{m-1}^{-1} & \text{if } k < m \end{cases}$$

since, for example, $\Phi(k,k-1) = Q_k A_{k-1}$ and if $k > m+1$, then

$$\Phi(k,m) = \Phi(k,k-1)\Phi(k-1,m)$$

so that if we assume $\Phi(k-1,m) = Q_{k-1}A_{k-2}\cdots A_m$ then, using (11),

$$\Phi(k,m) = Q_k A_{k-1} Q_{k-1} A_{k-2}\cdots A_m = Q_k A_{k-1}A_{k-2}\cdots A_m$$

so that the formula for $k > m$ follows by induction on k. Then if $k > m$

$$\begin{aligned}\bigl|\Phi(k,m)R_m\bigr| &= |Q_k A_{k-1}\cdots A_m Q_m P_m^+| \\ &= \bigl|Q_k X(s_k)X^{-1}(s_m)P_+(s_m)\bigr| \\ &\leqslant NK\mathrm{e}^{-\alpha(s_k-s_m)} \\ &\leqslant NK\mathrm{e}^{-\alpha s_{\min}(k-m)}.\end{aligned}$$

Similarly, if $k < m$,

$$\bigl|\Phi(k,m)(I-R_m)\bigr| = \bigl|Q_k X(s_k)X^{-1}(s_m)P_-(s_m)\bigr| \leqslant NK\mathrm{e}^{-\alpha s_{\min}(m-k)}.$$

Finally, if $k = m$,

$$\bigl|\Phi(k,m)R_m\bigr| = |R_m| \leqslant NK.$$

Hence (10) has an exponential dichotomy with projections R_k and constants NK and $\mathrm{e}^{-\alpha s_{\min}}$.

Now, conversely, suppose (10) has an exponential dichotomy with projections $R_k : \mathcal{R}(Q_k) \mapsto \mathcal{R}(Q_k)$ and constants K, λ. Then a solution u_k of Eq. (10) with $u_0 \in \mathcal{R}(R_0)$ satisfies

$$|u_k| \leqslant K\lambda^{k-m}|u_m|$$

for $m \leqslant k$. Define

$$\eta_k = -\sum_{\ell=k}^{\infty} \alpha_{\ell+1}(A_\ell u_\ell),$$

where the linear functional $\alpha_k : \mathbf{R}^n \mapsto \mathbf{R}$ is defined by

$$(I - Q_k)v = \alpha_k(v)x_0(s_k).$$

Note that

$$|\alpha_k| \leqslant N\Delta^{-1}$$

so that if $k \geqslant m$

$$|\eta_k| \leqslant \sum_{\ell=k}^{\infty} N\Delta^{-1}\mathrm{e}^{Ms_{\max}}|u_\ell| \leqslant \sum_{\ell=k}^{\infty} N\Delta^{-1}\mathrm{e}^{Ms_{\max}}K\lambda^{\ell-m}|u_m| \leqslant K_1\lambda^{k-m}|u_m|,$$

where

$$K_1 = NK\mathrm{e}^{Ms_{\max}}\Delta^{-1}(1-\lambda)^{-1}.$$

Then we define

$$v_k = \eta_k x_0(s_k) + u_k.$$

Note that for all k

$$\begin{aligned}
v_{k+1} &= \eta_{k+1}x_0(s_{k+1}) + u_{k+1} \\
&= \big[\eta_k + \alpha_{k+1}(A_k u_k)\big]x_0(s_{k+1}) + Q_{k+1}A_k u_k \\
&= \eta_k x_0(s_{k+1}) + (I - Q_{k+1})A_k u_k + Q_{k+1}A_k u_k \\
&= \eta_k A_k x_0(s_k) + A_k u_k \\
&= X(s_{k+1})X^{-1}(s_k)v_k
\end{aligned}$$

and that for $k \geqslant m$

$$|v_k| \leqslant K_1 L\lambda^{k-m}|u_m| + K\lambda^{k-m}|u_m| = (K_1 L + K)|u_m|\lambda^{k-m},$$

where we see that

$$|u_m| = |Q_m v_m| \leqslant N|v_m|$$

so that for $k \geqslant m$

$$|v_k| \leqslant K_2|v_m|\lambda^{k-m},$$

where

$$K_2 = (K_1 L + K)N.$$

Next note that

$$\eta_0 = -\sum_{k=0}^{\infty}\alpha_{k+1}(A_k u_k) = -\sum_{k=0}^{\infty}\alpha_{k+1}(A_k Q_k A_{k-1}\cdots A_0 u_0).$$

So what we have shown is that if we define the subspace

$$E^s = \left\{v_0 = \beta(u_0)x_0(s_0) + u_0\colon\ u_0 \in \mathcal{R}(R_0)\right\}$$

of $\mathbf{R}^n$, where

$$\beta(u_0) = -\sum_{k=0}^{\infty} \alpha_{k+1}(A_k Q_k A_{k-1} \cdots A_0 u_0),$$

then for $v_0 \in E^s$,

$$\left|X(s_k)X^{-1}(s_0)v_0\right| \leqslant K_2\lambda^{k-m}\left|X(s_m)X^{-1}(s_0)v_0\right|$$

for $m \leqslant k$. Now if $s \leqslant t$, we can find $m \leqslant k$ such that

$$s_m \leqslant s < s_{m+1}, \qquad s_k \leqslant t < s_{k+1}.$$

Then

$$\begin{aligned}
\left|X(t)X^{-1}(s_0)v_0\right| &\leqslant \mathrm{e}^{Ms_{\max}}\left|X(s_k)X^{-1}(s_0)v_0\right| \\
&\leqslant \mathrm{e}^{Ms_{\max}}K_2\lambda^{k-m}\left|X(s_m)X^{-1}(s_0)v_0\right| \\
&\leqslant \mathrm{e}^{2Ms_{\max}}K_2\lambda^{k-m}\left|X(s)X^{-1}(s_0)v_0\right| \\
&\leqslant K_2\mathrm{e}^{2Ms_{\max}}\lambda^{\frac{t-s}{s_{\max}}-1}\left|X(s)X^{-1}(s_0)v_0\right| \\
&= K_2\mathrm{e}^{2Ms_{\max}}\lambda^{-1}\mathrm{e}^{\frac{\ln\lambda}{s_{\max}}(t-s)}\left|X(s)X^{-1}(s_0)v_0\right|.
\end{aligned}$$

So if $v_0 \in E^s$, then for $s \leqslant t$

$$\left|X(t)X^{-1}(s_0)v_0\right| \leqslant K_3\mathrm{e}^{-\alpha(t-s)}\left|X(s)X^{-1}(s_0)v_0\right|, \tag{12}$$

where

$$K_3 = K_2\mathrm{e}^{2Ms_{\max}}/\lambda, \qquad \alpha = -\ln\lambda/s_{\max}.$$

Similarly, there is a subspace

$$E^u = \left\{v_0 = \gamma(v_0)x_0(s_0) + u_0\colon\ u_0 \in \mathcal{N}(R_0)\right\}$$

of $\mathbf{R}^n$ such that if $v_0 \in E^u$,

$$\left|X(t)X^{-1}(s_0)v_0\right| \leqslant K_3\mathrm{e}^{-\alpha(s-t)}\left|X(s)X^{-1}(s_0)v_0\right| \tag{13}$$

for $s \geqslant t$.

Next we show that $E^s \cap E^u = \{0\}$. Let v_0 be in this intersection. Then there exist $u_0 \in \mathcal{R}(R_0)$ and $\tilde{u}_0 \in \mathcal{N}(R_0)$ such that

$$v_0 = \beta(u_0)x_0(s_0) + u_0 = \gamma(\tilde{u}_0)x_0(s_0) + \tilde{u}_0.$$

Multiplying through by $I - Q_0$, we see that $\beta(u_0) = \gamma(\tilde{u}_0)$ which implies $u_0 = \tilde{u}_0$ and so both are zero. Thus $v_0 = 0$ and our assertion is proved.

Next note that $x_0(s_0)$ is not in $E^s \oplus E^u$, since if there exist $u_0 \in \mathcal{R}(R_0)$ and $\tilde{u}_0 \in \mathcal{N}(R_0)$ such that

$$x_0(s_0) = \beta(u_0)x_0(s_0) + u_0 + \gamma(\tilde{u}_0)x_0(s_0) + \tilde{u}_0,$$

then

$$x_0(s_0) = \big(\beta(u_0) + \gamma(\tilde{u}_0)\big)x_0(s_0) + u_0 + \tilde{u}_0.$$

Since $x_0(s_0)$ and $u_0 + \tilde{u}_0$ are linearly independent, it follows that $u_0 + \tilde{u}_0 = 0$ and hence $u_0 = \tilde{u}_0 = 0$ and so $x_0(s_0) = 0$, which is not true. So $x_0(s_0)$ is not in $E^s \oplus E^u$. Then since

$$\dim E^s + \dim E^u = \operatorname{rank} R_0 + (n - 1 - \operatorname{rank} R_0) = n - 1,$$

it follows that

$$\mathbf{R}^n = \operatorname{span}\big\{x_0(s_0)\big\} \oplus E^s \oplus E^u.$$

Let P_0, P_+, P_- be the projections associated with this splitting. Then it follows from (12) and (13) that for all ξ

$$\big|X(t)X^{-1}(s_0)P_+\xi\big| \leqslant K_3 \mathrm{e}^{-\alpha(t-s)}\big|X(t)X^{-1}(s_0)P_+\xi\big| \quad (s \leqslant t)$$

and

$$\big|X(t)X^{-1}(t_0)P_-\xi\big| \leqslant K_3 \mathrm{e}^{-\alpha(s-t)}\big|X(s)X^{-1}(s_0)P_+\xi\big| \quad (s \geqslant t).$$

Also since $|x_0(t)| \leqslant L\Delta^{-1}|x_0(s)|$ for all s, t, it follows for all t, s, ξ that

$$\big|X(t)X^{-1}(s_0)P_0\xi\big| \leqslant L\Delta^{-1}\big|X(s)X^{-1}(s_0)P_0\xi\big|.$$

Then, in view of Remark 2.5, it follows that (1) has a trichotomy on $\mathbf{R}$. This completes the proof of the proposition. □

3. Hyperbolic periodic orbits and their stable and unstable manifolds

In this section we study an autonomous system in $\mathbf{R}^n$

$$\dot{x} = f(x), \tag{14}$$

where f is C^1 and for all x

$$\left|f(x)\right| \leqslant M_0, \qquad \left|f'(x)\right| \leqslant M_1.$$

We denote by $\phi^t(x)$ the corresponding flow which, under these conditions, is defined for all t and x. Frequently we shall use the fact that

$$\left|D\phi^t(x)\right| \leqslant \mathrm{e}^{M_1|t|},$$

which follows from Gronwall's inequality, and also the fact that

$$D\phi^t(x) f(x) = f\big(\phi^t(x)\big).$$

We first define the Poincaré map associated with a periodic orbit of this system. Then we construct the local and global stable and unstable manifolds of a hyperbolic periodic orbit, thereby introducing the concept of asymptotic phase.

3.1. *Poincaré map*

First we define a more general Poincaré map. For this, we need the following lemma.

LEMMA 3.1. *Let $f(x)$ be a C^1 vectorfield in $\mathbf{R}^n$ with $|f(x)| \leqslant M_0$ and $|f'(x)| \leqslant M_1$. Let x be a point in $\mathbf{R}^n$, v a unit vector and T a real number such that $\langle f(y), v\rangle > 0$, where $y = \phi^T(x)$. Set*

$$\alpha = \frac{\langle f(y), v\rangle}{4M_0M_1}, \qquad \delta = \frac{\mathrm{e}^{-M_1T}\langle f(y), v\rangle}{4M_1}\min\left\{1, \frac{\langle f(y), v\rangle}{2M_0}\right\}.$$

Then if $|z - x| < \delta$, there exists a unique $\tau = \tau(z)$ such that

$$\big\langle \phi^\tau(z) - y, v\big\rangle = 0$$

and $|\tau - T| \leqslant \alpha$. Moreover,

$$\left|\tau(z) - T\right| \leqslant 2\big\langle f(y), v\big\rangle^{-1}\mathrm{e}^{M_1T}|z - x|, \tag{15}$$

$\tau(z)$ is a C^1 function with

$$\left|\tau'(z)\right| \leqslant 2\big\langle f(y), v\big\rangle^{-1}\mathrm{e}^{M_1(T+\alpha)}$$

and

$$\big\langle f\big(\phi^{\tau(z)}(z)\big), v\big\rangle \geqslant \frac{1}{2}\big\langle f(y), v\big\rangle.$$

PROOF. We need to solve the equation

$$g(\tau, z) = \langle \phi^\tau(z) - y, v \rangle = 0$$

for τ when $|z - x| < \delta$. Note that

$$g(T, x) = 0, \qquad g_\tau(T, x) = \langle f(y), v \rangle.$$

Then the equation holds if and only if

$$S\tau = \tau,$$

where

$$S\tau = T - g_\tau(T, x)^{-1}\eta(\tau, z),$$

with

$$\eta(\tau, z) = g(\tau, z) - g(T, x) - g_\tau(T, x)(\tau - T).$$

Note that if $|\tau_i - T| \leqslant \alpha$ and $|z - x| < \delta$, then

$$\begin{aligned} \bigl|\eta(\tau_1, z) - \eta(\tau_2, z)\bigr| &= \bigl|g(\tau_1, z) - g(\tau_2, z) - g_\tau(T, x)(\tau_1 - \tau_2)\bigr| \\ &= \left|\int_0^1 g_\tau\bigl(\theta\tau_1 + (1-\theta)\tau_2, z\bigr) - g_\tau(T, x)\,\mathrm{d}\theta(\tau_1 - \tau_2)\right|. \end{aligned}$$

Now if $|\tau - T| \leqslant \alpha$ and $|z - x| < \delta$,

$$\begin{aligned} \bigl|g_\tau(\tau, z) - g_\tau(T, x)\bigr| &\leqslant \bigl|f\bigl(\phi^\tau(z)\bigr) - f\bigl(\phi^T(x)\bigr)\bigr| \\ &\leqslant M_1\bigl|\phi^\tau(z) - \phi^T(x)\bigr| \\ &\leqslant M_1\bigl[\bigl|\phi^\tau(z) - \phi^T(z)\bigr| + \bigl|\phi^T(z) - \phi^T(x)\bigr|\bigr] \\ &\leqslant M_1\bigl[M_0|\tau - T| + \mathrm{e}^{M_1 T}|z - x|\bigr] \\ &\leqslant M_1\bigl[M_0\alpha + \mathrm{e}^{M_1 T}\delta\bigr]. \end{aligned} \tag{16}$$

It follows that if $|\tau_i - T| \leqslant \alpha$ and $|z - x| < \delta$, then

$$\bigl|\eta(\tau_1, z) - \eta(\tau_2, z)\bigr| \leqslant M_1[M_0\alpha + \mathrm{e}^{M_1 T}\delta]|\tau_1 - \tau_2|.$$

Next note that

$$\bigl|\eta(T, z)\bigr| = \bigl|g(T, z) - g(T, x)\bigr| \leqslant \bigl|\phi^T(z) - \phi^T(x)\bigr| \leqslant \mathrm{e}^{M_1 T}|z - x|.$$

Using these we see that if $|\tau - T| \leqslant \alpha$ and $|z - x| < \delta$, then

$$|S\tau - T| \leqslant \langle f(y), v\rangle^{-1}\big[\big|\eta(\tau, z) - \eta(T, z)\big| + \big|\eta(T, z)\big|\big]$$
$$\leqslant \langle f(y), v\rangle^{-1}\big[M_1[M_0\alpha + \mathrm{e}^{M_1 T}\delta]|\tau - T| + \mathrm{e}^{M_1 T}|z - x|\big]$$

so that

$$|S\tau - T| \leqslant \frac{1}{2}|\tau - T| + \langle f(y), v\rangle^{-1}\mathrm{e}^{M_1 T}|z - x|. \tag{17}$$

In particular, this means that

$$|S\tau - T| \leqslant \frac{1}{2}\alpha + \langle f(y), v\rangle^{-1}\mathrm{e}^{M_1 T}\delta \leqslant \alpha.$$

Next, by similar reasoning, if $|\tau_i - T| \leqslant \alpha$ and $|z - x| < \delta$, then

$$|S\tau_1 - S\tau_2| \leqslant \langle f(y), v\rangle^{-1}M_1[M_0\alpha + \mathrm{e}^{M_1 T}\delta]|\tau_1 - \tau_2| \leqslant \frac{1}{2}|\tau_1 - \tau_2|.$$

It follows by the contraction mapping principle that if $|z - x| < \delta$, S has a unique fixed point τ in $|\tau - T| \leqslant \alpha$. The inequality (15) follows from (17) by putting $S\tau = \tau$. This completes the proof of the first part of the lemma.

That $\tau(x)$ is C^1 follows from standard results concerning smoothness of fixed point of a uniform contraction (see, for example, Lemma 3.8 below). Now

$$\tau'(z)h = -\frac{g_z(\tau(z), z)h}{g_\tau(\tau(z), z)} = -\frac{\langle D\phi^{\tau(z)}(z)h, v\rangle}{\langle f(\phi^{\tau(z)}(z)), v\rangle}. \tag{18}$$

To estimate this, we use (16) which implies that

$$\big|g_\tau\big(\tau(z), z\big) - g_\tau(T, x)\big| \leqslant M_1[M_0\alpha + \mathrm{e}^{M_1 T}\delta] \leqslant \frac{1}{2}\langle f(y), v\rangle.$$

Hence, since $g_\tau(T, x) = \langle f(y), v\rangle$,

$$g_\tau\big(\tau(z), z\big) \geqslant \frac{1}{2}\langle f(y), v\rangle.$$

Next

$$\big|\langle D\phi^{\tau(z)}(z)h, v\rangle\big| \leqslant \big|D\phi^{\tau(z)}(z)\big||h| \leqslant \mathrm{e}^{M_1\tau(z)}|h| \leqslant \mathrm{e}^{M_1(T+\alpha)}|h|.$$

Thus we obtain the inequality for $|\tau'(z)|$ and we have also shown in passing the last inequality in the statement of the lemma. Thus we complete the proof of the lemma. $\square$

REMARK 3.2. We can write the τ found in the lemma as $\tau(z, x, v, T)$. We observe here that τ is a C^1 function of (z, x, v, T) on the open set of (z, x, v, T) for which $|z-x| < \delta = \delta(x, v, T)$. This follows from the smoothness of the fixed point of a uniform contraction (see, for example, Lemma 3.8 below).

We use this lemma to define a general Poincaré map. Consider a system (14) with a nonconstant solution $u(t)$. Let $t_1 < t_2$ be two times and let v_i $(i = 1, 2)$ be a unit vector such that $\langle f(u(t_i)), v_i\rangle > 0$. Let $W_i = \{v\colon \langle v, v_i\rangle = 0\}$ and set $C_i = u(t_i) + W_i$. Also set

$$\alpha = \bigl\langle f\bigl(u(t_2)\bigr), v_2\bigr\rangle / 4M_0M_1$$

and

$$\delta = \min\left\{\frac{\mathrm{e}^{-M_2(t_2-t_1)}\langle f(u(t_2)), v_2\rangle}{4M_1}\min\left\{1, \frac{\langle f(u(t_2)), v_2\rangle}{2M_0}\right\}, \frac{\langle f(u(t_1)), v_1\rangle}{M_1}\right\}.$$

Then, by Lemma 3.1, if $x \in C_1$ with $|x - u(t_1)| < \delta$, there exists a unique $\tau = \tau(x)$ in $|\tau - (t_2 - t_1)| < \alpha$ such that $\phi^\tau(x) \in C_2$. Moreover, $\tau(x)$ is a C^1 function, $\tau(u(t_1)) = t_2 - t_1$ and

$$\bigl|\tau(x) - (t_2 - t_1)\bigr| \leqslant 2\bigl\langle f\bigl(u(t_2)\bigr), v_2\bigr\rangle^{-1}\mathrm{e}^{M_1(t_2-t_1)}\bigl|x - u(t_1)\bigr|.$$

Then for $x \in C_1$ with $|x - u(t_1)| < \delta$ we define the Poincaré map

$$P(x) = \phi^{\tau(x)}(x).$$

Note that

$$P\bigl(u(t_1)\bigr) = u(t_2).$$

Also, as in (18), for $x \in C_1$ and $\xi \in W_1$,

$$\tau'(x)\xi = -\frac{\langle D\phi^{\tau(x)}(x)\xi, v_2\rangle}{\langle f(\phi^{\tau(x)}(x)), v_2\rangle}$$

so that

$$DP(x)\xi = f\bigl(P(x)\bigr)\tau'(x)\xi + D\phi^{\tau(x)}(x)\xi = Q_x D\phi^{\tau(x)}(x)\xi, \tag{19}$$

where Q_x is the projection on to W_2 along $f(P(x))$. Note that if $DP(x)\xi = 0$, then $D\phi^{\tau(x)}(x)\xi = \lambda f(P(x))$, where $\lambda = -\tau'(x)\xi$, and so

$$D\phi^{\tau(x)}(x)\bigl(-\lambda f(x) + \xi\bigr) = 0$$

from which it follows that $-\lambda f(x) + \xi = 0$. Now since $|x - u(t_1)| < \delta$,

$$\bigl\langle f(x), v_1\bigr\rangle > \bigl\langle f\bigl(u(t_1)\bigr), v_1\bigr\rangle - M_1\delta \geqslant 0$$

and so we must have $\lambda = 0$ and $\xi = 0$. So $DP(x)$ is injective and hence invertible. It follows that P is a diffeomorphism on $x \in C_1$, $|x - u(t_1)| < \delta$.

We use the notation $P(u(t_1), t_1, t_2, v_1, v_2)$ for the Poincaré map just defined.

DEFINITION 3.3. We say a T-periodic orbit $u_0(t)$ of (14) is *hyperbolic* if all but one of its Floquet multipliers lie off the unit circle.

PROPOSITION 3.4. *A T-periodic orbit $u_0(t)$ of* (14) *is hyperbolic if and only if $u_0(0)$ is hyperbolic as a fixed point of the Poincaré map*

$$P\bigl(u_0(0), 0, T, v, v\bigr), \qquad v = \frac{u_0'(0)}{|u_0'(0)|}.$$

PROOF. Write $P = P(u_0(0), 0, T, v, v)$. First we note that

$$P\bigl(u_0(0)\bigr) = u_0(T) = u_0(0).$$

Next we know from (19) that

$$DP\bigl(u_0(0)\bigr) = QD\phi^T\bigl(u_0(0)\bigr),$$

where Q is the orthogonal projection onto the orthogonal complement W (the domain of $DP(u_0(0))$) of $u_0'(0) = f(u_0(0))$. Also we know that

$$D\phi^T\bigl(u_0(0)\bigr)f\bigl(u_0(0)\bigr) = f\bigl(u_0(0)\bigr).$$

We choose a basis of $\mathbf{R}^n$ with $f(u_0(0))$ as the first basis vector and with the other basis vectors in W. Then the matrix of $D\phi^T(u_0(0))$ with respect to this basis has the partitioned form

$$\begin{bmatrix} 1 & b \\ 0 & B \end{bmatrix},$$

where B is the matrix of $DP(u_0(0))$. Then the proposition follows immediately since the characteristic polynomial is $(1 - \lambda)\det(B - \lambda I)$. □

3.2. *Local stable and unstable manifolds*

In this subsection we study the local stable and unstable manifolds of a hyperbolic periodic orbit. For later use, we consider a system

$$\dot{x} = f(x, \mu) \tag{20}$$

depending on a parameter μ. We assume that when $\mu = 0$ the system has a hyperbolic T-periodic orbit $u_0(t)$. By standard theorems (see [10]) the perturbed system (20) has a

nearby hyperbolic periodic orbit $u(t,\mu)$ such that $u(t,0) = u_0(t)$. This orbit will have period $T(\mu)$ with $T(0) = T$. Note that if $f(x,\mu)$ is C^r, then $u(t,\mu)$ and $T(\mu)$ are C^r also.

In the following proposition, we find the local stable manifold near $u(0,\mu)$ with asymptotic phase 0, that is, we find the solutions $x(t)$ with $x(0)$ near $u(0,\mu)$ such that $|x(t) - u(t,\mu)| \to 0$ as $t \to \infty$.

PROPOSITION 3.5. *Let the function $f(x,\mu)$ be C^{r+2} $(r \geqslant 0)$ and suppose*

$$\dot{x} = f(x,0)$$

has a hyperbolic T-periodic solution $u_0(t)$. Then

(i) *the system*

$$\dot{x} = f_x\big(u_0(t),0\big)x$$

has a trichotomy with projections P_0, P_+ and P_- and constants K, α;

(ii) *the perturbed system* (20) *has a hyperbolic $T(\mu)$-periodic solution $u(t,\mu)$ where $u(t,\mu)$ and $T(\mu)$ are C^{r+2} functions with $u(t,0) = u_0(t)$ and $T(0) = T$;*

(iii) *suppose $0 < \gamma < \alpha$. For $\Delta > 0$ sufficiently small, there are positive numbers μ_0, ξ_0 depending on Δ such that if $|\mu| < \mu_0$ and $\xi \in \mathcal{R}(P_+)$ satisfies $|\xi| < \xi_0$, there exists a unique solution $x(t) = x^+(t,\xi,\mu)$ of* (20) *such that*

$$\big|x(t) - u(t,\mu)\big|\mathrm{e}^{\gamma t} \leqslant \Delta \quad (t \geqslant 0), \qquad P_+\big[x(0) - u(0,\mu)\big] = \xi.$$

Moreover,

$$x^+(t,0,\mu) = u(t,\mu),$$

$x^+(t,\xi,\mu)$ is C^r in its arguments and $x^+_\xi(t,\xi,\mu)$ is the unique solution restricted to $\mathcal{R}(P_+)$ of

$$\dot{Z} = f_x\big(x^+(t,\xi,\mu),\mu\big)Z$$

such that $\sup_{t\geqslant 0} \mathrm{e}^{\gamma t}|Z(t)| < \infty$ *and $P_+Z(0) = P_+$ with, in particular,*

$$x^+_\xi(t,0,0) = X(t)P_+,$$

and there is a constant N such that when $r \geqslant 1$,

$$\big|x^+_\xi(t,\xi,\mu)\big| \leqslant N\mathrm{e}^{-\gamma t}, \quad \big|x^+_\mu(t,\xi,\mu) - u_\mu(t,\mu)\big| \leqslant N\mathrm{e}^{-\gamma t} \quad (t \geqslant 0)$$

and when $r \geqslant 2$,

$$\big|x^+_{\xi\xi}(t,\xi,\mu)\big| \leqslant N\mathrm{e}^{-\gamma t} \quad (t \geqslant 0).$$

PROOF. For the linear system

$$\dot{x} = f_x\big(u_0(t), 0\big)x, \tag{21}$$

1 is a Floquet multiplier of algebraic multiplicity one and the others lie off the unit circle. Now by Floquet theory there is a linear invertible periodic transformation $x = S(t)y$ taking (21) into an autonomous system $\dot{y} = Ay$ with A having zero as an eigenvalue of multiplicity one and the other eigenvalues having nonzero real parts. We define P_0, P_+, P_- to be the supplementary projections corresponding to the splitting of $\mathbf{R}^n$ into the generalized eigenspaces corresponding respectively to the zero eigenvalue, the eigenvalues with negative real parts and the eigenvalues with positive real parts. Note that the projections P_σ commute with A and there exist positive constants L, α (note that α can be taken as any positive number such that $|\mathrm{Re}(\lambda)| > \alpha$ for all eigenvalues λ of A with nonzero real part) such that

$$\begin{aligned}
&|\mathrm{e}^{tA}P_0| \leqslant L \quad \text{for all } t,\\
&|\mathrm{e}^{tA}P_+| \leqslant L\mathrm{e}^{-\alpha t} \quad \text{for } t \geqslant 0,\\
&|\mathrm{e}^{tA}P_-| \leqslant L\mathrm{e}^{\alpha t} \quad \text{for } t \leqslant 0.
\end{aligned}$$

Then $X(t) = S(t)\mathrm{e}^{tA}$ is a fundamental matrix for (21) and, since $X(t)P_\sigma X^{-1}(s) = S(t)\mathrm{e}^{(t-s)A}P_\sigma S^{-1}(s)$, we obtain the inequalities

$$\begin{aligned}
&\big|X(t)P_0X^{-1}(s)\big| \leqslant K \quad \text{for all } t,\\
&\big|X(t)P_+X^{-1}(s)\big| \leqslant K\mathrm{e}^{-\alpha(t-s)} \quad \text{for } t \geqslant s,\\
&\big|X(t)P_-X^{-1}(s)\big| \leqslant K\mathrm{e}^{-\alpha(s-t)} \quad \text{for } t \leqslant s,
\end{aligned}$$

where $K = LM^2$ with M an upper bound for $|S(t)|$ and $|S^{-1}(t)|$. Now if we replace P_σ by $X(0)P_\sigma X^{-1}(0)$, it follows that (21) has a trichotomy on $\mathbf{R}$ with these projections, which we rename as P_σ, and constants K and α. So (i) is proved.

As noted before the proposition, (ii) follows from standard theorems. Now if we change the time scale $\tau = \omega(\mu)t$, where $\omega(\mu) = T/T(\mu)$, then the new system

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \omega(\mu)^{-1}f(x,\mu)$$

has the periodic orbit $u(\omega(\mu)^{-1}\tau, \mu)$ which has period T for all μ. So we can assume without loss of generality that in the original system the period of $u(t,\mu)$ is T, independent of μ. The advantage of this is that now $\sup_{t\in\mathbf{R}}|u(t,\mu) - u_0(t)| \to 0$ as $\mu \to 0$.

To prove (iii), we look for solutions $x(t)$ of (20) such that $|x(t) - u(t,\mu)| \to 0$ at an exponential rate as $t \to \infty$. We write

$$x = u(t,\mu) + z.$$

Then

$$\dot{z} = f_x\big(u_0(t), 0\big)z + h(t, z, \mu), \tag{22}$$

where

$$h(t, z, \mu) = \big[f_x\big(u(t, \mu), \mu\big) - f_x\big(u_0(t), 0\big)\big]z + f\big(u(t, \mu) + z, \mu\big) - f\big(u(t, \mu), \mu\big) - f_x\big(u(t, \mu), \mu\big)z.$$

Note that $h(t, 0, \mu) = 0$ and

$$\big|h_z(t, z, \mu)\big| = \big|f_x\big(u(t, \mu) + z, \mu\big) - f_x\big(u_0(t), 0\big)\big| \leqslant \delta\big(|z|, |\mu|\big),$$

where $\delta(|z|, |\mu|)$ is a continuous function nondecreasing in both arguments which $\to 0$ as $(|z|, |\mu|) \to (0, 0)$.

Given small $\xi \in \mathcal{R}(P_+)$, we look for solutions $z(t)$ of (22) such that

$$P_+z(0) = \xi, \qquad \big|z(t)\big| \leqslant \Delta e^{-\gamma t} \quad (t \geqslant 0),$$

where Δ is chosen suitably small. First we prove the following lemma.

LEMMA 3.6. *Suppose the linear system*

$$\dot{x} = A(t)x \tag{23}$$

has a trichotomy on $[0, \infty)$ *with projections* P_0, P_+, P_- *and constants* K, α *and let* $0 < \gamma < \alpha$. *Suppose* $\xi \in \mathcal{R}(P_+)$ *and* $h(t)$ *is a continuous function with* $\|h\|_\gamma = \sup_{t \geqslant 0} |h(t)| e^{\gamma t} < \infty$. *Then*

$$x(t) = X(t)\xi + \int_0^t X(t)P_+X^{-1}(s)h(s)\,ds - \int_t^\infty X(t)(I - P_+)X^{-1}(s)h(s)\,ds$$

is the unique solution of

$$\dot{x} = A(t)x + h(t) \tag{24}$$

such that $\|x\|_\gamma < \infty$ *and* $P_+x(0) = \xi$. *Moreover*

$$\|x\|_\gamma \leqslant K|\xi| + K\big[(\alpha - \gamma)^{-1} + 2\gamma^{-1}\big]\|h\|_\gamma.$$

PROOF. To prove the uniqueness first, note that the difference $x(t)$ between any two such solutions would be a solution of (23) such that $x(t) \to 0$ as $t \to \infty$ and $P_+x(0) = 0$. Now from Remark 2.2 it follows that $x(0) \in \mathcal{R}(P_+)$. So $x(0) = P_+x(0) = 0$. This gives the uniqueness.

To prove the existence, note that

$$\begin{aligned}
&\int_0^t \left|X(t)P_+X^{-1}(s)h(s)\right| \mathrm{d}s + \int_t^\infty \left|X(t)(I-P_+)X^{-1}(s)h(s)\right| \mathrm{d}s \\
&\quad \leqslant \int_0^t K\mathrm{e}^{-\alpha(t-s)}\|h\|_\gamma \mathrm{e}^{-\gamma s}\,\mathrm{d}s + \int_t^\infty 2K\|h\|_\gamma \mathrm{e}^{-\gamma s}\,\mathrm{d}s \\
&\quad = K\big[(\alpha-\gamma)^{-1} + 2\gamma^{-1}\big]\|h\|_\gamma \mathrm{e}^{-\gamma t}.
\end{aligned}$$

Hence the integrals defining $x(t)$ converge and we find by differentiation that $x(t)$ is a solution of (24). Moreover, using the fact that $|X(t)P_+| \leqslant K\mathrm{e}^{-\gamma t}$ and the previous estimate, we find that $\|x\|_\gamma < \infty$ and that the inequality given in the statement of the lemma holds. □

From this lemma, we deduce another.

LEMMA 3.7. *Suppose the linear system* (23) *satisfies the conditions of Lemma* 3.6 *and let* $0 < \gamma < \alpha$. *Let* $h(t,x,\mu)$ *be a continuous function, T-periodic in t, satisfying*

$$h(t,0,\mu) = 0, \qquad \left|h(t,x_1,\mu) - h(t,x_2,\mu)\right| \leqslant L|x_1 - x_2|$$

for all $t \geqslant 0$, $|x_1| \leqslant \Delta$, $|x_2| \leqslant \Delta$, $|\mu| < \mu_0$. *Suppose that*

$$K(\gamma)L = K\big[(\alpha-\gamma)^{-1} + 2\gamma^{-1}\big]L < 1.$$

Then if $|\mu| < \mu_0$ *and* $\xi \in \mathcal{R}(P_+)$ *with*

$$K|\xi| < \big(1 - K(\gamma)L\big)\Delta, \tag{25}$$

the equation

$$\dot{x} = A(t)x + h(t,x,\mu)$$

has a unique solution $x(t) = x(t,\xi,\mu)$ *such that* $\|x\|_\gamma \leqslant \Delta$ *and* $P_+x(0) = \xi$. *Moreover*

$$x(t,0,\mu) = 0$$

and

$$\|x\|_\gamma \leqslant K\big(1 - K(\gamma)L\big)^{-1}|\xi|. \tag{26}$$

Also, if $h(t,x,\mu)$ *is* C^{r+2} $(r \geqslant 0)$ *in* (x,μ) *with all derivatives continuous in* (t,x,μ), *then* $x(t,\xi,\mu)$ *is* C^r *in* (x,μ) *and is continuous in* (t,x,μ) *together with all its derivatives. Moreover,* $x_\xi(t,\xi,\mu)$ *is the unique solution restricted to* $\mathcal{R}(P_+)$ *of*

$$\dot{Z} = \big[A(t) + h_x\big(t, x(t,\xi,\mu),\mu\big)\big]Z$$

such that $\|Z\|_\gamma < \infty$ *and* $P_+Z(0) = P_+$,

$$x_\xi(t, 0, 0) = X(t)P_+$$

and there exists a constant N *such that for* $t \geqslant 0$

$$\bigl|x_\xi(t, \xi, \mu)\bigr| \leqslant N\mathrm{e}^{-\gamma t}, \quad \bigl|x_\mu(t, \xi, \mu)\bigr| \leqslant N\mathrm{e}^{-\gamma t}, \quad \bigl|x_{\xi\xi}(t, \xi, \mu)\bigr| \leqslant N\mathrm{e}^{-\gamma t}$$

when these derivatives exist.

PROOF. Consider the Banach space E of continuous functions $x(t)$ on $[0, \infty)$ with $\|x\|_\gamma < \infty$, with $\|x\|_\gamma$ as norm. Let B be the ball of radius Δ and centre 0. Then B is a complete metric space. Next suppose $|\mu| < \mu_0$ and $\xi \in \mathcal{R}(P_+)$ satisfies (25). Then we define a mapping $T : B \mapsto B$ as follows: if $x \in B$, we let $T(x)(t)$ be the unique solution of

$$\dot{x} = A(t)x + h\bigl(t, x(t), \mu\bigr)$$

such that $P_+T(x)(0) = \xi$ and $\|T(x)\|_\gamma < \infty$. This exists by Lemma 3.6 and it also follows from that lemma that

$$\bigl\|T(x)\bigr\|_\gamma \leqslant K|\xi| + K(\gamma)L\|x\|_\gamma \leqslant K|\xi| + K(\gamma)L\Delta \leqslant \Delta. \tag{27}$$

Next if x_1 and x_2 are in B, it follows by similar reasoning that

$$\bigl\|T(x_1) - T(x_2)\bigr\|_\gamma \leqslant K(\gamma)L\|x_1 - x_2\|_\gamma. \tag{28}$$

Since $K(\gamma)L < 1$, T is a contraction and so has a unique fixed point $x(t)$. That $x(t) = 0$ when $\xi = 0$ follows by uniqueness and the inequality (26) follows from (27) by putting $Tx = x$.

To prove the smoothness, we write $T(x)$ as $T(x, \xi, \mu)$ to indicate the dependence on ξ and μ. Note that if $x \in B$

$$\begin{aligned} T(x, \xi, \mu)(t) = X(t)\xi &+ \int_0^t X(t)P_+X^{-1}(s)h\bigl(s, x(s), \mu\bigr)\,\mathrm{d}s \\ &- \int_t^\infty X(t)(I - P_+)X^{-1}(s)h\bigl(s, x(s), \mu\bigr)\,\mathrm{d}s. \end{aligned} \tag{29}$$

It is not hard to see that for $1 \leqslant m + p + q \leqslant r$, $p \geqslant 1$, the partial derivatives are given by the formulae

$$\bigl(T_\xi(x, \xi, \mu)\xi_1\bigr)(t) = X(t)\xi_1, \quad p = 1,\ m + q = 0,$$

$$T_{x^m\xi^p\mu^q}(x, \xi, \mu) = 0, \quad p = 1,\ m + q \geqslant 1 \text{ or } p \geqslant 2$$

and for $p=0$, $1 \leqslant m+q \leqslant r$ by

$$\begin{aligned}
&\big(T_{x^m\mu^q}(x,\xi,\mu)u_1\cdots u_m\big)(t)\\
&\quad=\int_0^t X(t)P_+X^{-1}(s)h_{x^m\mu^q}\big(s,x(s),\mu\big)u_1(s)\cdots u_m(s)\,\mathrm{d}s\\
&\qquad-\int_t^\infty X(t)(I-P_+)X^{-1}(s)h_{x^m\mu^q}\big(s,x(s),\mu\big)u_1(s)\cdots u_m(s)\,\mathrm{d}s,
\end{aligned}$$

where we use the notation $T_{x^m\xi^p\mu^q}$ for $\frac{\partial^{m+p+q}T}{\partial x^m\partial\xi^p\partial\mu^q}$, $\xi_1\in\mathcal{R}(P_+)$ and $u_1,\dots,u_m$ are in E. (Note when $m=0$ and $q=r$, we use the facts that $h(t,0,\mu)=0$ and the boundedness of the derivatives of h to verify that this formula yields the derivative.) These derivatives are also continuous functions of (x,ξ,μ) for $x\in B$, $\xi\in\mathcal{R}(P_+)$ satisfying (25) and μ in $|\mu|<\mu_0$. Thus $(x,\xi,\mu)\mapsto T(x,\xi,\mu)$ is a C^r map. Now we use the following well-known lemma:

LEMMA 3.8. *Let B be a closed ball in a Banach space E and let V be an open set in another Banach space Λ. Suppose $T: B\times V\mapsto B$ is a mapping such that for all x_1, $x_2\in B$ and $\lambda\in V$*

$$\big\|T(x_1,\lambda)-T(x_2,\lambda)\big\|\leqslant\sigma\|x_1-x_2\|$$

where $0\leqslant\sigma<1$. Then for each $\lambda\in V$ the equation

$$T(x,\lambda)=x$$

has a unique solution $x=x(\lambda)$ in B. Moreover, if T is C^r $(0\leqslant r\leqslant\infty)$, then so also is $x(\lambda)$, and the derivative $Dx(\lambda)$ satisfies the equation

$$Dx(\lambda)=\frac{\partial T}{\partial x}\big(x(\lambda),\lambda\big)Dx(\lambda)+\frac{\partial T}{\partial\lambda}\big(x(\lambda),\lambda\big).$$

(*Note*: *T is C^r means that T is defined and C^r in an open set containing $B\times V$.*)

We apply this lemma with V as the set of $(\xi,\mu)\in\Lambda=\mathcal{R}(P_+)\times\mathbf{R}$ satisfying (25) and $|\mu|<\mu_0$ and with $T(x,\xi,\mu)$ as defined in (29). In view of (28), we can take $\sigma=K(\gamma)L$. This means $x(\xi,\mu)$ defined by $x(\xi,\mu)(t)=x(t,\xi,\mu)$, which maps the (ξ,μ) space into the Banach space E is C^r.

In particular, since $x(\xi,\mu)$ is continuous as a function into E, it follows that $x(t,\xi,\mu)$ is continuous in (ξ,μ) uniformly with respect to t. Then since $x(t,\xi,\mu)$ is continuous in t for each fixed (ξ,μ), it follows that $x(t,\xi,\mu)$ is continuous in (t,ξ,μ).

Next, if $r\geqslant 1$, we know that

$$\sup_{t\geqslant 0}\mathrm{e}^{\gamma t}\big|x(t,\xi+\xi_1,\mu)-x(t,\xi,\mu)-\big(x_\xi(\xi,\mu)\xi_1\big)(t)\big|=\mathrm{o}\big(|\xi_1|\big)$$

as $\xi_1 \to 0$, where as follows from Lemma 3.8,

$$x_\xi(\xi, \mu) = T_x\big(x(\xi, \mu), \xi, \mu\big)x_\xi(\xi, \mu) + T_\xi\big(x(\xi, \mu), \xi, \mu\big)$$

so that, if we write $(x_\xi(\xi, \mu)\xi_1)(t) = Z(t)\xi_1$, we see that

$$\begin{aligned} Z(t)\xi_1 = X(t)\xi_1 &+ \int_0^t X(t)P_+X^{-1}(s)h_x\big(s, x(s, \xi, \mu), \mu\big)Z(s)\xi_1 \, \mathrm{d}s \\ &- \int_t^\infty X(t)(I - P_+)X^{-1}(s)h_x\big(s, x(s, \xi, \mu), \mu\big)Z(s)\xi_1 \, \mathrm{d}s. \end{aligned}$$

This means that $x_\xi(t, \xi, \mu)$ exists and equals $Z(t)$ restricted to $\mathcal{R}(P_+)$, where $Z(t)$ is the unique solution of

$$\dot{Z} = \big[A(t) + h_x\big(t, x(t, \xi, \mu), \mu\big)\big]Z$$

such that $\|Z\|_\gamma < \infty$ and $P_+Z(0) = P_+$. Moreover, since $x_\xi(\xi, \mu)$ is a continuous function of (ξ, μ), it follows that $x_\xi(t, \xi, \mu)$ is a continuous function of (ξ, μ), uniformly with respect to t. Since $x_\xi(t, \xi, \mu)$ is continuous in t for each fixed (ξ, μ), it follows that $x_\xi(t, \xi, \mu)$ is continuous in (t, ξ, μ). Furthermore from the equation for $Z(t)$ above and Lemma 3.6, we have the estimate

$$\|Z\|_\gamma \leqslant K + K(\gamma)L\|Z\|_\gamma$$

so that for $t \geqslant 0$

$$\big|x_\xi(t, \xi, \mu)\big| \leqslant \big(1 - K(\gamma)L\big)^{-1}K\mathrm{e}^{-\gamma t}.$$

Also, since $h(t, 0, 0) = 0$, we see that

$$x_\xi(t, 0, 0) = X(t)P_+.$$

Similarly, we show that $x_\mu(t, \xi, \mu)$ is the unique solution of

$$\dot{x} = \big[A(t) + h_x\big(t, x(t, \xi, \mu), \mu\big)\big]x + h_\mu\big(t, x(t, \xi, \mu), \mu\big)$$

such that $\|x\|_\gamma < \infty$ and $P_+x(0) = 0$ and, if $r \geqslant 2$, that $x_{\xi\xi}(t, \xi, \mu)$ is the unique solution restricted to $\mathcal{R}(P_+) \times \mathcal{R}(P_+)$ of

$$\dot{Z} = \big[A(t) + h_x\big(t, x(t, \xi, \mu), \mu\big)\big]Z + h_{xx}\big(t, x(t, \xi, \mu), \mu\big)x_\xi(t, \xi, \mu)x_\xi(t, \xi, \mu)$$

such that $\|Z\|_\gamma < \infty$ and $P_+Z(0) = 0$. Also we get the required estimates from Lemma 3.6. Thus we complete the proof of Lemma 3.7. $\square$

We apply Lemma 3.7 to Eq. (22). Suppose Δ satisfies

$$K(\gamma)\delta(\Delta,0) = K\big[(\alpha-\gamma)^{-1} + 2\gamma^{-1}\big]\delta(\Delta,0) < 1.$$

Then we take μ_0 as the smallest μ such that

$$K(\gamma)\delta(\Delta,\mu) = 1$$

and set

$$\xi_0 = K^{-1}\big(1 - K(\gamma)L\big)\Delta.$$

Then, using Lemma 3.7, if $|\mu| < \mu_0$ and if $\xi \in \mathcal{R}(P_+)$ with $|\xi| < \xi_0$, we get a unique solution $z(t) = z(t,\xi,\mu)$ of (22) with $P_+z(0) = \xi$ and $\|z\|_\gamma \leqslant \Delta$. Then we define

$$x^+(t,\xi,\mu) = u(t,\mu) + z(t,\xi,\mu)$$

and we see that (iii) and hence Proposition 3.5 has been proved. $\square$

Similarly we find the local unstable manifold near $u(0,\mu)$ with asymptotic phase 0, that is, we find the solutions $x(t)$ with $x(0)$ near $u(0,\mu)$ such that $|x(t) - u(t,\mu)| \to 0$ as $t \to -\infty$. We state the proposition without proof as it is similar to that of Proposition 3.5.

PROPOSITION 3.9. *As in Proposition* 3.5, *let the function* $f(x,\mu)$ *be* C^{r+2} $(r \geqslant 0)$ *and suppose*

$$\dot{x} = f(x,0)$$

has a hyperbolic T*-periodic solution* $u_0(t)$. *Then* (i) *and* (ii) *hold as in Proposition* 3.5. *Next suppose* $0 < \gamma < \alpha$. *Then for* $\Delta > 0$ *sufficiently small, there are positive numbers* μ_0, η_0 *depending on* Δ *such that if* $|\mu| < \mu_0$ *and* $\eta \in \mathcal{R}(P_-)$ *satisfies* $|\eta| < \eta_0$, *there exists a unique solution* $x(t) = x^-(t,\eta,\mu)$ *of* (20) *such that*

$$\big|x(t) - u(t,\mu)\big|e^{-\gamma t} \leqslant \Delta \quad (t \leqslant 0), \qquad P_-\big[x(0) - u(0,\mu)\big] = \eta.$$

Moreover,

$$x^-(t,0,\mu) = u(t,\mu),$$

$x^-(t,\xi,\mu)$ *is* C^r *in its arguments with* $x^-_\eta(t,\eta,\mu)$ *as the unique solution restricted to* $\mathcal{R}(P_-)$ *of*

$$\dot{Z} = f_x\big(x^+(t,\eta,\mu),\mu\big)Z$$

such that $\sup_{t\leqslant 0} e^{-\gamma t}|Z(t)| < \infty$ *and* $P_-Z(0) = P_-$ *with, in particular,*

$$x^-_\eta(t,0,0) = X(t)P_-$$

and there is a constant N such that when $r \geqslant 1$,

$$\left|x_\eta^-(t,\eta,\mu)\right| \leqslant N\mathrm{e}^{\gamma t}, \quad \left|x_\mu^-(t,\xi,\mu) - u_\mu(t,\mu)\right| \leqslant N\mathrm{e}^{\gamma t} \quad (t \leqslant 0)$$

and when $r \geqslant 2$,

$$\left|x_{\eta\eta}^-(t,\eta,\mu)\right| \leqslant N\mathrm{e}^{\gamma t} \quad (t \leqslant 0).$$

REMARK 3.10. It is clear from the proof that when $f(x,\mu)$ does not depend on μ the relevant conclusions of Propositions 3.5 and 3.9 hold under the weaker assumption that $f(x)$ is C^r $(r \geqslant 1)$.

3.3. *Global stable and unstable manifolds and asymptotic phase*

Consider now the system

$$\dot{x} = f(x), \tag{30}$$

with flow ϕ^t, where f is C^r $(r \geqslant 1)$ with $|f(x)| \leqslant M_0$ and $|f'(x)| \leqslant M_1$ for all $x \in \mathbf{R}^n$. We suppose the system has a hyperbolic periodic orbit $u_0(t)$ with minimal period T.

From Proposition 3.5(i), we know that

$$\dot{x} = f'\big(u_0(t)\big)x$$

has a trichotomy with projections P_0, P_+, P_- and constants K, α, where α can be any positive number such that $|\lambda| > \mathrm{e}^{\alpha T}$ for all Floquet multipliers λ with $|\lambda| > 1$ and $|\lambda| < \mathrm{e}^{-\alpha T}$ for those with $|\lambda| < 1$.

Also, given γ such that $0 < \gamma < \alpha$, there exist $\Delta > 0$ and ξ_0 such that if $\xi \in \mathcal{R}(P_+)$ with $|\xi| < \xi_0$ then (30) has a unique solution $x(t) = x^+(t,\xi)$ such that

$$\left|x(t) - u_0(t)\right| \leqslant \Delta\mathrm{e}^{-\gamma t} \quad \text{for } t \geqslant 0, \qquad P_+\big[x(0) - u_0(0)\big] = \xi.$$

Moreover, in view of Remark 3.10, $x_\xi^+(t,\xi)$ exists, is continuous and is the unique solution restricted to $\mathcal{R}(P_+)$ of

$$\dot{Z} = f'\big(x^+(t,\xi)\big)Z \tag{31}$$

such that $\|Z\|_\gamma < \infty$ and $P_+Z(0) = P_+$.

Similarly, if $\eta \in \mathcal{R}(P_-)$ with $|\eta| < \eta_0$ then (30) has a unique solution $x(t) = x^-(t,\eta)$ such that

$$\left|x(t) - u_0(t)\right| \leqslant \Delta\mathrm{e}^{\gamma t} \quad \text{for } t \leqslant 0, \qquad P_-\big[x(0) - u_0(0)\big] = \eta.$$

Moreover $x_\eta^-(t,\eta)$ exists, is continuous and is the unique solution restricted to $\mathcal{R}(P_-)$ of

$$\dot{Z} = f'\bigl(x^-(t,\eta)\bigr)Z$$

such that $\sup_{t\leqslant 0}|Z(t)|\mathrm{e}^{-\gamma t} < \infty$ and $P_- Z(0) = P_-$.

Note that $x^+(t,\xi)$ (resp. $x_-(t,\eta)$) can be continued backwards (resp. forwards) so as to be defined for all t.

Now we define the global stable and unstable manifolds.

DEFINITION 3.11. The *global stable manifold* is defined as

$$W^s(u_0) = \bigl\{x \in \mathbf{R}^n\colon\ \operatorname{dist}\bigl(\phi^t(x), u_0(\cdot)\bigr) \to 0 \text{ as } t \to \infty\bigr\}$$

and the *global unstable manifold* as

$$W^u(u_0) = \bigl\{x \in \mathbf{R}^n\colon\ \operatorname{dist}\bigl(\phi^t(x), u_0(\cdot)\bigr) \to 0 \text{ as } t \to -\infty\bigr\},$$

where

$$\operatorname{dist}\bigl(y, u_0(\cdot)\bigr) = \min_{0\leqslant t\leqslant T}\bigl|y - u_0(t)\bigr|.$$

In the next proposition, we show that the global stable and unstable manifolds can be defined in terms of the functions $x^+(t,\xi)$ and $x^-(t,\eta)$.

PROPOSITION 3.12. *$x \in W^s(u_0)$ if and only if there exists γ^+ and ξ such that*

$$\phi^t(x) = x^+(t+\gamma^+, \xi)$$

for all t. This γ^+ has the property that $|\phi^t(x) - u_0(t+\gamma^+)| \to 0$ as $t\to\infty$ and is called the (forward) asymptotic phase of x. It is unique up to a multiple of T. It also follows that

$$W^s(u_0) = \bigl\{x^+(t,\xi)\colon\ -\infty < t < \infty, |\xi| < \xi_0\bigr\}$$

and is an immersed submanifold of $\mathbf{R}^n$. Moreover, if $x \in W^s(u_0)$, then

$$T_x W^s(u_0) = \Bigl\{\xi \in \mathbf{R}^n\colon\ \sup_{t\geqslant 0}\bigl|D\phi^t(x)\xi\bigr| < \infty\Bigr\}.$$

PROOF. Denote by C the hyperplane passing through $u_0(0)$ and orthogonal to $f(u_0(0))$. We choose an open neighbourhood N of $u_0(0)$ in C with the following properties:

(a) $|f(x) - f(u_0(0))| < |f(u_0(0))|$ if $x \in N$;

(b) the Poincaré map $P: N \to P(N) \subset C$ associated with the cross-section C as in Proposition 3.4 is a well-defined C^1 diffeomorphism such that

$$P(x) = \phi^{\tau(x)}(x) \quad \text{with } \bigl|\tau(x) - T\bigr| \leqslant T/2$$

and

$$\phi^t(x) \notin N \quad \text{if } 0 < t < \tau(x);$$

(c) there exists a constant L such that if $P^k(x) \in N$ for $k \geqslant 0$, then $|P^k(x) - u_0(0)| \leqslant L\mathrm{e}^{-k\gamma T}|x - u_0(0)|$ for $k \geqslant 0$.

(c) can be arranged by Proposition 1.8 in [19, p. 5], since the eigenvalues λ of $DP(0)$ satisfy $|\lambda| > \mathrm{e}^{\alpha T} > \mathrm{e}^{\gamma T}$ if $|\lambda| > 1$ and $|\lambda| < \mathrm{e}^{-\alpha T} < \mathrm{e}^{-\gamma T}$ if $|\lambda| < 1$. Later another condition will be added to N.

Now consider the intersections of the orbit $x(t) = \phi^t(x)$ with N. By (a) the tangent vector $\dot{x}(t) = f(x(t))$ is transverse to C when $x(t) \in N$ and so the set $\{t\colon\ x(t) \in N\}$ is isolated. It follows from (b) and the fact that

$$\operatorname{dist}\bigl(x(t), u_0(\cdot)\bigr) \to 0$$

as $t \to \infty$ that the set $\{t \geqslant 0\colon\ x(t) \in N\}$ is an increasing sequence $\{t_k\}_{k=0}^{\infty}$, where $t_k \to \infty$ as $k \to \infty$, and there is a positive integer k_0 such that

$$x(t_{k+1}) = P\bigl(x(t_k)\bigr)$$

for $k \geqslant k_0$. Then from (c) it follows that

$$\bigl|x(t_k) - u_0(0)\bigr| \leqslant L\mathrm{e}^{-\gamma(k-k_0)T}\bigl|x(t_{k_0}) - u_0(0)\bigr|$$

for $k \geqslant k_0$.

Now we define

$$s_k = t_k - kT$$

so that for $k \geqslant k_0$

$$\begin{aligned} |s_{k+1} - s_k| &= |t_{k+1} - t_k - T| \\ &= \bigl|\tau\bigl(x(t_k)\bigr) - \tau\bigl(u_0(0)\bigr)\bigr| \\ &\leqslant L_1 L\bigl|x(t_{k_0}) - u_0(0)\bigr|\mathrm{e}^{-\gamma(k-k_0)T}, \end{aligned}$$

where L_1 is a Lipschitz constant for $\tau(x)$ in N. It follows that s_k is a Cauchy sequence. Let the limit be $-\gamma_1$. Then if $k \geqslant k_0$

$$|s_k + \gamma_1| = \left|\sum_{\ell=k+1}^{\infty}(s_\ell - s_{\ell-1})\right| \leqslant L_2\bigl|x(t_{k_0}) - u_0(0)\bigr|\mathrm{e}^{-\gamma(k-k_0)T},$$

where

$$L_2 = L_1 L(1 - \mathrm{e}^{-\gamma T})^{-1}.$$

Now it follows from Gronwall's lemma that

$$\left|x(t)-u_0(t+\beta)\right| \leqslant \mathrm{e}^{M_1|t-s|}\left|x(s)-u_0(s+\beta)\right|$$

for all t, s and β. Then if $k \geqslant k_0$

$$\begin{aligned}
\left|x(kT-\gamma_1)-u_0(0)\right| &\leqslant \mathrm{e}^{M_1|s_k+\gamma_1|}\left|x(t_k)-u_0(s_k+\gamma_1)\right| \\
&\leqslant \mathrm{e}^{M_1|s_k+\gamma_1|}\left[\left|x(t_k)-u_0(0)\right|+\left|u_0(0)-u_0(s_k+\gamma_1)\right|\right] \\
&\leqslant \mathrm{e}^{M_1|s_k+\gamma_1|}\left[\left|x(t_k)-u_0(0)\right|+M_0|s_k+\gamma_1|\right] \\
&\leqslant \mathrm{e}^{M_1|s_k+\gamma_1|}[L+M_0L_2]\left|x(t_{k_0})-u_0(0)\right|\mathrm{e}^{-\gamma(k-k_0)T} \\
&\leqslant L_3\left|x(t_{k_0})-u_0(0)\right|\mathrm{e}^{-\gamma kT},
\end{aligned}$$

where

$$L_3=\mathrm{e}^{M_1L_2|x(t_{k_0})-u_0(0)|}[L+M_0L_2]\mathrm{e}^{\gamma k_0T}.$$

Suppose now $t \geqslant k_0T-\gamma_1$. Then there exists $k \geqslant k_0$ such that

$$kT-\gamma_1 \leqslant t < (k+1)T-\gamma_1.$$

Then

$$\begin{aligned}
\left|x(t)-u_0(t+\gamma_1)\right| &\leqslant \mathrm{e}^{M_1(t-kT+\gamma_1)}\left|x(kT-\gamma_1)-u_0(kT)\right| \\
&= \mathrm{e}^{M_1(t-kT+\gamma_1)}\left|x(kT-\gamma_1)-u_0(0)\right| \\
&\leqslant \mathrm{e}^{M_1T}L_3\left|x(t_{k_0})-u_0(0)\right|\mathrm{e}^{-\gamma kT} \\
&\leqslant \mathrm{e}^{M_1T}L_3\left|x(t_{k_0})-u_0(0)\right|\mathrm{e}^{-\gamma(t-T+\gamma_1)}
\end{aligned}$$

so that for $t \geqslant 0$

$$\begin{aligned}
\left|x(t+k_0T-\gamma_1)-u_0(t)\right| &= \left|x(t+k_0T-\gamma_1)-u_0(t+k_0T)\right| \\
&\leqslant \mathrm{e}^{M_1T}L_3\left|x(t_{k_0})-u_0(0)\right|\mathrm{e}^{-\gamma(t+(k_0-1)T)} \\
&= \mathrm{e}^{(M_1-\gamma(k_0-1))T}L_3\left|x(t_{k_0})-u_0(0)\right|\mathrm{e}^{-\gamma t},
\end{aligned}$$

where

$$\mathrm{e}^{(M_1-\gamma(k_0-1))T}L_3=\mathrm{e}^{M_1L_2|x(t_{k_0})-u_0(0)|}[L+M_0L_2]\mathrm{e}^{(M_1+\gamma)T}.$$

Then if N is also chosen so that for all $x \in N$

$$\mathrm{e}^{M_1L_2|x-u_0(0)|}[L+M_0L_2]\mathrm{e}^{(M_1+\gamma)T}\left|x-u_0(0)\right| < \min\{\Delta, K^{-1}\xi_0\},$$

it follows by uniqueness in Proposition 3.5 (see the remarks at the beginning of this subsection) that

$$x(t + k_0 T - \gamma_1) = x^+(t, \xi),$$

with

$$\xi = P_+\big[x(k_0 T - \gamma_1) - u_0(0)\big].$$

It follows that

$$x(t) = x^+(t + \gamma_1 - k_0 T, \xi)$$

for $t \geqslant k_0 T - \gamma_1$ and hence for all t. Taking $\gamma^+ = k_0 T - \gamma_1$, this proves the first part of the proposition.

Now we examine the extent to which the asymptotic phase is unique. Suppose x has asymptotic phases γ_1 and γ_2. Then

$$\big|u(t + \gamma_1) - u(t + \gamma_2)\big| \to 0 \quad \text{as } t \to \infty. \tag{32}$$

Let

$$\gamma_1 = \bar{\gamma}_1 + k_1 T, \qquad \gamma_2 = \bar{\gamma}_2 + k_2 T$$

where $0 \leqslant \bar{\gamma}_1, \bar{\gamma}_2 < T$ and k_1, k_2 are integers. Then it follows from (32) that

$$\big|u(\bar{\gamma}_1) - u(\bar{\gamma}_2)\big| = \big|u\big((k - k_1)T + \gamma_1\big) - u\big((k - k_1)T + \gamma_2\big)\big| \to 0 \quad \text{as } k \to \infty.$$

That is, $u(\bar{\gamma}_1) = u(\bar{\gamma}_2)$ and so $\bar{\gamma}_1 = \bar{\gamma}_2$. Hence if γ_1, γ_2 are two asymptotic phases, $\gamma_1 - \gamma_2$ is a multiple of T. Conversely, it is obvious that if γ^+ is an asymptotic phase, then so also is $\gamma^+ + kT$ for any integer k.

It is clear from the first part of the proposition that

$$W^s(u_0) = \big\{x^+(t, \xi)\colon -\infty < t < \infty, |\xi| < \xi_0\big\}.$$

Now we know that $x_t^+(t, \xi) = f(x(t, \xi))$ is a solution of (31) bounded on $\mathcal{R}_+$ whose norm is bounded below by a positive number as $t \to \infty$ and $x(t) = x_\xi^+(t, \xi)\xi_1$ for $\xi_1 \in \mathcal{R}(P_+)$ is a solution $\to 0$ as $t \to \infty$ such that $P_+ x(0) = \xi_1$. It follows that the matrix

$$\big[\, x_t^+(t, \xi) \quad x_\xi^+(t, \xi) \,\big]$$

is of full rank so that $(t, \xi) \mapsto x^+(t, \xi)$ is an immersion.

If $x = x^+(t, \xi) \in W^s(u_0)$, then we define the tangent space

$$T_x W^s(u_0) = \text{the range of } \big[\, x_t^+(t, \xi) \quad x_\xi^+(t, \xi) \,\big].$$

We show this is independent of (t,ξ) such that $x = x^+(t,\xi)$. Suppose

$$x = x^+(t_1,\xi_1) = x^+(t_2,\xi_2)$$

with $t_1 \geqslant t_2$. Then, for all t,

$$\phi^t(x) = x^+(t+t_1,\xi_1) = x^+(t+t_2,\xi_2) \tag{33}$$

so that t_1 and t_2 are both asymptotic phases for x. It follows that $t_1 = t_2 + kT$ for some nonnegative integer $k \geqslant 0$ and taking $t = -t_2$ in (33) we find that

$$x^+(0,\xi_2) = x^+(kT,\xi_1).$$

It follows that

$$\left|P_+\left[x^+(kT,\xi_1) - u_0(0)\right]\right| = \left|P_+\left[x^+(0,\xi_2) - u_0(0)\right]\right| = |\xi_2| < \xi_0$$

so that

$$\left|P_+\left[x^+(kT,\xi) - u_0(0)\right]\right| < \xi_0$$

if ξ is sufficiently close to ξ_1. Thus, by uniqueness in Proposition 3.5

$$x^+(t+kT,\xi) = x^+\big(t,\psi(\xi)\big)$$

for ξ close to ξ_1, where

$$\psi(\xi) = P_+\left[x^+(kT,\xi) - u_0(0)\right].$$

It follows that

$$x_t^+(t_1,\xi_1) = x_t^+(t_2+kT,\xi_1) = x_t^+\big(t_2,\psi(\xi_1)\big) = x_t^+(t_2,\xi_2)$$

and

$$\begin{aligned} x_\xi^+(t_1,\xi_1) &= x_\xi^+(t_2+kT,\xi_1) \\ &= \frac{\partial}{\partial\xi} x^+\big(t_2,\psi(\xi)\big)\Big|_{\xi=\xi_1} \\ &= x_\xi^+\big(t_2,\psi(\xi_1)\big)\psi'(\xi_1) \\ &= x_\xi^+(t_2,\xi_2)\psi'(\xi_1), \end{aligned}$$

which show that the range of the matrix $[\,x_t^+(t_1,\xi_1) \quad x_\xi^+(t_1,\xi_1)\,]$ is contained in and hence coincides with the range of $[\,x_t^+(t_2,\xi_2) \quad x_\xi^+(t_2,\xi_2)\,]$. So $T_x W^s(u_0)$ is well-defined.

Now, as we mentioned earlier, $Z(t) = x_\xi^+(t, \xi_1)$ is the unique solution restricted to $\mathcal{R}(P_+)$ of (31) with $\xi = \xi_1$ such that $\|Z\|_\gamma < \infty$ and $P_+ Z(0) = P_+$. However, since $|f'(x^+(t,\xi_1)) - f'(u_0(t))| \to 0$ as $t \to \infty$, it follows from Remarks 2.7 and 2.3 that

$$\dot{y} = f'\big(x^+(t, \xi_1)\big) y \tag{34}$$

has a trichotomy on $[t_1, \infty)$ with projections having the same rank as those for $\dot{x} = f'(u_0(t))x$. So this equation has a subspace W_t of solutions bounded on $[t_1, \infty)$ with dimension equal to $\operatorname{rank}(P_0 + P_+)$. It follows that

$$\operatorname{span}\big\{f(x^+(t,\xi_1)\big\} \oplus \mathcal{R}\big(x_\xi^+(t, \xi_1)\big),$$

which has dimension equal to $\operatorname{rank}(P_0 + P_+)$, coincides with W_t. Next note that (34) can be written as $\dot{y} = f'(\phi^{t-t_1}(x))y$. So, by a time shift, it follows that $\operatorname{span}\{f(x^+(t+t_1,\xi_1)\} \oplus \mathcal{R}(x_\xi^+(t+t_1, \xi_1))$ is the subspace of solutions of

$$\dot{y} = f'\big(\phi^t(x)\big) y \tag{35}$$

which are bounded on $[0, \infty)$. Hence

$$T_x W^s(u_0) = \operatorname{span}\big\{f\big(x^+(t_1,\xi_1)\big)\big\} \oplus \mathcal{R}\big(x_\xi^+(t_1, \xi_1)\big)$$

consists of the values at $t = 0$ of the subspace of solutions of (35) which are bounded on $[0, \infty)$, that is,

$$\Big\{\xi \in \mathbf{R}^n \colon \sup_{t \geqslant 0} \big|D\phi^t(x)\xi\big| < \infty\Big\}.$$

This completes the proof of the proposition. $\square$

Similarly, we can prove the following proposition.

PROPOSITION 3.13. *$x \in W^u(u_0)$ if and only if there exists γ^- and η such that*

$$\phi^t(x) = x^-(t + \gamma^-, \eta)$$

for all t. This γ^- has the property that $|\phi^t(x) - u_0(t+\gamma^-)| \to 0$ as $t \to -\infty$ and is called the (*backward*) *asymptotic phase of x. It is unique up to a multiple of T. It also follows that*

$$W^u(u_0) = \big\{x^-(t, \eta) \colon -\infty < t < \infty, |\eta| < \eta_0\big\}$$

and is an immersed submanifold of $\mathbf{R}^n$. Moreover, if $x \in W^u(u_0)$, then

$$T_x W^u(u_0) = \Big\{\eta \in \mathbf{R}^n \colon \sup_{t \leqslant 0} \big|D\phi^t(x)\eta\big| < \infty\Big\}.$$

Finally we define the local stable and unstable fibres.

DEFINITION 3.14. Corresponding to γ^+ in $0 \leqslant \gamma^+ < T$, we define the *local stable fibre*:

$$W^{s,\gamma^+}_{\mathrm{loc}}(u_0) = \left\{x^+(\gamma^+, \xi)\colon\ \xi \in \mathcal{R}(P_+),\ |\xi| < \xi_0\right\}$$

and we define the *global stable fibre* $W^{s,\gamma^+}(u_0)$ as

$$\bigcup_{m\in Z} \phi^{mT}\left(W^{s,\gamma^+}_{\mathrm{loc}}(u_0)\right) = \left\{x^+(\gamma^+ + mT, \xi)\colon\ \xi \in \mathcal{R}(P_+), |\xi| < \xi_0, m \in Z\right\}.$$

It is easy to see that $x \in W^{s,\gamma^+}(u_0)$ if and only if $|\phi^t(x) - u_0(t + \gamma^+)| \to 0$ as $t \to \infty$ and that

$$W^s(u_0) = \bigcup_{0\leqslant\gamma^+<T} W^{s,\gamma^+}(u_0).$$

DEFINITION 3.15. Corresponding to γ^- in $0 \leqslant \gamma^- < T$, we define the *local unstable fibre*:

$$W^{u,\gamma^-}_{\mathrm{loc}}(u_0) = \left\{x^-(\gamma^-, \eta)\colon\ \eta \in \mathcal{R}(P_-), |\eta| < \eta_0\right\}$$

and we define the *global unstable fibre* $W^{u,\gamma^-}(u_0)$ as

$$\bigcup_{m\in Z} \phi^{mT}\left(W^{u,\gamma^-}_{\mathrm{loc}}(u_0)\right) = \left\{x^-(\gamma^- + mT, \eta)\colon\ \eta \in \mathcal{R}(P_-), |\eta| < \eta_0, m \in Z\right\}.$$

It is easy to see that $x \in W^{u,\gamma^-}(u_0)$ if and only if $|\phi^t(x) - u_0(t + \gamma^-)| \to 0$ as $t \to -\infty$ and that

$$W^u(u_0) = \bigcup_{0\leqslant\gamma^-<T} W^{u,\gamma^-}(u_0).$$

4. Homoclinic orbits

In this section we study properties of periodic-to-periodic homoclinic orbits. First we define a Poincaré map associated with such an orbit. Then we give equivalent conditions for transversality of the periodic-to-periodic homoclinic orbits including its equivalence with the transversality of the corresponding homoclinic orbit of the Poincaré map. Then Smale's theorem enables us to deduce the existence of chaotic behaviour in the neighbourhood of a transversal periodic-to-periodic homoclinic orbit.

So let $u_0(t)$ be a hyperbolic T-periodic orbit of the C^1 system

$$\dot{x} = f(x), \tag{36}$$

where, as usual, we assume that $|f(x)| \leqslant M_0$ and $|f'(x)| \leqslant M_1$. We make the following definition.

DEFINITION 4.1. If $u_0(t)$ is a hyperbolic T-periodic orbit of (36), then a solution $p(t)$ is said to be a *homoclinic orbit* if dist $(p(t), u_0(\cdot)) \to 0$ as $|t| \to \infty$ or, equivalently, $p(t) \in W^s(u_0) \cap W^u(u_0)$.

4.1. *Poincaré map associated with a homoclinic orbit*

Consider system (36) with hyperbolic T-periodic orbit $u_0(t)$ and associated homoclinic orbit $p(t)$. Denote by C the hyperplane passing through $u_0(0)$ and orthogonal to $f(u_0(0))$. In a neighbourhood of $u_0(0)$ in C, we will define a map F which has $u_0(0)$ as a hyperbolic fixed point and an associated homoclinic orbit which consists of points on the solution $p(t)$ which lie on C.

First we choose an open neighbourhood N of $u_0(0)$ in C with the following properties:

(a) $|f(y) - f(u_0(0))| \leqslant \frac{1}{2}|f(u_0(0))|$ if $y \in N$;

(b) the Poincaré map $P : N \mapsto C$ (see Proposition 3.4) has the properties that for $x \in N$

$$\phi^t(x) \notin N \quad \text{if } 0 < t < \tau_0(x)$$

and

$$\left|\tau_0(x) - T\right| \leqslant T/2,$$

where $P(x) = \phi^{\tau_0(x)}(x)$;

(c) $P^k(x) \in N$ for all integers k if and only if $x = u_0(0)$.

That N can be chosen so that (c) is satisfied follows from the fact that $u_0(0)$ is an isolated invariant set for P (see [19, p. 44]).

Consider the intersections of the homoclinic orbit $p(t)$ with N. Similarly to the beginning of the proof of Proposition 3.12, it follows that the set $\{t \in \mathbf{R}\colon\ p(t) \in N\}$ is an increasing sequence $\{t_k\}_{k=-\infty}^{\infty}$, where $t_k \to \pm\infty$ as $k \to \pm\infty$ and

$$p(t_{k+1}) = P\big(p(t_k)\big)$$

if $|k|$ is sufficiently large. It is also clear that $p(t_k) \to u_0(0)$ as $k \to \pm\infty$. For short, we write

$$p_k = p(t_k).$$

Because of property (c) of P, there exists an integer K^- such that

$$p_{k+1} = P(p_k) \quad \text{if } k < K^-$$

but

$$P(p_{K^-}) \neq p_{K^-+1}.$$

We set

$$v_1 = P(p_{K^-}).$$

It follows from (b) that $\phi^t(p_{K^-}) \notin N$ if $0 < t < \tau_0(p_{K^-})$ and so if v_1 were in N, we would have $v_1 = p_{K^-+1}$. So $v_1 \notin N$.

Next we take K^+ to be an integer greater than K^- such that $p_{k+1} = P(p_k)$ if $k \geqslant K^+$. Then we set

$$v_2 = p_{K^+} = \phi^{T_1}(v_1).$$

Now consider the subset

$$I_0 = \{\ldots, p_{K^- -1}, p_{K^-}\} \cup \{u(0)\} \cup \{p_{K^+ +1}, p_{K^+ +2}, \ldots\}$$

of N and choose an open neighbourhood $V_2 \subset N$ of $v_2 = p_{K^+}$ such that the closure of V_2 does not intersect $I_0 \cup \{v_1\}$. Then we can choose an open neighbourhood $V_0 \subset N$ of I_0 such that v_1 is not in the closure of V_0 and $P(V_0)$ does not intersect V_2.

Next we let $V_1 \subset C$ be an open ball with centre v_1 such that $V_0 \cap V_1$ is empty and the Poincaré map $Q = P(v_1, 0, T_1, v, v)$ with $v = u_0'(0)/|u_0'(0)|$ is defined on V_1, has range in V_2 and satisfies $|\tau_1(x) - T_1| \leqslant T_1/2$, where $Q(x) = \phi^{\tau_1(x)}(x)$. Note that $Q(v_1) = \phi^{T_1}(v_1) = v_2 = p_{K^+}$.

Now we define a diffeomorphism $F : V_0 \cup V_1 \mapsto C$ by

$$F(x) = \begin{cases} P(x) & \text{if } x \in V_0, \\ Q(x) & \text{if } x \in V_1. \end{cases}$$

F is well-defined since $V_0 \cap V_1$ is empty and is one to one since

$$P(V_0) \cap Q(V_1) \subset P(V_0) \cap V_2,$$

which is empty, and it is a diffeomorphism on to an open set since P and Q are diffeomorphisms. Since $u_0(0)$ is a hyperbolic fixed point for P, it is also a hyperbolic fixed point for F. Next we see that

$$\{\ldots, p_{K^- -1}, p_{K^-}, v_1 = P(p_{K^-}), v_2 = Q(v_1) = p_{K^+}, p_{K^+ +1}, p_{K^+ +2}, \ldots\}$$

is a homoclinic orbit to $u_0(0)$.

4.2. *Transversal periodic-to-periodic homoclinic orbits and hyperbolicity*

We now define the notion of transversality for a periodic-to-periodic homoclinic orbit and show it is equivalent to hyperbolicity and also to transversality of the corresponding homoclinic orbit of the Poincaré map defined in the previous subsection.

DEFINITION 4.2. A homoclinic orbit $p(t)$ of (36) is said to be *transversal* if the tangent spaces $T_{p(t)}W^s(u_0)$ and $T_{p(t)}W^u(u_0)$ intersect in the subspace spanned by $p'(t) = f(p(t))$.

DEFINITION 4.3. A bounded solution $p(t)$ of (36) is said to be *hyperbolic* if $|f(p(t))|$ is bounded below by a positive number and there is a splitting

$$\mathbf{R}^n = \operatorname{span}\{f\big(p(t)\big)\} \oplus E^s_{p(t)} \oplus E^u_{p(t)} \quad \text{for } t \in \mathbf{R}$$

with the invariance property

$$D\phi^\tau\big(p(t)\big)(E^s_{p(t)}) = E^s_{p(t+\tau)}, \quad D\phi^\tau\big(p(t)\big)(E^u_{p(t)}) = E^u_{p(t+\tau)}, \quad \text{for } \tau, t \in \mathbf{R},$$

and there are positive constants K and α such that for all t and $\tau \geqslant 0$

$$\big|D\phi^\tau\big(p(t)\big)\xi\big| \leqslant K\mathrm{e}^{-\alpha\tau}|\xi|, \quad \xi \in E^s_{p(t)},$$

and

$$\big|D\phi^{-\tau}\big(p(t)\big)\xi\big| \leqslant K\mathrm{e}^{-\alpha\tau}|\xi|, \quad \xi \in E^u_{p(t)}.$$

Note that a periodic solution is hyperbolic in the sense of Definition 3.3 if and only if it is hyperbolic in the sense of Definition 4.3 (see, for example, [19]). Now we give equivalent conditions for transversality of a periodic-to-periodic homoclinic orbit.

PROPOSITION 4.4. *Suppose $u_0(t)$ is a hyperbolic T-periodic orbit of* (36) *and $p(t)$ is an associated homoclinic orbit. Then the following conditions are equivalent*:

(i) *$p(t)$ is transversal*;

(ii) *$p(t)$ is hyperbolic*;

(iii) *$p'(t)$ is, up to a scalar multiple, the unique bounded solution of*

$$\dot{x} = f'\big(p(t)\big)x; \tag{37}$$

(iv) *the variational system* (37) *has a trichotomy on* $\mathbf{R}$;

(v) *the corresponding homoclinic orbit of the associated Poincaré map F is transversal*.

PROOF. Note that by Proposition 3.12, $T_{p(t)}W^s(u_0)$ is the subspace of solutions of (37) bounded on $[0, \infty)$ and by Proposition 3.13, $T_{p(t)}W^u(u_0)$ is the subspace of solutions of

(37) bounded on $(-\infty, 0]$. Hence their intersection is the subspace of solutions of (37) bounded on $(-\infty, \infty)$. It follows at once that (i) and (iii) are equivalent.

By Remark 2.4, (iv) implies (iii). To prove that (iii) implies (iv), let γ^+ and γ^- be the forward and backward asymptotic phases of $p(0)$ so that $|p(t) - u_0(t+\gamma^+)| \to 0$ as $t \to \infty$ and $|p(t) - u_0(t+\gamma^-)| \to 0$ as $t \to -\infty$. Then since $\dot{x} = f'(u_0(t))x$ has a trichotomy on $\mathbf{R}$, it follows from Remarks 2.7 and 2.3 that (37) has trichotomies on both half-lines with projections of the same rank. By Remark 2.4 this together with (iii) implies (iv). So (iii) and (iv) are equivalent.

To show that (iv) and (v) are equivalent, we apply Proposition 2.9 to (37). Note that $|f'(p(t))| \leqslant M_1$ and $p'(t) = f(p(t))$ is a bounded solution of (37) such that $\inf|p'(t)| > 0$ since $\operatorname{dist}(p(t), u_0(\cdot)) \to 0$ as $|t| \to \infty$. Now, as in 4.1, the corresponding homoclinic orbit of the Poincaré map F can be written as a sequence $p(s_k)$ where $p(s_{-1}) = p_{K^-}$, $p(s_0) = v_1 = P(p_{K^-})$, $p(s_1) = v_2 = p_{K^+}$, etc., where we see from the properties of $\tau_0(x)$ and $\tau_1(x)$ that $T/2 \leqslant s_{k+1} - s_k \leqslant 3T/2$ if $k \neq 0$ and $s_1 - s_0 > 0$. So

$$0 < \inf(s_{k+1} - s_k) \leqslant \sup(s_{k+1} - s_k) < \infty.$$

We define the projection Q_k to be the projection with range W, the orthogonal complement of $\dot{u}_0(0) = f(u_0(0))$, and nullspace spanned by $f(p(s_k))$. Since $f(p(s_k)) \to f(u_0(0))$ as $|k| \to \infty$, Q_k converges as $|k| \to \infty$ to an orthogonal projection and so $|Q_k|$ is bounded. Next we see from the definition of F and (19) that

$$DF\big(p(s_k)\big) = Q_k D\phi^{s_{k+1}-s_k}\big(p(s_k)\big) = Q_k X(s_{k+1}) X^{-1}(s_k),$$

where $X(t) = D\phi^t(p(0))$ is the fundamental matrix of (37) with $X(0) = I$.

Then it follows from Proposition 2.9 that (37) has a trichotomy on $\mathbf{R}$ if and only if the difference equation

$$u_{k+1} = DF\big(p(s_k)\big)u_k, \quad u_k \in W,$$

has an exponential dichotomy on Z. However, this last statement is equivalent to the statement that $\{p(s_k)\}_{k=-\infty}^{\infty}$ is a transversal homoclinic orbit for F (see [19]). So (iv) and (v) are equivalent.

If (iv) holds, then there exist supplementary projections P_0, P_+, P_- with rank $P_0 = 1$ and positive constants K, α such that

$$\big|X(t)P_0X^{-1}(s)\big| \leqslant K \quad \text{for all } t, s,$$
$$\big|X(t)P_+X^{-1}(s)\big| \leqslant K\mathrm{e}^{-\alpha(t-s)} \quad \text{for } t \geqslant s,$$
$$\big|X(t)P_-X^{-1}(s)\big| \leqslant K\mathrm{e}^{-\alpha(s-t)} \quad \text{for } t \leqslant s,$$

where $X(t) = D\phi^t(p(0))$. The last two inequalities can be rewritten as

$$\big|D\phi^{\tau}\big(p(s)\big)P_+(s)\big| \leqslant K\mathrm{e}^{-\alpha\tau} \quad \text{for } \tau \geqslant 0,$$
$$\big|D\phi^{-\tau}\big(p(s)\big)P_-(s)\big| \leqslant K\mathrm{e}^{-\alpha\tau} \quad \text{for } \tau \geqslant 0,$$

where $P_\sigma(t) = X(t)P_\sigma X^{-1}(t)$. Note also since $p'(t)$ is a bounded solution of (37), it follows that the range of $P_0(t)$ is spanned by $p'(t) = f(p(t))$. Note also, as observed above, that $\inf|p'(t)| > 0$. Hence if we define $E^s_{p(t)} = \mathcal{R}(P_+(t))$ and $E^u_{p(t)} = \mathcal{R}(P_-(t))$, we see that $p(t)$ is hyperbolic. Thus (iv) implies (ii).

If (ii) holds, then any solution $x(t)$ of (37) can be written as $x(t) = \lambda p'(t) + x_+(t) + x_-(t)$, where $x_\sigma(t) \in E^\sigma_{p(t)}$. If $x_+(t) \neq 0$, then $|x_+(t)| \to \infty$ as $t \to -\infty$ whereas $p'(t) + x_-(t)$ is bounded so that $|x(t)| \to \infty$. Hence, if $x(t)$ is bounded, we must have $x_+(t) = 0$. Similarly, $x_-(t) = 0$. So $x(t)$ is a multiple of $p'(t)$. Thus (ii) implies (iii) and the proof is completed. □

4.3. *Chaos near a transversal periodic-to-periodic homoclinic orbit*

In this subsection we examine the chaotic behaviour in the neighbourhood of a transversal periodic-to-periodic homoclinic orbit.

Consider system (36) with hyperbolic T-periodic orbit $u_0(t)$ and associated transversal homoclinic orbit $p(t)$. In the previous two subsections, we defined an associated Poincaré map F and showed that the sequence of points

$$\left\{\ldots, p_{K^- -1}, p_{K^-}, v_1 = P(p_{K^-}), v_2 = Q(v_1) = p_{K^+}, p_{K^+ +1}, p_{K^+ +2}, \ldots\right\}$$

is a transversal homoclinic orbit bi-asymptotic to the hyperbolic fixed point $u_0(0)$.

Now we prove the following

PROPOSITION 4.5. *Suppose system* (36) *has a hyperbolic T-periodic orbit $u_0(t)$ with associated transversal homoclinic orbit $p(t)$. Then there exists a compact invariant set $\mathcal{S}$ such that*

(a) *the periodic points are dense*;
(b) *there is a dense orbit*;
(c) *there exists $\delta > 0$ such that if x is in $\mathcal{S}$ there exists y in $\mathcal{S}$ arbitrarily close to x such that for any continuous nonnegative function $\alpha(t)$ with $\alpha(0) = 0$ there exists $t \geqslant 0$ with $|\phi^{\alpha(t)}(y) - \phi^t(x)| > \delta$.*

REMARK 4.6. (c) is sensitive dependence on initial conditions in the flow context. To the author's knowledge, there is no definition of sensitive dependence directly for flows in the literature since usually chaos for flows is described in terms of a return map.

PROOF OF PROPOSITION 4.5. As shown, for example, in [19], there is a compact invariant subset $S \subset V_0 \cup V_1$ on which the Poincaré map F is chaotic in the sense of Devaney (see [8]), that is

(i) the periodic points are dense;
(ii) there is a dense orbit;
(iii) there exists $\delta > 0$ such that if $x \in S$ there is $y \in S$ arbitrarily close to x such that $|F^k(x) - F^k(y)| > \delta$ for some $k \geqslant 0$.

Then we define

$$\mathcal{S}=\left\{\phi^t(x)\colon\ x\in S, -\infty<t<\infty\right\}.$$

Clearly this set is invariant for system (36). To prove the compactness, first note that if $x\in S$, then there exist times $t_k(x)$ such that $\phi^{t_k}(x)=F^k(x)$ for all integers k. Now, using the notation of Section 4.1, $t_{k+1}(x)-t_k(x)=\tau_0(F^k(x))$ if $F^k(x)\in V_0$ or $\tau_1(F^k(x))$ if $F^k(x)\in V_1$ and so

$$c_1\leqslant t_{k+1}(x)-t_k(x)\leqslant c_2,$$

where by properties of τ_0 and τ_1

$$c_1=\frac{1}{2}\min\{T,T_1\},\qquad c_2=\frac{3}{2}\max\{T,T_1\}.$$

Now let $\phi^{s_m}(x_m)$ be a sequence in $\mathcal{S}$. Then there exists k_m such that $t_{k_m}(x_m)\leqslant s_m<t_{k_m+1}(x_m)$ and so it follows that $\phi^{s_m}(x_m)=\phi^{\sigma_m}(z_m)$, where $0\leqslant\sigma_m=s_m-t_{k_m}(x_m)\leqslant c_2$ and $z_m=F^{k_m}(x_m)\in S$. Then there are subsequences $\sigma_{j_m}\to\sigma$ and $z_{j_m}\to z\in S$. By continuity of $\phi^t(x)$, it follows that $\phi^{s_{j_m}}(x_{j_m})\to\phi^{\sigma}(z)\in\mathcal{S}$. Thus $\mathcal{S}$ is compact.

To prove (a) and (b), let $x=\phi^t(y)$ be in $\mathcal{S}$, where $y\in S$. Then there exists $t_k=t_k(y)$ such that $\phi^{t_k}(y)=F^k(y)$ and $t_k\leqslant t<t_{k+1}$. So $x=\phi^{t-t_k}(F^k(y))$. Given $\varepsilon>0$, choose $\delta>0$ so that $|z-F^k(y)|<\delta$ implies $|\phi^t(z)-\phi^t(F^k(y))|<\varepsilon$ for $0\leqslant t\leqslant c_2$. Then we see that $|x-\phi^{t-t_k}(z)|<\varepsilon$. Since we can choose $z\in S$ to be periodic for F so that $\phi^{t-t_k}(z)$ is periodic for the flow, it follows that the periodic points are dense in $\mathcal{S}$. Since we can choose $z\in S$ to be on the orbit of F which is dense in S, it follows that the orbit of z in the flow is dense in $\mathcal{S}$.

To prove (c), suppose $x\in S$. Then we know there exists $y\in S$ arbitrarily close to x such that $|F^k(y)-F^k(x)|>\delta$ for some $k\geqslant 0$. In the following δ_1 will be a positive number which will be determined as we proceed. Also we write

$$\Delta=\inf_{x\in\mathcal{S}}\left|f(x)\right|>0.$$

Let $\alpha(t)$ be a continuous nonnegative function with $\alpha(0)=0$. Suppose

$$\left|\phi^{\alpha(t)}(y)-\phi^t(x)\right|\leqslant\delta_1$$

for all $t\geqslant 0$. Now we apply Lemma 3.1 with $\phi^{\alpha(t)}(y)$ as z, $\phi^t(x)$ as x, $v=f(\phi^t(x))/|f(\phi^t(x))|$ and $T=0$ to deduce that if

$$\delta_1<\min\left\{\frac{\Delta}{4M_1},\frac{\Delta^2}{8M_0M_1}\right\},$$

there exists $\tau=\tau(t)$ with $|\tau|\leqslant 2\delta_1/\Delta$ such that

$$\left\langle\phi^{\alpha(t)+\tau}(y)-\phi^t(x),\,f\left(\phi^t(x)\right)\right\rangle=0.$$

Then if we define

$$\beta(t) = \alpha(t) + \tau(t),$$

we see that

$$\left\langle \phi^{\beta(t)}(y) - \phi^t(x), f\left(\phi^t(x)\right)\right\rangle = 0$$

and

$$\begin{aligned}\left|\phi^{\beta(t)}(y) - \phi^t(x)\right| &\leqslant \left|\phi^{\beta(t)}(y) - \phi^{\alpha(t)}(y)\right| + \left|\phi^{\alpha(t)}(y) - \phi^t(x)\right| \\ &\leqslant M_0\left|\tau(t)\right| + \delta_1 \leqslant \delta_2,\end{aligned} \tag{38}$$

where

$$\delta_2 = (2M_0/\Delta + 1)\delta_1.$$

Note it also follows from Remark 3.2 that $\tau(t)$ and hence $\beta(t)$ is continuous. Now we show that $\beta(t)$ is, in fact, differentiable provided that

$$M_1\delta_2 < \Delta.$$

To this end, consider the continuously differentiable real function

$$g(\beta, t) = \left\langle \phi^{\beta}(y) - \phi^t(x), f\left(\phi^t(x)\right)\right\rangle. \tag{39}$$

We estimate

$$\begin{aligned}g_\beta\left(\beta(t), t\right) &= \left\langle f\left(\phi^{\beta(t)}(y)\right), f\left(\phi^t(x)\right)\right\rangle \\ &\geqslant \left|f\left(\phi^t(x)\right)\right|^2 - \left|f\left(\phi^{\beta(t)}(y)\right) - f\left(\phi^t(x)\right)\right|\left|f\left(\phi^t(x)\right)\right| \\ &\geqslant \left|f\left(\phi^t(x)\right)\right|[\Delta - M_1\delta_2] \\ &> 0.\end{aligned}$$

Then it follows from the continuity of $\beta(t)$ and the implicit function theorem that $\beta(t)$ is continuously differentiable. Moreover, if we differentiate the equation $g(\beta(t), t) = 0$ with respect to t, we obtain the equation

$$\begin{aligned}&\left\langle f\left(\phi^{\beta(t)}(y)\right), f\left(\phi^t(x)\right)\right\rangle \beta'(t) \\ &\quad - \left|f\left(\phi^t(x)\right)\right|^2 + \left\langle \phi^{\beta(t)}(y) - \phi^t(x), f'\left(\phi^t(x)\right) f\left(\phi^t(x)\right)\right\rangle = 0\end{aligned}$$

so that $\beta'(t)$ is given by

$$\begin{aligned}&1 - \left[\left\langle \phi^{\beta(t)}(y) - \phi^t(x), f'\left(\phi^t(x)\right) f\left(\phi^t(x)\right)\right\rangle \right. \\ &\quad \left. + \left\langle f\left(\phi^{\beta(t)}(y)\right) - f\left(\phi^t(x)\right), f\left(\phi^t(x)\right)\right\rangle\right] \Big/ \left\langle f\left(\phi^{\beta(t)}(y)\right), f\left(\phi^t(x)\right)\right\rangle\end{aligned}$$

from which we see, using the estimate after (39), that

$$\left|\beta'(t)-1\right| \leqslant 2(\Delta - M_1\delta_2)^{-1}M_1\delta_2. \tag{40}$$

Now there is a sequence $t_k = t_k(x)$ such that $\phi^{t_k}(x) = F^k(x)$ for $k \geqslant 0$. We apply Lemma 3.1 with $\phi^{\beta(t_k)}(y)$ as z, $\phi^{t_k}(x)$ as x, $T = 0$ and $v = f(u_0(0))/|f(u_0(0))|$. Note that by (38), $|\phi^{\beta(t_k)}(y) - \phi^{t_k}(x)| \leqslant \delta_2$ and that since $\phi^{t_k}(x) \in N$,

$$\left|\left\langle f\left(\phi^{t_k}(x)\right), f\left(u_0(0)\right)\right\rangle\right| \geqslant \frac{1}{2}\left|f\left(u_0(0)\right)\right|^2$$

so that

$$\left|\left\langle f\left(\phi^{t_k}(x)\right), v\right\rangle\right| \geqslant \frac{1}{2}\Delta.$$

Then if

$$8M_1\delta_2 < \Delta, \qquad 32M_0M_1\delta_2 < \Delta^2,$$

it follows from Lemma 3.1 that there exists a h_k satisfying $|h_k| \leqslant 4\Delta^{-1}\delta_2$ which is unique in $|h_k| \leqslant \Delta/(8M_0M_1)$ such that

$$\left\langle \phi^{\beta(t_k)+h_k}(y) - \phi^{t_k}(x), f\left(u_0(0)\right)\right\rangle = 0.$$

Now we show that for $k \geqslant 0$, $\phi^{\beta(t_k)+h_k}(y) \in S$ and

$$\phi^{\beta(t_k)+h_k}(y) = F^k(y). \tag{41}$$

Note when $k = 0$ that the equation

$$\left\langle \phi^{\beta(t_0)+h}(y) - \phi^{t_0}(x), f\left(u_0(0)\right)\right\rangle = 0$$

has the solution $h = -\beta(0)$ since $t_0 = 0$ and $x, y \in S$ which is in the hyperplane orthogonal to $f(u_0(0)$. Since

$$\left|\beta(0)\right| = \left|\tau(0)\right| \leqslant 2\Delta^{-1}\delta_1 \leqslant 2\Delta^{-1}\delta_2 < \Delta/(8M_0M_1),$$

it follows from uniqueness that $h_0 = -\beta(t_0)$ so that $\beta(t_0) + h_0 = 0$. Hence (41) holds for $k = 0$ and, of course, $y \in S$.

Now we use induction to complete the proof. We suppose for some $k \geqslant 0$ that $\phi^{\beta(t_k)+h_k}(y) \in S$ and (41) holds. Note, using (40), that

$$\begin{aligned}
&\left|\beta(t_{k+1}) + h_{k+1} - \left(\beta(t_k) + h_k\right) - (t_{k+1} - t_k)\right| \\
&\quad \leqslant \left|\beta(t_{k+1}) - \beta(t_k) - (t_{k+1} - t_k)\right| + |h_{k+1}| + |h_k|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2(\Delta - M_1\delta_2)^{-1}M_1\delta_2(t_{k+1}-t_k) + 8\Delta^{-1}\delta_2\\
&\leqslant 2(\Delta - M_1\delta_2)^{-1}M_1c_2\delta_2 + 8\Delta^{-1}\delta_2\\
&\leqslant \Delta/(16M_0M_1) + \Delta/(16M_0M_1)\\
&= \Delta/(8M_0M_1),
\end{aligned}
$$

provided that

$$64M_0M_1^2c_2\delta_2 < \Delta^2, \qquad 128M_0M_1\delta_2 < \Delta^2.$$

Note also that

$$\left|\phi^{\beta(t_k)+h_k}(y) - \phi^{t_k}(x)\right| \leqslant M_0|h_k| + \left|\phi^{\beta(t_k)}(y) - \phi^{t_k}(x)\right| \leqslant (4M_0\Delta^{-1}+1)\delta_2.$$

Next we assume δ_1 is so small that

$$(4M_0\Delta^{-1}+1)\delta_2 \leqslant \frac{e^{-M_1c_2}|f(u_0(0))|}{8M_1}\min\left\{1, \frac{|f(u_0(0))|}{4M_0}\right\}$$

and such that

$$\left|\tau(x) - \tau(y)\right| \leqslant \Delta/(8M_0M_1)$$

if x, y are in S and $|x-y| \leqslant (4M_0\Delta^{-1}+1)\delta_2$, where $\tau(x)$ is the function such that $F(x) = \phi^{\tau(x)}(x)$ for $x \in V_0 \cup V_1$. From the latter we conclude that

$$\left|\tau\left(\phi^{\beta(t_k)+h_k}(y)\right) - (t_{k+1}-t_k)\right| = \left|\tau\left(\phi^{\beta(t_k)+h_k}(y)\right) - \tau\left(\phi^{t_k}(x)\right)\right| \leqslant \Delta/(8M_0M_1).$$

Then it follows by the uniqueness in Lemma 3.1 with $\phi^{t_k}(x)$ as x, $z = \phi^{\beta(t_k)+h_k}(y)$, $T = t_{k+1} - t_k = \tau(\phi^{t_k}(x))$, $v = f(u_0(0))/|f(u_0(0))|$ and

$$\alpha = \frac{\langle f(\phi^{t_{k+1}}(x)), f(u_0(0))\rangle}{4M_0M_1|f(u_0(0))|} \quad \left(\geqslant \Delta/(8M_0M_1)\right)$$

that

$$\beta(t_{k+1}) + h_{k+1} - \left(\beta(t_k)+h_k\right) = \tau\left(\phi^{\beta(t_k)+h_k}(y)\right).$$

Hence

$$\phi^{\beta(t_{k+1})+h_{k+1}}(y) = F\left(F^k(y)\right) = F^{k+1}(y) \in S.$$

This finishes the induction proof and so we know that $\phi^{\beta(t_k)+h_k}(y) \in S$ and (41) holds for all $k \geqslant 0$.

Hence we conclude that for $k \geqslant 0$

$$\left|F^k(y) - F^k(x)\right| = \left|\phi^{\beta(t_k)+h_k}(y) - \phi^{t_k}(x)\right| \leqslant (4M_0/\Delta + 1)\delta_2.$$

This is a contradiction if

$$(4M_0/\Delta + 1)\delta_2 \leqslant \delta.$$

So we conclude that if $x \in S$ and for some $y \in S$ there exists $k \geqslant 0$ such that $|F^k(y) - F^k(x)| > \delta$, then if $\alpha(t)$ is a continuous nonnegative function with $\alpha(0) = 0$ we must have

$$\left|\phi^{\alpha(t)}(y) - \phi^t(x)\right| > \delta_1$$

for some $t \geqslant 0$, where δ_1 is a positive number satisfying all the conditions given above.

Now suppose $x \in \mathcal{S}$. Then, as at the beginning of this proof, $x = \phi^s(z)$ where $z \in S$ and $0 \leqslant s \leqslant c_2$. Let the nonnegative continuous function $\alpha(t)$ with $\alpha(0) = 0$ be given. We define the continuous function

$$\tilde{\alpha}(t) = \begin{cases} t & \text{if } 0 \leqslant t \leqslant s, \\ \alpha(t-s) + s, & \text{if } t \geqslant s. \end{cases}$$

Now, applying the first part of this proof to $\tilde{\alpha}(t)$ instead of $\alpha(t)$, we deduce that there exists $w \in S$ arbitrarily close to z such that

$$\left|\phi^{\tilde{\alpha}(t)}(w) - \phi^t(z)\right| > \delta_1$$

for some $t \geqslant 0$. Note that if w is sufficiently close to z then we can assume $t \geqslant s$. So we have

$$\left|\phi^{\alpha(t-s)+s}(w) - \phi^{t-s}(x)\right| > \delta_1$$

for some $t \geqslant s$. Hence

$$\left|\phi^{\alpha(t)}\big(\phi^s(w)\big) - \phi^t(x)\right| > \delta_1$$

for some $t \geqslant 0$, where $\phi^s(w)$ is in $\mathcal{S}$ and can be taken arbitrarily close to x. This completes the proof of (c) and hence of the proposition. □

5. Robustness of transversal periodic-to-periodic homoclinic orbits

In this section we consider a system depending on a parameter

$$\dot{x} = f(x, \mu) \tag{42}$$

where we assume the unperturbed system

$$\dot{x} = f(x, 0) \tag{43}$$

has a hyperbolic T-periodic orbit $u_0(t)$ with associated transversal homoclinic orbit $p_0(t)$. Our aim is to show that the perturbed system has a similar structure.

THEOREM 5.1. *Let $f(x, \mu)$ be a C^3 function. Suppose system* (43) *has a hyperbolic T-periodic orbit $u_0(t)$ and a transversal homoclinic orbit $p_0(t)$ to $u_0(t)$. Then*

(i) *for μ sufficiently small, the perturbed system* (42) *has a hyperbolic periodic orbit $u(t, \mu)$ with $u(t, 0) = u_0(t)$;*

(ii) *for μ sufficiently small,* (42) *has a transversal homoclinic orbit $x(t, \mu)$ associated with the hyperbolic periodic orbit $u(t, \mu)$ with $x(t, 0) = p_0(t)$.*

PROOF. (i) follows from standard theorems [10].

As in Proposition 3.5, the variational system

$$\dot{x} = f_x\big(u_0(t), 0\big)x$$

has a trichotomy on $(-\infty, \infty)$ with projections P_0, P_+, P_- and constants $K \geqslant 1$ and $\alpha > 0$.

Denote by γ^+ $(0 \leqslant \gamma^+ < T)$ the forward asymptotic phase for $p_0(t)$ and by γ^- $(0 \leqslant \gamma^- < T)$ the backward asymptotic phase. Note there exists a positive integer m such that

$$\big|p_0(t) - u_0(t + \gamma^+)\big| < \min\{\Delta, \xi_0/K\}$$

if $t \geqslant mT - \gamma^+$, where Δ and ξ_0 are as in Proposition 3.5. It follows by uniqueness that

$$p_0(t + mT - \gamma^+) = x^+(t, \xi_0, 0),$$

where $\xi_0 = P_+[p_0(mT - \gamma^+) - u_0(0)]$. Then

$$p_0(0) = x^+(-mT + \gamma^+, \xi_0, 0).$$

Similarly,

$$p_0(0) = x^-(mT + \gamma^-, \eta_0, 0)$$

where $\eta_0 = P_-[p_0(-mT - \gamma^-) - u_0(0)]$, provided that m also satisfies

$$\big|p_0(t) - u_0(t + \gamma^-)\big| < \min\{\Delta, \eta_0/K\}$$

if $t \leqslant -mT - \gamma^-$, where η_0 is as in Proposition 3.9.

Now it follows from Proposition 4.4 that the variational system

$$\dot{x} = f_x\big(p_0(t), 0\big)x \tag{44}$$

has a trichotomy on $\mathbf{R}$ with (say) projections Q_0, Q_+, Q_- where the range of Q_0 is spanned by $f(p_0(0), 0)$.

To find homoclinic orbits of (42) near $p_0(t)$, we solve the equation

$$F(\gamma, \xi, \eta, \mu) = x^+(-mT + \gamma, \xi, \mu) - x^-(mT + \gamma^-, \eta, \mu) = 0 \tag{45}$$

for $(\gamma, \xi, \eta) \in \mathbf{R} \times \mathcal{R}(P_+) \times \mathcal{R}(P_-)$ near $(\gamma^+, \xi_0, \eta_0)$, where we note that as a consequence of Proposition 3.5, F is a C^1 function. Note that

$$F(\gamma^+, \xi_0, \eta_0, 0) = p_0(0) - p_0(0) = 0.$$

We use the implicit function theorem. Write $z_0 = (\gamma^+, \xi_0, \eta_0)$. Then

$$F(z_0, 0) = 0.$$

Now we show $F_z(z_0, 0)$ is invertible. Note that if $z = (\gamma, \xi, \eta)$

$$\begin{aligned} F_z(z_0, 0)z &= \gamma x_t^+(-mT + \gamma^+, \xi_0, 0) + x_\xi^+(-mT + \gamma^+, \xi_0, 0)\xi \\ &\quad - x_\eta^-(mT + \gamma^-, \eta_0, 0)\eta \\ &= \gamma f\bigl(p_0(0), 0\bigr) + x_\xi^+(-mT + \gamma^+, \xi_0, 0)\xi - x_\eta^-(mT + \gamma^-, \eta_0, 0)\eta. \end{aligned}$$

Now it follows from Proposition 3.5 that $Z(t) = x_\xi^+(t - mT + \gamma^+, \xi_0, 0)$ is a solution of (44) restricted to $\mathcal{R}(P_+)$ such that $\|Z\|_\gamma < \infty$ and $P_+ Z(mT - \gamma^+) = P_+$. It follows that the range of $x_\xi^+(-mT + \gamma^+, \xi_0, 0)$ is contained in the range of Q_+. However the condition $P_+ X(mT - \gamma^+) = P_+$ implies that rank $x_\xi^+(-mT + \gamma^+, \xi_0, 0) = \text{rank } P_+ = \text{rank } Q_+$ so that the range of $x_\xi^+(-mT + \gamma^+, \xi_0, 0)$ coincides with the range of Q_+. Similarly, the range of $x_\eta^-(mT + \gamma^-, \eta_0, 0)$ coincides with the range of Q_-. Hence

$$\mathcal{R}\bigl(F_z(z_0, 0)\bigr) = \operatorname{span}\bigl\{f\bigl(p_0(0), 0\bigr)\bigr\} \oplus \mathcal{R}(Q_+) \oplus \mathcal{R}(Q_-) = \mathbf{R}^n$$

and so $F_z(z_0, 0)$ is invertible.

Then we deduce from the implicit function theorem that Eq. (45) has a C^1 solution $(\gamma(\mu), \xi(\mu), \eta(\mu))$ such that $(\gamma(0), \xi(0), \eta(0)) = (\gamma^+, \xi_0, \eta_0)$. Then

$$x(t, \mu) = \begin{cases} x^+(t - mT + \gamma(\mu), \xi(\mu), \mu) & \text{for } t \geqslant 0, \\ x^-(t + mT + \gamma^-, \eta(\mu), \mu) & \text{for } t \leqslant 0 \end{cases}$$

is a solution of (42) such that

$$\bigl|x(t, \mu) - u\bigl(t + \gamma(\mu), \mu\bigr)\bigr| \leqslant \Delta e^{-\gamma(t - mT + \gamma(\mu))} \quad \text{for } t \geqslant mT - \gamma(\mu)$$

and

$$\bigl|x(t, \mu) - u(t + \gamma^-, \mu)\bigr| \leqslant \Delta e^{\gamma(t + mT + \gamma^-)} \quad \text{for } t \leqslant -mT - \gamma^-.$$

So $x(t,\mu)$ is a homoclinic orbit associated with the hyperbolic periodic orbit $u_\mu(t) = u(t,\mu)$. Note also that

$$x(t,0) = p_0(t).$$

To verify that the homoclinic orbit is transversal, note it follows from the proofs of Propositions 3.12 and 3.13 that for

$$x(0,\mu) = x^+\bigl(-mT + \gamma(\mu), \xi(\mu), \mu\bigr) = x^-\bigl(mT + \gamma^-, \eta(\mu), \mu\bigr),$$

the tangent space $T_{x(0,\mu)}W^s(u_\mu)$ is spanned by $f(x(0,\mu),\mu)$ and the range of $x_\xi^+(-mT + \gamma(\mu), \xi(\mu), \mu)$, and the tangent space $T_{x(0,\mu)}W^u(u_\mu)$ is spanned by $f(x(0,\mu),\mu)$ and the range of $x_\eta^-(mT + \gamma^-, \eta(\mu), \mu)$. Then

$$w \in T_{x(0,\mu)}W^s(u_\mu) \cap T_{x(0,\mu)}W^u(u_\mu)$$

if and only if there exist $\gamma_1 \in \mathbf{R}$, $\xi \in \mathcal{R}(P_+)$, $\gamma_2 \in \mathbf{R}$ and $\eta \in \mathcal{R}(P_-)$ such that

$$\begin{aligned} w &= \gamma_1 f\bigl(x(0,\mu),\mu\bigr) + x_\xi^+\bigl(-mT + \gamma(\mu), \xi(\mu), \mu\bigr)\xi \\ &= \gamma_2 f\bigl(x(0,\mu),\mu\bigr) + x_\eta^-\bigl(mT + \gamma^-, \eta(\mu), \mu\bigr)\eta. \end{aligned}$$

Now note that

$$\begin{aligned} &F_z\bigl(\gamma(\mu), \xi(\mu), \eta(\mu), \mu\bigr)(\gamma_1 - \gamma_2, \xi, \eta) \\ &\quad = (\gamma_1 - \gamma_2) f\bigl(x(0,\mu),\mu\bigr) + x_\xi^+\bigl(-mT + \gamma(\mu), \xi(\mu), \mu\bigr)\xi \\ &\qquad - x_\eta^-\bigl(mT + \gamma^-, \eta(\mu), \mu\bigr)\eta \\ &\quad = w - w \\ &\quad = 0. \end{aligned}$$

However, when $\mu = 0$, this derivative is $F_z(z_0, 0)$, which is invertible, and hence still invertible for μ sufficiently small. So $(\gamma_1 - \gamma_2, \xi, \eta) = 0$ and $w = \gamma_1 f(x(0,\mu),\mu)$. Thus, for μ sufficiently small,

$$T_{x(0,\mu)}W^s(u_\mu) \cap T_{x(0,\mu)}W^u(u_\mu) = \operatorname{span}\bigl\{f\bigl(x(0,\mu),\mu\bigr)\bigr\}$$

and so $x(t,\mu)$ is a transversal homoclinic orbit. Thus we have proved the theorem. □

6. Finding transversal periodic-to-periodic homoclinic orbits through regular perturbation

Even though, as we found in the previous section, transversal periodic-to-periodic homoclinic orbits persist under perturbation, it does not seem possible to write down an explicit

system with an explicit transversal periodic-to-periodic homoclinic orbit. However, it turns out to be quite easy to write down an explicit system with an explicit nontransversal homoclinic orbit and obtain a transversal orbit by perturbation. To this end, as in the previous section, we consider a system depending on a parameter

$$\dot{x} = f(x, \mu). \tag{46}$$

We assume the unperturbed system

$$\dot{x} = f(x, 0) \tag{47}$$

has a hyperbolic T-periodic orbit $u_0(t)$ with associated homoclinic orbit $p_0(t)$. As in Proposition 3.5, this means that the variational system

$$\dot{x} = f_x\big(u_0(t), 0\big)x$$

has a trichotomy with projections P_0, P_+, P_- and constants K, α. Note, again as in Proposition 3.5, it follows by standard theorems that the perturbed system (46) has a periodic orbit $u(t, \mu)$ with $u(t, 0) = u_0(t)$.

Denote by γ^+ $(0 \leqslant \gamma^+ < T)$ the forward asymptotic phase for $p_0(t)$ and by γ^- $(0 \leqslant \gamma^- < T)$ the backward asymptotic phase. Then

$$p_0(0) \in W^{s,\gamma^+}(u_0) \cap W^{u,\gamma^-}(u_0).$$

What we suppose here is that $W^{s,\gamma^+}(u_0)$ and $W^{u,\gamma^-}(u_0)$ intersect in a curve C containing $p_0(0)$ which certainly implies that $p_0(t)$ cannot be transversal. So we are supposing there is a function $p(t, \beta)$ defined for all real t and β near 0 with $p(t, 0) = p_0(t)$ such that for each β, $p(t, \beta)$ is homoclinic to $u_0(t)$ with asymptotic phases γ^+ and γ^-.

THEOREM 6.1. *Let $f(x, \mu)$ be a C^4 function. Suppose system* (47) *has a hyperbolic T-periodic orbit $u_0(t)$ with associated homoclinic orbit $p_0(t)$ with asymptotic phases γ^+ and γ^- such that the variational system*

$$\dot{x} = f_x\big(p_0(t), 0\big)x \tag{48}$$

has a two-dimensional subspace of bounded solutions. Suppose $W^{s,\gamma^+}(u_0)$ and $W^{u,\gamma^-}(u_0)$ intersect in a C^1 curve $p(\beta)$ with $p(0) = p_0(0)$ and denote by $p(t, \beta)$ the solution of (47) *with $p(0, \beta) = p(\beta)$. Then*

(i) *for μ sufficiently small, the perturbed system* (46) *has a hyperbolic periodic orbit $u_\mu(t) = u(t, \mu)$ with $u(t, 0) = u_0(t)$;*

(ii) *the adjoint system*

$$\dot{x} = -f_x\big(p(t, \beta), 0\big)^* x \tag{49}$$

has, up to a scalar multiple, a unique nontrivial solution $\psi(t,\beta)$ *with* $\psi(t,\beta)\to 0$ *as* $|t|\to\infty$ *and if the Melnikov function*

$$\int_{-\infty}^{\infty}\psi^*(t,\beta)f_\mu\bigl(p(t,\beta),0\bigr)\,\mathrm{d}t$$

has a simple zero at $\beta=0$, *then for* $\mu\neq 0$ *sufficiently small, the perturbed system* (46) *has a transversal homoclinic orbit* $y(t,\mu)$ *associated with the hyperbolic periodic orbit* $u(t,\mu)$ *with* $y(t,0)=p_0(t)$.

PROOF. (i) follows from standard theorems as in Proposition 3.5.

Now, as in Proposition 3.5, the variational system

$$\dot{x}=f_x\bigl(u_0(t),0\bigr)x$$

has a trichotomy on **R** with projections P_0, P_+, P_- and constants $K\geqslant 1$ and $\alpha>0$.

As in the proof of Theorem 5.1, there exists a positive integer m such that

$$p_0(0)=x^+(-mT+\gamma^+,\xi_0,0),\quad \text{where } \xi_0=P_+\bigl[p_0(mT-\gamma^+)-u_0(0)\bigr].$$

So, in a neighbourhood of $p_0(0)$, $W^{s,\gamma^+}(u_\mu)$ can be parametrized as $x^+(-mT+\gamma^+,\xi,\mu)$, where $\xi\in\mathcal{R}(P_+)$ is near ξ_0. Similarly, in the neighbourhood of $p_0(0)$, $W^{u,\gamma^-}(u_\mu)$ can be parametrized as $x^-(mT+\gamma^-,\eta,\mu)$, where η is near

$$\eta_0=P_-\bigl[p_0(-mT-\gamma^-)-u_0(0)\bigr].$$

So we can write

$$p(\beta)=x^+\bigl(-mT+\gamma^+,\xi(\beta),0\bigr)=x^-\bigl(mT+\gamma^-,\eta(\beta),0\bigr),$$

where $\xi(\beta)$ and $\eta(\beta)$ are C^1 functions with $\xi(0)=\xi_0$ and $\eta(0)=\eta_0$.

Note that since

$$p_\beta(t,\beta)=x^+_\xi\bigl(t-mT+\gamma^+,\xi(\beta),0\bigr)\xi'(\beta)=x^-_\eta\bigl(t+mT+\gamma^-,\eta(\beta),0\bigr)\eta'(\beta),$$

it follows from Propositions 3.5 and 3.9 that $p_\beta(t,\beta)\to 0$ as $|t|\to\infty$. Next, by Proposition 3.5 again,

$$\begin{aligned}\bigl|p(t,\beta)-u_0(t+\gamma^+)\bigr|&=\bigl|x^+\bigl(t-mT+\gamma^+,\xi(\beta),0\bigr)-u_0(t-mT+\gamma^+)\bigr|\\&\leqslant \Delta\mathrm{e}^{-\gamma(t-mT+\gamma^+)}\end{aligned}\tag{50}$$

if $t\geqslant mT-\gamma^+$ and using Proposition 3.9 we get a similar inequality for $|p(t,\beta)-u_0(t+\gamma^-)|$ in $t\leqslant -mT-\gamma^-$. This implies that the norm of $p_t(t,\beta)=f(p(t,\beta))$ has a uniform

upper bound and positive lower bound independent of β. In particular, this means that $p_t(t,\beta)$ and $p_\beta(t,\beta)$ are independent bounded solutions of the variational system

$$\dot{x} = f_x\big(p(t,\beta),0\big)x. \tag{51}$$

Next the last inequality above implies that for $t \geqslant t_1$,

$$\big|f_x\big(p(t,\beta),0\big) - f_x\big(u_0(t+\gamma^+),0\big)\big| \leqslant \delta(t_1),$$

where $\delta(t_1) \to 0$ as $t_1 \to \infty$ uniformly with respect to β. It follows from the roughness result in Remark 2.7 that if t_1 is large enough, not depending on β, system (51) has a trichotomy on $[t_1,\infty)$ with projections $P_0(\beta)$, $P_+(\beta)$, $P_-(\beta)$ such that $\mathcal{R}(P_0(\beta)) = \operatorname{span}\{p_t(0,\beta)\}$ and with exponent and constant independent of β. Next by Remark 2.3, Eq. (51) has a trichotomy on $[0,\infty)$ with the same projections and exponent but a different constant perhaps.

Similarly, (51) has a trichotomy on $\mathbf{R}_-$ with projections $Q_0(\beta)$, $Q_+(\beta)$, $Q_-(\beta)$. Note that the range of $P_0(\beta)$ and $Q_0(\beta)$ coincide as the span of $p_t(0,\beta)$. Also from Remark 2.8, we know that the projections $P_\sigma(\beta)$ and $Q_\sigma(\beta)$ are C^1 in β. Now the subspace of initial values of bounded solutions of (51) is

$$\mathcal{R}\big(P_0(\beta)+P_+(\beta)\big) \cap \mathcal{R}\big(Q_0(\beta)+Q_-(\beta)\big).$$

By hypothesis, when $\beta = 0$ this subspace has dimension 2. Since the projections are continuous, for small β the dimension is at most 2. However, $p_t(0,\beta)$ and $p_\beta(t,\beta)$ are linearly independent bounded solutions. So the dimension is exactly 2.

Next the adjoint equation (49) has a trichotomy on $\mathbf{R}_+$ with projections $P_0^*(\beta)$, $P_-^*(\beta)$, $P_+^*(\beta)$ and on $\mathbf{R}_-$ with projections $Q_0^*(\beta)$, $Q_-^*(\beta)$, $Q_+^*(\beta)$. The subspace of initial values at $t = 0$ of solutions $x(t)$ with $|x(t)| \to 0$ as $|t| \to \infty$ is

$$\mathcal{R}\big(P_-^*(\beta)\big) \cap \mathcal{R}\big(Q_+^*(\beta)\big) = V(\beta)^\perp,$$

where

$$V(\beta) = \mathcal{R}\big(P_0(\beta)+P_+(\beta)\big) + \mathcal{R}\big(Q_0(\beta)+Q_-(\beta)\big).$$

$V(\beta)^\perp$ is one-dimensional since the subspaces $\mathcal{R}(P_0(\beta)+P_+(\beta))$ and $\mathcal{R}(Q_0(\beta)+Q_-(\beta))$ have a two-dimensional intersection and their dimensions sum to $n+1$.

To get a C^1 basis for $V(\beta)$, we begin with bases for $\mathcal{R}(P_0(0)+P_+(0))$ and $\mathcal{R}(Q_0(0)+Q_-(0))$ containing $p_t(0,0)$ and $p_\beta(t,0)$. Then we obtain a C^1 basis for $V(\beta)$ by taking $p_t(0,\beta)$ and $p_\beta(t,\beta)$ and by applying the projections $P_0(\beta)+P_+(\beta)$ and $Q_0(\beta)+Q_+(\beta)$ to the other vectors. We assemble these vectors into an $n \times (n-1)$ matrix. Then the vector whose ith component is $(-i)^i$ times the determinant of the matrix obtained by deleting the ith row is nonzero, depends C^1 on β and is orthogonal to all the vectors in the basis. Dividing by its norm, we obtain a C^1 unit vector $\psi(\beta)$ which spans $V(\beta)^\perp$. Denote by

$\psi(t,\beta)$ the solution of the adjoint equation (49) with $\psi(0,\beta)=\psi(\beta)$. Up to a scalar multiple, it is the unique solution of (49) which tends to zero as $|t|\to\infty$.

To find homoclinic orbits near $p_0(t)$, we solve the equation

$$F(\gamma,\xi,\eta,\mu)=x^+(-mT+\gamma,\xi,\mu)-x^-(mT+\gamma^-,\eta,\mu)=0$$

for (γ,ξ,η) near (γ^+,ξ_0,η_0). Note that by Propositions 3.5 and 3.9, F is a C^2 function. Also we have

$$F\big(\gamma^+,\xi(\beta),\eta(\beta),0\big)=p(0,\beta)-p(0,\beta)=0.$$

We use the Crandall–Rabinowitz theorem [6] as given in Theorem 4.1 in [17], a suitable version of which can be stated as follows:

Let $F:E\times\mathbf{R}\mapsto G$, $(z,\mu)\mapsto F(z,\mu)$, be a C^2 mapping, where E and G are Banach spaces. Suppose that there exists a C^2 function $\phi(\beta)$ defined on an interval I such that $\phi'(\beta)\neq 0$ and such that

$$F\big(\phi(\beta),0\big)=0$$

and $L(\beta)=F_z(\phi(\beta),0)$ is Fredholm of index zero with nullspace spanned by $\phi'(\beta)$. Define

$$d(\beta)=\psi(\beta)\big(F_\mu\big(\phi(\beta),0\big)\big),$$

where $\psi(\beta)\in G^*$ is a C^1 function such that $\mathcal{N}(\psi(\beta))=\mathcal{R}(L(\beta))$. Then if $d(\beta)$ has a simple zero at β_0, the equation

$$F(z,\mu)=0$$

has a solution $z(\mu)$ for μ sufficiently small. Moreover, $z(0)=\phi(\beta_0)$, $z(\mu)$ is a C^1 function and $F_z(z(\mu),\mu)$ is invertible if $\mu\neq 0$.

We apply this theorem with $E=\mathbf{R}\times\mathcal{R}(P_+)\times\mathcal{R}(P_-)$ and $G=\mathbf{R}^n$. Write $z=(\gamma,\xi,\eta)$ and $\phi(\beta)=(\gamma^+,\xi(\beta),\eta(\beta))$. Then

$$F\big(\phi(\beta),0\big)=0.$$

Now we examine $F_z(\phi(\beta),0)$. Note if $z=(\gamma,\xi,\eta)$ that

$$\begin{aligned}F_z\big(\phi(\beta),0\big)z&=\gamma x_t^+\big(-mT+\gamma^+,\xi(\beta),0\big)+x_\xi^+\big(-mT+\gamma^+,\xi(\beta),0\big)\xi\\&\quad-x_\eta^-\big(mT+\gamma^-,\eta(\beta),0\big)\eta\\&=\gamma f\big(p(0,\beta),0\big)+x_\xi^+\big(-mT+\gamma^+,\xi(\beta),0\big)\xi\\&\quad-x_\eta^-\big(mT+\gamma^-,\eta(\beta),0\big)\eta.\end{aligned}$$

Reasoning as in the proof of Theorem 5.1, we find that

$$\mathcal{R}\big(F_z\big(\phi(\beta),0\big)\big)=\operatorname{span}\big\{f\big(p(0,\beta),0\big)\big\}\oplus\big[\mathcal{R}\big(P_+(\beta)\big)+\mathcal{R}\big(Q_-(\beta)\big)\big]$$

so that

$$\mathcal{R}\big(F_z\big(\phi(\beta),0\big)\big) = \operatorname{span}\big\{\psi(\beta)\big\}^{\perp}.$$

(Note that $\psi(\beta)$ can be regarded as the linear functional $x \mapsto \psi(\beta)^* x$ and the last subspace is its nullspace.) From this it follows that dim $\mathcal{N}(F_z(\phi(\beta),0)) = 1$ and so

$$\mathcal{N}\big(F_z\big(\phi(\beta),0\big)\big) = \operatorname{span}\big\{\phi'(\beta)\big\}.$$

Next we calculate the Melnikov function

$$d(\beta) = \psi(\beta)^* F_\mu\big(\phi(\beta),0\big).$$

Now

$$F_\mu\big(\phi(\beta),0\big) = z_\mu^+\big(-mT+\gamma^+,\xi(\beta),0\big) - z_\mu^-\big(mT+\gamma^-,\eta(\beta),0\big),$$

where $z^+(t,\xi,\mu) = x^+(t,\xi,\mu) - u(t,\mu)$ and $z^-(t,\eta,\mu) = x^-(t,\eta,\mu) - u(t,\mu)$. By differentiation with respect to μ at $\mu = 0$ and using the fact that $x^+(t,\xi(\beta),0) = p(t+mT-\gamma^+,\beta)$, we find that $w(t) = z_\mu^+(t-mT+\gamma^+,\xi(\beta),0)$ is a solution of

$$\dot{w} = f_x\big(p(t,\beta),0\big)w + h(t),$$

where

$$\begin{aligned} h(t) &= f_x\big(p(t,\beta),0\big)u_\mu(t+\gamma^+,0) - u_{\mu t}(t+\gamma^+,0) + f_\mu\big(p(t,\beta),0\big) \\ &= \big[f_x\big(p(t,\beta),0\big) - f_x\big(u_0(t+\gamma^+),0\big)\big]u_\mu(t+\gamma^+,0) \\ &\quad + f_\mu\big(p(t,\beta),0\big) - f_\mu\big(u_0(t+\gamma^+),0\big) \end{aligned}$$

so that, by (50), $\|h\|_\gamma < \infty$. We also know from Proposition 3.5 that $\|w\|_\gamma < \infty$. So, by Lemma 3.6,

$$\begin{aligned} w(t) &= Y(t,\beta)P_+(\beta)w(0) + \int_0^t Y(t,\beta)P_+(\beta)Y^{-1}(s,\beta)h(s)\,\mathrm{d}s \\ &\quad - \int_t^\infty Y(t,\beta)\big(I - P_+(\beta)\big)Y^{-1}(s,\beta)h(s)\,\mathrm{d}s, \end{aligned}$$

where $Y(t,\beta)$ is the fundamental matrix of (51) with $Y(0,\beta) = I$. Thus

$$z_\mu^+\big(-mT+\gamma^+,\xi(\beta),0\big) = P_+(\beta)w(0) - \int_0^\infty \big(I - P_+(\beta)\big)Y^{-1}(t,\beta)h(t)\,\mathrm{d}t$$

and so since $\psi(\beta)^* P_+(\beta) = 0$

$$\psi(\beta)^* z_\mu^+\big(-mT+\gamma^+,\xi(\beta),0\big) = -\int_0^\infty \psi(t,\beta)^* h(t)\,\mathrm{d}t.$$

Similarly,

$$\psi(\beta)^* z_\mu^-\bigl(mT + \gamma^+, \eta(\beta), 0\bigr) = \int_{-\infty}^{0} \psi(t, \beta)^* h(t)\,\mathrm{d}t$$

and hence

$$d(\beta) = -\int_{-\infty}^{\infty} \psi(t, \beta)^* h(t)\,\mathrm{d}t.$$

However, integrating by parts,

$$\begin{aligned}\int_{-\infty}^{\infty} \psi(t, \beta)^* u_{\mu t}(t + \gamma^+, 0)\,\mathrm{d}t &= -\int_{-\infty}^{\infty} \psi_t(t, \beta)^* u_\mu(t + \gamma^+, 0)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \psi(t, \beta)^* f_x\bigl(p(t, \beta), 0\bigr) u_\mu(t + \gamma^+, 0)\,\mathrm{d}t\end{aligned}$$

and so

$$d(\beta) = -\int_{-\infty}^{\infty} \psi(t, \beta)^* f_\mu\bigl(p(t, \beta), 0\bigr)\,\mathrm{d}t.$$

Since, by hypothesis, $d(\beta)$ has a simple zero at $\beta = 0$ it follows from the Crandall–Rabinowitz theorem as stated above that the equation

$$F(\gamma, \xi, \eta, \mu) = 0$$

has a solution $(\gamma(\mu), \xi(\mu), \eta(\mu))$ such that $(\gamma(0), \xi(0), \eta(0)) = (\gamma^+, \xi_0, \eta_0)$. Then

$$x(t, \mu) = \begin{cases} x^+(t - mT + \gamma(\mu), \xi(\mu), \mu) & \text{for } t \geqslant 0, \\ x^-(t + mT + \gamma^-, \eta(\mu), \mu) & \text{for } t \leqslant 0 \end{cases}$$

is a solution of (46) such that

$$\bigl|x(t, \mu) - u\bigl(t + \gamma(\mu), \mu\bigr)\bigr| \leqslant \Delta \mathrm{e}^{-\beta(t - mT + \gamma(\mu))} \quad \text{for } t \geqslant mT - \gamma(\mu)$$

and

$$\bigl|x(t, \mu) - u(t + \gamma^-, \mu)\bigr| \leqslant \Delta \mathrm{e}^{\beta(t + mT + \gamma^-)} \quad \text{for } t \leqslant -mT - \gamma^-.$$

So $x(t, \mu)$ is a homoclinic orbit associated with the hyperbolic periodic orbit $u(t, \mu)$.

To verify that the homoclinic orbit is transversal when $\mu \neq 0$, we use the fact that the Crandall–Rabinowitz theorem also tells us that the operator $F_z(\gamma(\mu), \xi(\mu), \eta(\mu), \mu)$ is invertible when $\mu \neq 0$.

As in the proof of Theorem 5.1, we find that

$$w \in T_{x(0,\mu)} W^s(u_\mu) \cap T_{x(0,\mu)} W^u(u_\mu)$$

if and only if there exist $(\gamma_1, \xi) \in \mathbf{R} \times \mathcal{R}(P_+)$ and $(\gamma_2, \eta) \in \mathbf{R} \times \mathcal{R}(Q_-)$ such that

$$\begin{aligned} w &= \gamma_1 f\big(x(0,\mu),\mu\big) + x_\xi^+\big(-mT + \gamma(\mu), \xi(\mu), \mu\big)\xi \\ &= \gamma_2 f\big(x(0,\mu),\mu\big) + x_\eta^-\big(mT + \gamma^-, \eta(\mu), \mu\big)\eta \end{aligned}$$

and that then

$$F_z\big(\gamma(\mu), \xi(\mu), \eta(\mu), \mu\big)(\gamma_1 - \gamma_2, \xi, \eta) = 0.$$

However, when $\mu \neq 0$, this derivative is invertible and hence one to one. So $(\gamma_1 - \gamma_2, \xi, \eta) = 0$ and $w = \gamma_1 f(x(0,\mu),\mu)$. Thus, when $\mu \neq 0$,

$$T_{x(0,\mu)} W^s(u_\mu) \cap T_{x(0,\mu)} W^u(u_\mu) = \operatorname{span}\big\{ f\big(x(0,\mu),\mu\big)\big\}$$

and so $x(t,\mu)$ is a transversal homoclinic orbit. Thus we have proved the theorem. □

EXAMPLE 6.2. Suppose $F : U \to \mathbf{R}^2$ is a C^4 vector field such that

$$\dot{x} = F(x)$$

has a hyperbolic periodic solution $r(t)$ with minimal period T and $g : \mathbf{R}^2 \to \mathbf{R}^2$ is a C^4 vector field such that 0 is a saddle point of

$$\dot{y} = g(y)$$

and $\zeta(t)$ is an associated homoclinic orbit. Then $(r(t), 0)$ is a hyperbolic periodic orbit of the system

$$\dot{x} = F(x), \qquad \dot{y} = g(y)$$

and $p(t,\beta) = (r(t), \zeta(t+\beta))$ is a nontransversal homoclinic orbit with asymptotic phases $\gamma^+ = \gamma^- = 0$ such that the subspace of bounded solutions of the variational system

$$\dot{x} = F'\big(r(t)\big)x, \qquad \dot{y} = g'\big(\zeta(t)\big)y$$

is spanned by $(r'(t), 0)$ and $(0, \zeta'(t))$.

Let $\psi(t)$ be, up to a scalar multiple, the unique bounded solution of

$$\dot{y} = -g'\big(\zeta(t)\big)^* y.$$

Then $(0, \psi(t))$ is, up to a scalar multiple, the unique solution of

$$\dot{x} = -F'\big(r(t)\big)^* x, \qquad \dot{y} = -g'\big(\zeta(t)\big)^* y$$

which tends to 0 as $|t| \to \infty$.

Then it follows from Theorem 6.1 that if the Melnikov function

$$d(\beta) = \int_{-\infty}^{\infty} \psi^*(t+\beta) g_1\big(r(t), \zeta(t+\beta), 0\big)\,\mathrm{d}t$$

has a simple zero at $\beta = 0$, then for $\mu \neq 0$ sufficiently small, the perturbed C^4 system

$$\dot{x} = F(x) + \mu F_1(x, y, \mu), \qquad \dot{y} = g(y) + \mu g_1(x, y, \mu)$$

has a transversal homoclinic orbit associated with a hyperbolic periodic orbit.

Now we give an explicit example of this situation.

EXAMPLE 6.3. The planar autonomous system $\dot{x} = F(x)$ given by

$$\dot{x}_1 = -x_1 + x_2 + \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \qquad \dot{x}_2 = -x_1 - x_2 + \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$$

has the periodic solution $r(t) = (\cos t, -\sin t)$. One calculates the trace $\operatorname{Tr} DF(r(t)) = -1$. Then, by Jacobi's formula, the product of the Floquet multipliers is

$$\mathrm{e}^{\int_0^{2\pi} \operatorname{Tr} DF(r(t))\,\mathrm{d}t} = \mathrm{e}^{-2\pi}.$$

Since one Floquet multiplier is 1, the other must be $\mathrm{e}^{-2\pi}$. So $r(t)$ is hyperbolic.

Next $(0, 0)$ is a saddle for the system

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = y_1 - 2y_1^3$$

with associated saddle connexion $\zeta(t) = (\xi(t), \dot{\xi}(t))$ where $\xi(t) = \operatorname{sech} t$. Note that $\psi(t) = (\ddot{\xi}(t), -\dot{\xi}(t))$ is a bounded solution of the system adjoint to the variational equation along $\zeta(t)$.

Following Example 6.2, we apply Theorem 6.1 to the system

$$\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 + \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \\
\dot{x}_2 &= -x_1 - x_2 + \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \\
\dot{y}_1 &= y_2 + \mu x_1, \\
\dot{y}_2 &= y_1 - 2y_1^3 + \mu x_2.
\end{aligned} \tag{52}$$

The Melnikov function is

$$
\begin{aligned}
d(\beta) &= \int_{-\infty}^{\infty} \dot{\xi}(t+\beta)\sin t + \ddot{\xi}(t+\beta)\cos t \,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \dot{\xi}(t)\sin(t-\beta) + \ddot{\xi}(t)\cos(t-\beta)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \dot{\xi}(t)(\sin t\cos\beta - \cos t\sin\beta) + \ddot{\xi}(t)(\cos t\cos\beta + \sin t\sin\beta)\,\mathrm{d}t \\
&= I_1\cos\beta + I_2\sin\beta,
\end{aligned}
$$

where

$$
I_1 = \int_{-\infty}^{\infty} \dot{\xi}(t)\sin t + \ddot{\xi}(t)\cos t\,\mathrm{d}t \quad \text{and} \quad I_2 = \int_{-\infty}^{\infty} -\dot{\xi}(t)\cos t + \ddot{\xi}(t)\sin t\,\mathrm{d}t.
$$

Note that $I_2 = 0$ since its integrand is odd. We calculate

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} -\frac{\mathrm{d}}{\mathrm{d}t}\big(\dot{\xi}(t)\cos t\big) + 2\ddot{\xi}(t)\cos t\,\mathrm{d}t \\
&= 2\int_{-\infty}^{\infty} \ddot{\xi}(t)\cos t\,\mathrm{d}t \\
&= 2\left(\dot{\xi}(t)\cos t\,|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \dot{\xi}(t)(-\sin t)\,\mathrm{d}t\right) \\
&= 2\int_{-\infty}^{\infty} \dot{\xi}(t)\sin t\,\mathrm{d}t \\
&= 2\left(\xi(t)\sin t\,|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \xi(t)\cos t\,\mathrm{d}t\right) \\
&= -2\int_{-\infty}^{\infty} \xi(t)\cos t\,\mathrm{d}t
\end{aligned}
$$

and so

$$
I_1 = -2\int_{-\infty}^{\infty} \operatorname{sech} t\cos t\,\mathrm{d}t \neq 0.
$$

Hence we have shown that

$$
d(\beta) = I_1\cos\beta,
$$

where $I_1 \neq 0$. It is clear that this function has simple zeros and hence we may conclude that, for μ nonzero and sufficiently small, system (52) has a hyperbolic periodic orbit with associated transversal homoclinic orbit.

7. Finding transversal periodic-to-periodic homoclinic orbits through numerical shadowing

In this last section, which is based on [4], we show how transversal periodic-to-periodic homoclinic orbits can be constructed via numerical shadowing. As usual, we consider the differential equation

$$\dot{x} = f(x) \tag{53}$$

where now f is a C^2 vector field in $\mathbf{R}^n$ with

$$\left|f(x)\right| \leqslant M_0, \quad \left|f'(x)\right| \leqslant M_1, \quad \left|f''(x)\right| \leqslant M_2.$$

Denote by $\phi^t(x)$ the corresponding flow.

First we define pseudo (or approximate) orbits, which are obtained when an orbit is computed numerically.

DEFINITION 7.1. A sequence $\{w_k\}_{k=-\infty}^{+\infty}$ with associated times $\{h_k\}_{k=-\infty}^{+\infty}$ is said to be a δ *pseudo orbit* of (53) if

$$\sup_{k\in Z} h_k < \infty, \qquad \inf_{k\in Z} h_k > 0$$

and for all k

$$f(w_k) \neq 0, \qquad \left|w_{k+1} - \phi^{h_k}(w_k)\right| \leqslant \delta.$$

We look for a true orbit near a pseudo orbit. Such a true orbit is said to shadow the pseudo orbit.

DEFINITION 7.2. A sequence $\{x_k\}_{k=-\infty}^{+\infty}$ with associated times $\{t_k\}_{k=-\infty}^{+\infty}$ is said to ε *shadow* a δ pseudo orbit $\{w_k\}_{k=-\infty}^{+\infty}$ with associated times $\{h_k\}_{k=-\infty}^{+\infty}$ if for all k, $x_{k+1} = \phi^{t_k}(x_k)$ and

$$|x_k - w_k| \leqslant \varepsilon, \qquad |t_k - h_k| \leqslant \varepsilon.$$

We are interested in finding homoclinic orbits associated to periodic orbits. There are particular kinds of pseudo orbits associated with these. First we define pseudo periodic orbits.

DEFINITION 7.3. A δ pseudo orbit $\{y_k\}_{k=-\infty}^{+\infty}$ with associated times $\{\ell_k\}_{k=-\infty}^{+\infty}$ is said to be a δ *pseudo periodic orbit* with period N of (53) if for all k

$$y_{k+N} = y_k, \qquad \ell_{k+N} = \ell_k.$$

Normally we expect pseudo periodic orbits to be shadowed by discrete true periodic orbits, that is, we expect x_k and t_k to satisfy

$$x_{k+N} = x_k, \qquad t_{k+N} = t_k$$

for all k.

Next we define pseudo homoclinic orbits.

DEFINITION 7.4. Let $\{y_k\}_{k=-\infty}^{+\infty}$ be a δ pseudo periodic orbit with associated times $\{\ell_k\}_{k=-\infty}^{+\infty}$ and period N. A δ pseudo orbit $\{w_k\}_{k=-\infty}^{+\infty}$ with associated times $\{h_k\}_{k=-\infty}^{+\infty}$ is said to be a δ *pseudo homoclinic orbit* if for some τ, $0 \leqslant \tau < N$,

$$w_k = y_{k+\tau}, \quad h_k = \ell_{k+\tau} \quad \text{for } k \leqslant p, \qquad w_k = y_k, \quad h_k = \ell_k \quad \text{for } k \geqslant q$$

for some integers $p < q$.

Note that τ allows for the possibility of homoclinic orbits with different forward and backward asymptotic phases.

In preparation for the statement of the main theorem, we define a linear operator associated with a pseudo orbit. Consider a fixed pseudo orbit $\{w_k\}_{k=-\infty}^{+\infty}$ with associated times $\{h_k\}_{k=-\infty}^{+\infty}$. For each k, let Y_{w_k} be the subspace of $\mathbf{R}^n$ consisting of the vectors orthogonal to $f(w_k)$. Let Y_w be the Banach space of bounded sequences $\mathbf{v} = \{v_k\}_{k\in Z}$ with $v_k \in Y_{w_k}$, and equip Y_w with the norm

$$\|\mathbf{v}\| = \sup_{k\in Z} |v_k|.$$

Also, let $\tilde{Y}_w$ be a similar Banach space except that $v_k \in Y_{w_{k+1}}$.

Then let $L_w : Y_w \to \tilde{Y}_w$ be the linear operator defined by

$$(L_w \mathbf{v})_k = v_{k+1} - P_{w_{k+1}} D\phi^{h_k}(w_k) v_k,$$

where if $f(x) \neq 0$, the operator $P_x : \mathbf{R}^n \to \mathbf{R}^n$ is the orthogonal projection defined by

$$P_x v = v - \frac{f(x)^* v}{|f(x)|^2} f(x).$$

L_w represents the action of the derivative of the flow along the pseudo orbit projected on to the subspace orthogonal to the vector field.

Now we are ready to state the main theorem in this section, which gives conditions under which a pseudo homoclinic orbit is shadowed by a true homoclinic orbit.

THEOREM 7.5. *Suppose* $f : \mathbf{R}^n \to \mathbf{R}^n$ *is a* C^2 *vectorfield. Let* $\{y_k\}_{k=-\infty}^{+\infty}$ *be a* δ *pseudo periodic orbit with period* N *of* (53) *and let* $\{w_k\}_{k=-\infty}^{+\infty}$ *be a* δ *pseudo homoclinic*

orbit with associated times $\{h_k\}_{k=-\infty}^{+\infty}$ *connecting* $\{y_{k+\tau}\}_{k=-\infty}^{\infty}$ *to* $\{y_k\}_{k=-\infty}^{+\infty}$ *such that* $\Delta = \inf_{k\in Z} |f(w_k)| > 0$ *and the operator* L_w *is invertible.*

Then there exist constants C *and* δ_0 *depending only on* f, $\|L_w^{-1}\|$, Δ, $h_{\max}$ *and* $h_{\min}$, *where*

$$h_{\max} = \sup_{k\in Z} h_k, \qquad h_{\min} = \inf_{k\in Z} h_k,$$

such that if $\delta < \delta_0$,

(i) *the pseudo periodic orbit* $\{y_k\}_{k=-\infty}^{+\infty}$ *is shadowed by a true discrete periodic orbit* $\{x_k\}_{k=-\infty}^{\infty}$ *such that for all* k

$$f(y_k)^*(x_k - y_k) = 0, \quad |x_k - y_k| \leqslant C\delta.$$

Moreover, the periodic orbit $\phi^t(x_0)$ *is hyperbolic*;

(ii) *The pseudo connecting orbit* $\{w_k\}_{k=-\infty}^{+\infty}$ *is shadowed by a true orbit* $\{z_k\}_{k=-\infty}^{\infty}$ *such that for all* k

$$f(w_k)^*(z_k - w_k) = 0, \quad |z_k - w_k| \leqslant C\delta.$$

Moreover, $\phi^t(z_0)$ *is hyperbolic as in Definition* 4.3 *and* $\operatorname{dist}(\phi^t(z_0), \mathrm{O}(x_0)) \to 0$ *as* $|t| \to \infty$, *where* $\mathrm{O}(x_0)$ *is the orbit of* x_0.

Furthermore, suppose there is r *such that for* $k = 0, \ldots, N-1$

$$|w_r - y_k| > \left(\left|f(y_k)\right| + M_1 C\delta\right)\frac{\mathrm{e}^{M_1(h_{\max}+C\delta)} - 1}{M_1} + 2C\delta, \tag{54}$$

where $M_1 = \sup_{x\in\mathbf{R}^n} |f'(x)|$. *Then* z_r *and* x_r *lie on distinct orbits and so we may conclude that* $\phi^t(z_0)$ *is a transversal homoclinic orbit associated with the periodic orbit* $\phi^t(x_0)$.

To prove the theorem, we need the following lemmas. The first is a Newton–Kantorovich type of result.

LEMMA 7.6. *Let* X, Y *be Banach spaces and* $\mathcal{G}: X \to Y$ *a* C^2 *function. Suppose* $\mathbf{y}$ *is an element of* X *for which*

$$\left\|\mathcal{G}(\mathbf{y})\right\| \leqslant \delta$$

and $D\mathcal{G}(\mathbf{y}) = L$ *is invertible with* $\|L^{-1}\| \leqslant K$. *Set*

$$M = \sup\left\{\left\|D^2\mathcal{G}(\mathbf{x})\right\|: \mathbf{x} \in X,\ \|\mathbf{x} - \mathbf{y}\| \leqslant 2K\delta\right\}.$$

Then if

$$2MK^2\delta < 1,$$

there is a unique solution $\mathbf{x}$ *of the equation*

$$\mathcal{G}(\mathbf{x}) = \mathbf{0}$$

satisfying $\|\mathbf{x} - \mathbf{y}\| \leqslant 2K\delta$.

PROOF. Define the operator $F : X \to X$ by

$$F(\mathbf{x}) = \mathbf{y} - L^{-1}\big[\mathcal{G}(\mathbf{x}) - D\mathcal{G}(\mathbf{y})(\mathbf{x} - \mathbf{y})\big].$$

Clearly $\mathcal{G}(\mathbf{x}) = \mathbf{0}$ if and only if $F(\mathbf{x}) = \mathbf{x}$. Moreover, if $\|\mathbf{x} - \mathbf{y}\| \leqslant \varepsilon = 2K\delta$,

$$\begin{aligned}\big\|F(\mathbf{x}) - \mathbf{y}\big\| &\leqslant \|L^{-1}\|\big\|\mathcal{G}(\mathbf{x}) - \mathcal{G}(\mathbf{y}) - D\mathcal{G}(\mathbf{y})(\mathbf{x} - \mathbf{y}) + \mathcal{G}(\mathbf{y})\big\| \\ &\leqslant K\left[\frac{1}{2}M\|\mathbf{x} - \mathbf{y}\|^2 + \delta\right] \\ &\leqslant MK\varepsilon \cdot \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \\ &< \varepsilon.\end{aligned}$$

Furthermore, if $\|\mathbf{x} - \mathbf{y}\| \leqslant \varepsilon$ and $\|\mathbf{z} - \mathbf{y}\| \leqslant \varepsilon$, then

$$\big\|F(\mathbf{x}) - F(\mathbf{z})\big\| \leqslant \|L^{-1}\|\big\|\mathcal{G}(\mathbf{x}) - \mathcal{G}(\mathbf{z}) - D\mathcal{G}(\mathbf{y})(\mathbf{x} - \mathbf{z})\big\| \leqslant KM\varepsilon\|\mathbf{x} - \mathbf{z}\|.$$

Since $KM\varepsilon < 1$, F is a contraction on the closed ball of radius ε, centre $\mathbf{y}$, and thus the lemma follows from the contraction mapping principle. □

We use this lemma to prove a second lemma, which gives a condition under which a pseudo orbit is shadowed by a true orbit.

LEMMA 7.7. *Let* $\{w_k\}_{k=-\infty}^{+\infty}$ *be a bounded* δ *pseudo orbit of* (53) *with associated times* $\{h_k\}_{k=-\infty}^{+\infty}$ *such that* $\Delta = \inf_{k \in Z} |f(w_k)| > 0$ *and* L_w *is invertible with* $\|L_w^{-1}\| \leqslant K$. *Then there exist constants* C, δ_0 *depending only on* f, K, Δ, $h_{\max}$ *and* $h_{\min}$, *where*

$$h_{\max} = \sup_{k \in Z} h_k, \qquad h_{\min} = \inf_{k \in Z} h_k,$$

such that if $\delta < \delta_0$ *there is a unique true orbit* $\{x_k\}_{k=-\infty}^{+\infty}$ *of* (53) *with associated times* $\{t_k\}_{k=-\infty}^{+\infty}$ *such that for* $k \in Z$

$$f(w_k)^*(x_k - w_k) = 0, \quad |x_k - w_k| \leqslant C\delta, \quad |t_k - h_k| \leqslant C\delta.$$

Moreover, the orbit $\phi^t(x_0)$ *is not an equilibrium and is hyperbolic in the sense of Definition* 4.3.

PROOF. In order to prove the existence and uniqueness of the orbit $\{x_k\}_{k=-\infty}^{+\infty}$, let X be the Banach space $Y_w \times \ell^\infty(Z, \mathbf{R})$ with the maximum norm (note: we use the Euclidean norm in $\mathbf{R}^n$). So we define

$$\|(\mathbf{z}, \mathbf{t})\| = \max\{\|\mathbf{z}\|, \|\mathbf{t}\|\},$$

for $(\mathbf{z}, \mathbf{t}) = (\{z_k\}_{k=-\infty}^{+\infty}, \{t_k\}_{k=-\infty}^{+\infty}) \in X$, where

$$\|\mathbf{z}\| = \sup_{k\in Z} |z_k|, \qquad \|\mathbf{t}\| = \sup_{k\in Z} |t_k|.$$

Then we define $\mathcal{G}: X \to \ell^\infty(Z, \mathbf{R}^n)$ by

$$[\mathcal{G}(\mathbf{z}, \mathbf{t})]_k = w_{k+1} + z_{k+1} - \phi^{t_k}(w_k + z_k).$$

We seek a solution $(\mathbf{z}, \mathbf{t})$ of the equation

$$\mathcal{G}(\mathbf{z}, \mathbf{t}) = \mathbf{0}$$

such that

$$\|\mathbf{z}\| \leqslant C\delta, \qquad \|\mathbf{t} - \mathbf{h}\| \leqslant C\delta,$$

where $\mathbf{h} = \{h_k\}_{k=-\infty}^{\infty}$ and C is to be determined. We apply Lemma 7.6. To this end, note that

$$\|\mathcal{G}(\mathbf{0}, \mathbf{h})\| \leqslant \delta.$$

Next note that the derivative of $\mathcal{G}$ at $(\mathbf{0}, \mathbf{h})$ is given for $(\mathbf{u}, \mathbf{s}) \in X$ by

$$[D\mathbf{G}(\mathbf{0}, \mathbf{h})(\mathbf{u}, \mathbf{s})]_k = u_{k+1} - f(\phi^{h_k}(w_k))s_k - D\phi^{h_k}(w_k)u_k.$$

We approximate this operator by the operator T given by

$$[T(\mathbf{u}, \mathbf{s})]_k = u_{k+1} - f(w_{k+1})s_k - D\phi^{h_k}(w_k)u_k.$$

First we show that T is invertible. To do this, we must show that for all $\mathbf{g} = \{g_k\}_{k=-\infty}^{+\infty}$ in $\ell^\infty(Z, \mathbf{R}^n)$, there is a unique solution of the equation

$$T(\mathbf{u}, \mathbf{s}) = \mathbf{g},$$

that is,

$$u_{k+1} - f(w_{k+1})s_k - D\phi^{h_k}(w_k)u_k = g_k, \quad k \in Z. \tag{55}$$

We can solve (55) for s_k by multiplying both sides by $f(w_{k+1})^*$ to get

$$s_k = -\frac{f(w_{k+1})^*}{|f(w_{k+1})|^2}\left\{D\phi^{h_k}(w_k)u_k + g_k\right\}, \quad k \in Z \tag{56}$$

and we obtain the estimate

$$\|\mathbf{s}\| \leqslant \Delta^{-1}\left[\mathrm{e}^{M_1 h_{\max}}\|\mathbf{u}\| + \|\mathbf{g}\|\right], \tag{57}$$

where

$$M_1 = \sup_{x\in\mathbf{R}^n}\left|f'(x)\right|.$$

Substituting (56) in (55) and rearranging, we obtain the equation

$$(L_w\mathbf{u})_k = \bar{g}_k, \tag{58}$$

where $\mathbf{u} = \{u_k\}_{k=-\infty}^{\infty} \in Y_w$ and the sequence $\bar{\mathbf{g}} = \{\bar{g}_k\}_{k=-\infty}^{\infty}$ is given by

$$\bar{g}_k = g_k - \frac{f(w_{k+1})^* g_k}{|f(w_{k+1})|^2} f(w_{k+1}) = P_{w_{k+1}} g_k, \quad k \in Z,$$

and so is in $\tilde{Y}_w$. Since L_w is invertible, we may solve (58) to get

$$\mathbf{u} = L_w^{-1}\bar{\mathbf{g}}. \tag{59}$$

So (55) has a unique solution $(\mathbf{u}, \mathbf{s})$ given by (59) and (56) with

$$\|\mathbf{u}\| \leqslant \|L_w^{-1}\|\|\bar{\mathbf{g}}\| \leqslant K\|\mathbf{g}\|$$

and $\|\mathbf{s}\|$ satisfying the inequality in (57). It follows that T is invertible with

$$\|T^{-1}\| \leqslant C_1 = \max\left\{\Delta^{-1}(\mathrm{e}^{M_1 h_{\max}}K + 1), K\right\}.$$

Now observe that

$$\left\|D\mathcal{G}(\mathbf{0}, \mathbf{h}) - T\right\| \leqslant M_1\delta.$$

Then if

$$M_1 C_1 \delta \leqslant 1/2,$$

it follows that $D\mathcal{G}(\mathbf{0}, \mathbf{h})$ is also invertible and

$$\left\|D\mathcal{G}(\mathbf{0}, \mathbf{h})^{-1}\right\| \leqslant C/2,$$

where

$$C = 4C_1.$$

Next if $\mathbf{v} = (\{z_k\}_{k=-\infty}^{+\infty}, \{t_k\}_{k=-\infty}^{+\infty})$, $\mathbf{u} = (\{\tau_k\}_{k=-\infty}^{\infty}, \{\xi_k\}_{k=-\infty}^{\infty})$, and $\bar{\mathbf{u}} = (\{\sigma_k\}_{k=-\infty}^{\infty}, \{\eta_k\}_{k=-\infty}^{\infty})$, one calculates, for $k = 0, \dots, N$, that

$$\begin{aligned}\left[D^2\mathcal{G}(\mathbf{v})\mathbf{u}\bar{\mathbf{u}}\right]_k = & -\tau_k\sigma_k f'\big(\varphi^{t_k}(w_k + z_k)\big) f\big(\varphi^{t_k}(w_k + z_k)\big) \\ & - \tau_k f'\big(\varphi^{t_k}(w_k + z_k)\big) D\varphi^{t_k}(w_k + z_k)\eta_k \\ & - \sigma_k f'\big(\varphi^{t_k}(w_k + z_k)\big) D\varphi^{t_k}(w_k + z_k)\xi_k \\ & - D^2\varphi^{t_k}(w_k + z_k)\xi_k\eta_k.\end{aligned}$$

We use the following estimates, which follow easily from Gronwall's lemma and the variation of constants formula

$$\left\|D\varphi^t(\mathbf{x})\right\| \leqslant \mathrm{e}^{M_1 t}, \qquad \left\|D^2\varphi^t(\mathbf{x})\right\| \leqslant M_2 t\mathrm{e}^{2M_1 t}$$

where

$$M_2 = \sup_{x\in\mathbf{R}^n}\left|f''(x)\right|.$$

It follows that

$$\left\|D^2\mathcal{G}(\mathbf{z}, \mathbf{t})\right\| \leqslant M_3, \tag{60}$$

where

$$M_3 = M_0 M_1 + 2M_1\mathrm{e}^{M_1(h_{\max}+C\delta)} + M_2(h_{\max} + C\delta)\mathrm{e}^{2M_1(h_{\max}+C\delta)}.$$

Then provided

$$M_3 C^2\delta < 2,$$

Lemma 7.6 can be applied and we get a unique solution $(\mathbf{z}, \mathbf{t})$ of $\mathcal{G}(\mathbf{z}, \mathbf{t}) = 0$ with $\|\mathbf{z}\| \leqslant C\delta$, $\|\mathbf{t}\| \leqslant C\delta$. Then with $x_k = w_k + z_k$, the first part of the lemma follows.

In the rest of the proof we also assume that

$$M_1 C\delta < \Delta.$$

Then we observe that

$$\left|f(x_k)\right| \geqslant \left|f(w_k)\right| - \left|f(w_k) - f(x_k)\right| \geqslant \Delta - M_1 C\delta > 0$$

so that the orbit $\phi^t(x_0)$ is not an equilibrium.

Now we show that $\phi^t(x_0)$ is hyperbolic. First note that the derivative of $\mathcal{G}$ at $(\mathbf{z}, \mathbf{t})$, where $\mathbf{z} = \{x_k - w_k\}_{k=-\infty}^{\infty}$, is given for $(\mathbf{u}, \mathbf{s}) \in X$ by

$$\left[D\mathbf{G}(\mathbf{z}, \mathbf{t})(\mathbf{u}, \mathbf{s})\right]_k = u_{k+1} - f(x_{k+1})s_k - D\phi^{t_k}(x_k)u_k.$$

In view of (60),

$$\left\| D\mathbf{G}(\mathbf{z}, \mathbf{t}) - D\mathbf{G}(\mathbf{0}, \mathbf{h})\right\| \leqslant M_3 \left\|(\mathbf{z}, \mathbf{t}) - (\mathbf{0}, \mathbf{h})\right\| \leqslant M_3 C\delta.$$

Since $\|D\mathbf{G}(\mathbf{0}, \mathbf{h})^{-1}\| \leqslant C/2$ and $M_3 C\delta(C/2) < 1$, it follows that $D\mathbf{G}(\mathbf{z}, \mathbf{t})$ is invertible. In particular, this implies that for all bounded sequences $g_k \in Y_{w_{k+1}}$, the equation

$$v_{k+1} - f(x_{k+1})s_k - D\phi^{t_k}(x_k)v_k = g_k, \quad k \in Z \tag{61}$$

has a unique bounded solution $v_k \in Y_{w_k}$ and $s_k \in \mathbf{R}$. Note that

$$f(w_{k+1})^* f(x_{k+1}) \geqslant \left|f(w_{k+1})\right|\left[\left|f(w_{k+1})\right| - M_1 C\delta\right] > 0.$$

Multiplying (61) by $f(w_{k+1})^*$, we obtain

$$s_k = -\frac{f(w_{k+1})^* D\phi^{t_k}(x_k)v_k}{f(w_{k+1})^* f(x_{k+1})}, \quad k \in Z. \tag{62}$$

Then substituting this back in (61), we obtain

$$v_{k+1} = P_{x_{k+1}} D\phi^{t_k}(x_k)v_k + g_k, \quad k \in Z.$$

The upshot is that for all bounded sequences $g_k \in Y_{w_{k+1}}$, the equation

$$v_{k+1} - P_{x_{k+1}} D\phi^{t_k}(x_k)v_k = g_k \tag{63}$$

has a bounded solution $v_k \in Y_{w_k}$. Moreover, it must be unique for if v_k is a bounded solution of (63) and we define s_k by (62), then we see that s_k and v_k form a bounded solution of (61). So it follows that the difference equation

$$v_{k+1} = P_{x_{k+1}} D\phi^{t_k}(x_k)v_k \tag{64}$$

has an exponential dichotomy on Z (see [12,24]).

Now we want to apply Proposition 2.9 with $A(t) = f'(\phi^t(x_0))$ so that $X(t) = D\phi^t(x_0)$, $x_0(t) = f(\phi^t(x_0))$ and with $Q_k = P_{x_k}$ and s_k defined recursively by $s_0 = 0$ and $s_{k+1} = s_k + t_k$. Now, provided that

$$C\delta < h_{\min},$$

we have

$$0 < h_{\min} - C\delta \leqslant t_k \leqslant h_{\max} + C\delta < \infty.$$

Also note that $|f(\phi^t(x_0))| \leqslant M_0$ for all t. Next given any real t there exists k such that $s_k \leqslant t < s_{k+1}$. Then

$$\begin{aligned}\left|f(x_k)\right| = \left|f\left(\phi^{s_k}(x_0)\right)\right| &= \left|D\phi^{s_k - t}\left(\phi^t(x_0)\right)f\left(\phi^t(x_0)\right)\right| \\ &\leqslant \mathrm{e}^{M_1(h_{\max}+C\delta)}\left|f\left(\phi^t(x_0)\right)\right|\end{aligned}$$

so that since $0 < \Delta - M_1C\delta \leqslant |f(x_k)|$ it follows that for all t

$$\left|f\left(\phi^t(x_0)\right)\right| \geqslant \mathrm{e}^{-M_1(h_{\max}+C\delta)}(\Delta - M_1C\delta) > 0.$$

Then, since (64) can be written as $v_{k+1} = Q_{k+1}X(s_{k+1})X^{-1}(s_k)v_k$ with $v_k \in \mathcal{R}(Q_k)$, it follows from Proposition 2.9 that

$$\dot{x} = f'\left(\phi^t(x_0)\right)x$$

has a trichotomy and so by Proposition 4.4 the solution $\phi^t(x_0)$ is hyperbolic.

This completes the proof of the lemma with δ_0 as the largest $\delta > 0$ such that

$$M_3C^2\delta \leqslant 2, \quad M_1C_1\delta \leqslant 1/2, \quad M_1C\delta \leqslant \Delta, \quad C\delta \leqslant h_{\min}. \tag{65}$$

□

In the next lemma we show under a uniqueness condition that an orbit which shadows a pseudo homoclinic orbit is asymptotic to it.

LEMMA 7.8. *Let $\{w_k\}_{k=-\infty}^{+\infty}$ be a pseudo orbit of* (53) *with associated times $\{h_k\}_{k=-\infty}^{+\infty}$ and suppose it is ε shadowed by a true orbit $\{z_k\}_{k=-\infty}^{+\infty}$ with associated times $\{s_k\}_{k=-\infty}^{+\infty}$ such that for all k*

$$f(w_k)^*(z_k - w_k) = 0.$$

Next let $\{y_k\}_{k=-\infty}^{+\infty}$ be a pseudo periodic orbit with associated times $\{\ell_k\}_{k=-\infty}^{+\infty}$ and suppose it is ε shadowed by a unique true orbit $\{x_k\}_{k=-\infty}^{+\infty}$ with associated times $\{t_k\}_{k=-\infty}^{+\infty}$ such that for all k

$$f(y_k)^*(x_k - y_k) = 0.$$

Suppose also that

$$w_k = y_k, \quad h_k = \ell_k \quad \textit{for } k \geqslant q.$$

Then

$$|z_k - x_k| + |s_k - t_k| \to 0 \quad \textit{as } k \to \infty.$$

PROOF. Let $\{y_k\}_{k=-\infty}^{+\infty}$ have period N. Since for all k

$$f(y_k)^*(x_{k+N} - y_k) = f(y_{k+N})^*(x_{k+N} - y_{k+N}) = 0,$$

and

$$|x_{k+N} - y_k| = |x_{k+N} - y_{k+N}| \leqslant \varepsilon, \qquad |t_{k+N} - \ell_k| = |t_{k+N} - \ell_{k+N}| \leqslant \varepsilon$$

also, it follows by uniqueness that $x_{k+N} = x_k$ and $t_{k+N} = t_k$ for all k.
Define for $j \geqslant 0$ and all k

$$z_k^{(j)} = z_{k+jN}, \qquad s_k^{(j)} = s_{k+jN}.$$

We claim that for any fixed k

$$z_k^{(j)} \to x_k, \qquad s_k^{(j)} \to t_k$$

as $j \to \infty$. Suppose not. Then there exists $i, \eta > 0$ and a sequence $j_r \to \infty$ as $r \to \infty$ such that

$$|z_i^{(j_r)} - x_i| + |s_i^{(j_r)} - t_i| \geqslant \eta \tag{66}$$

for all r. By taking a further subsequence if necessary, we can assume that for each k

$$z_k^{(j_r)} \to \tilde{x}_k, \qquad s_k^{(j_r)} \to \tilde{t}_k$$

as $r \to \infty$. Now, for all k and r, $\phi^{s_k^{(j_r)}}(z_k^{(j_r)}) = z_{k+1}^{(j_r)}$ and for fixed k, if r is large enough,

$$|z_k^{(j_r)} - y_k| = |z_k^{(j_r)} - y_{k+j_rN}| = |z_{k+j_rN} - w_{k+j_rN}| \leqslant \varepsilon$$

and, similarly,

$$|s_k^{(j_r)} - \ell_k| \leqslant \varepsilon.$$

Also for fixed k, if r is large enough,

$$f(y_k)^*(z_k^{(j_r)} - y_k) = 0.$$

So, letting $r \to \infty$, we obtain that for all k, $\phi^{\tilde{t}_k}(\tilde{x}_k) = \tilde{x}_{k+1}$ and

$$|\tilde{x}_k - y_k| \leqslant \varepsilon, \quad |\tilde{t}_k - \ell_k| \leqslant \varepsilon, \quad f(y_k)^*(\tilde{x}_k - y_k) = 0.$$

By uniqueness, it follows that for all k, $\tilde{x}_k = x_k$ and $\tilde{t}_k = t_k$. Hence for each k

$$z_k^{(j_r)} \to x_k, \qquad s_k^{(j_r)} \to t_k$$

as $r \to \infty$. This contradicts (66). So we deduce that for each k

$$z_{k+jN} = z_k^{(j)} \to x_k \quad \text{and} \quad s_{k+jN} = s_k^{(j)} \to t_k \quad \text{as } j \to \infty.$$

It follows that

$$z_k - x_k \to 0 \quad \text{and} \quad s_k - t_k \to 0 \quad \text{as } k \to \infty. \qquad \square$$

PROOF OF THEOREM 7.5. We take δ_0 to be the largest δ satisfying the inequalities in (65) with $K = \|L_w^{-1}\|$.

Now let Y_y be the Banach space of bounded sequences $\mathbf{v} = \{v_k\}_{k \in Z}$ with $v_k \in \mathbf{R}^n$ and orthogonal to $f(y_k)$, Y_y being equipped with the norm

$$\|\mathbf{v}\| = \sup_{k \in Z} |v_k|$$

and let $\tilde{Y}_y$ be a similar Banach space but with v_k orthogonal to $f(y_{k+1})$. Then let $L_y : Y_y \to \tilde{Y}_y$ be the linear operator defined by

$$(L_y \mathbf{v})_k = v_{k+1} - P_{y_{k+1}} D\phi^{\ell_k}(y_k) v_k,$$

where $\{\ell_k\}_{k=-\infty}^{+\infty}$ are the times associated with $\{y_k\}_{k=-\infty}^{+\infty}$. By hypothesis, L_w is invertible. We now show that L_y is invertible and that

$$\|L_y^{-1}\| \leqslant \|L_w^{-1}\|.$$

Let $\mathbf{g} = \{g_k\}_{-\infty}^{\infty} \in \tilde{Y}_y$ and r a natural number. Then, since L_w is invertible, the difference equation

$$v_{k+1} = P_{w_{k+1}} D\phi^{h_k}(w_k) v_k + P_{w_{k+1}} g_{k-rN}, \quad k \in Z$$

has a solution $v_k^{(r)} \in Y_{w_k}$ such that for all k

$$|v_k^{(r)}| \leqslant \|L_w^{-1}\| \|\mathbf{g}\|.$$

Take

$$u_k^{(r)} = v_{k+rN}^{(r)}.$$

Then $u_k^{(r)} \in Y_{w_{k+rN}}$ is a solution of

$$u_{k+1}^{(r)} = P_{w_{k+rN+1}} D\phi^{h_{k+rN}}(w_{k+rN}) u_k^{(r)} + P_{w_{k+rN+1}} g_k, \quad k \in Z \tag{67}$$

such that for all k

$$|u_k^{(r)}| \leqslant \|L_w^{-1}\| \|\mathbf{g}\|. \tag{68}$$

Next, by Cantor's diagonalization procedure, we can find a subsequence $u_k^{(j_r)} \to \bar{u}_k$, where $\bar{u}_k$ is orthogonal to $f(y_k)$, as $r \to \infty$ for each k. Then letting $r \to \infty$ in (67) and (68) (with r replaced by j_r), we obtain

$$\bar{u}_{k+1} = P_{y_{k+1}} D\phi^{\ell_k}(y_k)\bar{u}_k + g_k, \quad k \in Z$$

and

$$|\bar{u}_k| \leqslant \|L_w^{-1}\| \|\mathbf{g}\| \tag{69}$$

for all k. In particular, since $\mathbf{g}$ is arbitrary, this means that the difference equation

$$u_{k+1} = P_{y_{k+1}} D\phi^{\ell_k}(y_k) u_k + g_k$$

has a solution u_k bounded on $[0, \infty)$, where u_k is orthogonal to $f(y_k)$, whenever g_k is a sequence bounded on $[0, \infty)$ with g_k orthogonal to $f(y_{k+1})$. It follows from the discrete analogue of Proposition 3 of Lecture 3 in [5] that the difference equation

$$u_{k+1} = P_{y_{k+1}} D\phi^{\ell_k}(y_k) u_k,$$

where u_k is orthogonal to $f(y_k)$, has an exponential dichotomy on $[0, \infty)$. However, then it follows from the discrete analogue of the periodic case of Proposition 3 in Lecture 8 of [5] that the equation has an exponential dichotomy on Z. This, in turn, means that L_y is invertible so that the equation

$$u_{k+1} = P_{y_{k+1}} D\phi^{\ell_k}(y_k) u_k + g_k, \quad k \in Z$$

has the unique bounded solution $(L_y^{-1}\mathbf{g})_k$ orthogonal to $f(y_k)$. Thus $\bar{u}_k = (L_y^{-1}\mathbf{g})_k$ and it follows from (69) that

$$\|L_y^{-1}\| \leqslant \|L_w^{-1}\|.$$

Then it follows from Lemma 7.7 applied to $\{y_k\}_{k=-\infty}^{+\infty}$ and $\{\ell_k\}_{k=-\infty}^{+\infty}$ (note that $h_{\min} \leqslant \ell_k \leqslant h_{\max}$ and that $|f(y_k)| \geqslant \Delta$) that provided $\delta < \delta_0$ there is a unique orbit $\{x_k\}_{k=-\infty}^{+\infty}$ of (53) with associated times $\{t_k\}_{k=-\infty}^{+\infty}$ such that for all k

$$f(y_k)^*(x_k - y_k) = 0, \quad |x_k - y_k| \leqslant C\delta, \quad |t_k - \ell_k| \leqslant C\delta.$$

However, since for all k

$$f(y_k)^*(x_{k+N} - y_k) = f(y_{k+N})^*(x_{k+N} - y_{k+N}) = 0,$$

and

$$|x_{k+N} - y_k| = |x_{k+N} - y_{k+N}| \leqslant C\delta, \qquad |t_{k+N} - \ell_k| = |t_{k+N} - \ell_{k+N}| \leqslant C\delta$$

also, it follows by uniqueness that $x_{k+N} = x_k$ and $t_{k+N} = t_k$ for all k. Hence $\phi^t(x_0)$ has period $t_0 + t_1 + \cdots + t_{N-1}$. Also note it follows from Lemma 7.7 that x_0 is not an equilibrium and that $\phi^t(x_0)$ is hyperbolic.

Next, by Lemma 7.7 applied to $\{w_k\}_{k=-\infty}^{+\infty}$ and $\{h_k\}_{k=-\infty}^{+\infty}$, provided $\delta < \delta_0$, there is a unique orbit $\{z_k\}_{k=-\infty}^{\infty}$ of (53) with associated times $\{s_k\}_{k=-\infty}^{+\infty}$ such that for all k

$$f(w_k)^*(z_k - w_k) = 0, \quad |z_k - w_k| \leqslant C\delta, \quad |s_k - h_k| \leqslant C\delta.$$

Again $\phi^t(z_0)$ is hyperbolic.

Next it follows from Lemma 7.8 that

$$z_k - x_k \to 0 \quad \text{and} \quad s_k - t_k \to 0 \quad \text{as } k \to \infty.$$

Now for all k

$$s_k = h_k + s_k - h_k \geqslant h_{\min} - C\delta > 0.$$

We denote by $O(x_0)$ the orbit of x_0. Then, if $0 \leqslant t \leqslant s_k$,

$$\begin{aligned} \operatorname{dist}\bigl(\phi^t(z_k), O(x_0)\bigr) &\leqslant \bigl|\phi^t(z_k) - \phi^t(x_k)\bigr| \\ &\leqslant \mathrm{e}^{M_1 t}|z_k - x_k| \\ &\leqslant \mathrm{e}^{M_1(h_{\max}+C\delta)}|z_k - x_k|. \end{aligned}$$

It follows that $\operatorname{dist}(\phi^t(z_0), O(x_0)) \to 0$ as $t \to \infty$.

Similarly, we prove that $\operatorname{dist}(\phi^t(z_0), O(x_0)) \to 0$ as $t \to -\infty$.

Finally to show that z_0 does not lie on the periodic orbit, we first note that for $0 \leqslant t \leqslant t_k$

$$\phi^t(x_k) - x_k = \int_0^t f\bigl(\phi^s(x_k)\bigr)\,\mathrm{d}s$$

so that

$$\bigl|\phi^t(x_k) - x_k\bigr| \leqslant \bigl|f(x_k)\bigr|t + \int_0^t \bigl|f\bigl(\phi^s(x_k)\bigr) - f(x_k)\bigr|\,\mathrm{d}s$$

and hence for $0 \leqslant t \leqslant t_k$

$$\left|\phi^t(x_k) - x_k\right| \leqslant \left|f(x_k)\right|t + M_1 \int_0^t \left|\phi^s(x_k) - x_k\right| \mathrm{d}s.$$

Then it follows from Gronwall's lemma that for $0 \leqslant t \leqslant t_k$

$$\left|\phi^t(x_k) - x_k\right| \leqslant \left|f(x_k)\right| \frac{\mathrm{e}^{M_1 t} - 1}{M_1}.$$

Then for $0 \leqslant t \leqslant t_k$, $k = 0, \ldots, N-1$,

$$\begin{aligned}
\left|z_r - \phi^t(x_k)\right| &\geqslant |w_r - y_k| - \left|\phi^t(x_k) - x_k\right| - |x_k - y_k| - |w_r - z_r| \\
&> \left(\left|f(y_k)\right| + M_1 C\delta\right) \frac{\mathrm{e}^{M_1(h_{\max} + C\delta)} - 1}{M_1} + 2C\delta \\
&\quad - \left|f(x_k)\right| \frac{\mathrm{e}^{M_1 t} - 1}{M_1} - 2C\delta \\
&\geqslant 0,
\end{aligned}$$

where we have used (54). Hence z_r does not lie on the periodic orbit. So the point z_0 is indeed a transversal homoclinic point to the periodic orbit $\phi^t(x_0)$, where we note that the transversality follows from the hyperbolicity.

Thus the proof of the theorem is completed. $\square$

EXAMPLE 7.9. In [4], Theorem 7.5 (with some refinement of the estimates) is used to show that the *Lorenz Equations*

$$\begin{aligned}
\dot{x}_1 &= \sigma(x_2 - x_1), \\
\dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3, \\
\dot{x}_3 &= x_1 x_2 - \alpha x_3
\end{aligned}$$

with the classic parameter values

$$\sigma = 10.0, \quad \rho = 28.0, \quad \beta = 8/3$$

possess a hyperbolic periodic orbit with an associated transversal homoclinic orbit.

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CHAPTER 5

Successive Approximation Techniques in Non-Linear Boundary Value Problems for Ordinary Differential Equations

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Contents

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HANDBOOK OF DIFFERENTIAL EQUATIONS
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Abstract

In this work, we investigate the solvability and the approximate construction of solutions of certain types of regular non-linear boundary value problems for systems of ordinary differential equations on a compact interval. For this purpose, we construct analytically a uniformly convergent parametrised sequence of functions depending on the properties of the concrete boundary conditions and non-linearities in the given systems. The value of the parameter introduced artificially into the scheme is to be determined by solving a certain system of algebraic or transcendental equations.

The text is divided into 10 sections. Sections 1 and 2 contain the notation used in what follows and provide a short introduction. In Section 3, the successive approximation techniques are treated for the investigation of periodic solutions of non-autonomous periodic systems. In Section 4, we apply the method for the study the periodic solutions of autonomous systems by using the appropriate reduction to a non-autonomous system. In Section 5, we establish conditions under which a system of non-linear non-autonomous ordinary differential equations has a family of solutions that are periodic with a common period and possess a certain symmetry property. Sections 6 and 7 deal with the investigation of non-linear two-point boundary value conditions by using a parametrisation that leads one to a family of problem with linear two-point conditions considered together with certain additional algebraic or transcendental equations with respect to certain parameters. In Section 8, we use the parametrisation approach to study some three-point non-linear boundary value problem which, as a result, can be investigated through auxiliary two-point problems. Most of theoretical results are illustrated by examples. Section 9 contains some historical remarks concerning the development and application of the method. Finally, in Section 10, we give several exercises concerning the successive approximation technique under consideration.

1. Notation

(1) $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{N} = \{1, 2, \ldots\}$.
(2) C_T^n is the Banach space of continuous vector-functions $x = (x_k)_{k=1}^n : (-\infty, \infty) \to \mathbb{R}^n$ periodic with period T (i.e., $x(t) = x(t+T)$ for any $t \in (-\infty, \infty)$), with the usual norm

$$x \longmapsto \max_{k=1,2,\ldots,n} \max_{t \in [0,T]} \left| x_k(t) \right|. \tag{1.1}$$

(3) $C([0,T] \times D, \mathbb{R}^n)$, where $D \subset \mathbb{R}^n$ is a compact set, stands for the Banach space of continuous vector-functions $x = (x_k)_{k=1}^n : [0,T] \to \mathbb{R}^n$ with norm (1.1).
(4) $L_1([0,T], \mathbb{R}^n)$ is the Banach space of the Lebesgue integrable functions $x = (x_k)_{k=1}^n : (-\infty, \infty) \to \mathbb{R}^n$ with the norm

$$x \longmapsto \max_{k=1,2,\ldots,n} \int_0^T \left| x_k(t) \right| \mathrm{d}t.$$

(5) $\mathrm{GL}_n(\mathbb{R})$ is the algebra of square real matrices of dimension n.
(6) $\mathbb{1}_n$ is the n-dimensional unit matrix.
(7) The symbol $r(A)$ stands for the spectral radius of a square matrix A.
(8) I is the identity operator in various spaces.
(9) $\partial\Omega$ is the boundary of a set $\Omega \subset \mathbb{R}^n$.
(10) If $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^n$, and $\{\alpha_1, \alpha_2\} \subset \mathbb{R}$, then $\alpha_1\Omega_1 + \alpha_2\Omega_2 := \{\alpha_1 x_1 + \alpha_2 x_2 \mid x_1 \in \Omega_1,\ x_2 \in \Omega_2\}$.
(11) For a point $x = \mathrm{col}(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we denote by $|x|$ the vector $|x| = \mathrm{col}(|x_1|, |x_2|, \ldots, |x_n|)$. Similarly, the signs $|\cdot|$, max, min, and $\leqslant$ for vectors are understood componentwise.
(12) Let $\beta = \mathrm{col}(\beta_1, \ldots, \beta_n)$ be a vector from $\mathbb{R}_+^n$. By the β-neighbourhood of a point $y \in \mathbb{R}^n$ we understand the set $B(y, \beta)$ of points defined as follows:

$$B(y, \beta) := \left\{ x \in \mathbb{R}^n \mid |x - y| \leqslant \beta \right\}. \tag{1.2}$$

(13) For any vector $\beta \in \mathbb{R}_+^n$, by D_β, we mean the set

$$D_\beta := \left\{ x \in D \mid B(x, \beta) \subset D \right\} \tag{1.3}$$

that consists of points $x \in \mathbb{R}^n$ lying in D together with their β-neighbourhoods.

The symbol $\square$ denotes the end of a proof.

2. Introduction

In this work, we investigate several types of regular boundary value problems for the system of ordinary differential equations

$$x'(t) = f\big(t, x(t)\big), \quad t \in [0, T], \tag{2.1}$$

where $T \in (0, +\infty)$, $f \in C([0, T] \times D, \mathbb{R}^n)$. Here, D is the closure of a connected and bounded domain in $\mathbb{R}^n$. We are interested in continuously differentiable solutions of (2.1) which satisfy either the periodic conditions

$$x(0) = x(T) \tag{2.2}$$

or certain other, more complicated kinds of two or three-point boundary conditions, both linear and non-linear ones.

The singular problems are not dealt with in this text. We refer, in particular, to the work of Agarwal and O'Regan [2] for a survey of recent results on problems for Eqs. (2.1) having singularities with respect to the independent variable.

Boundary value problems for higher order scalar equations with both time and space singularities are discussed in the work of Rachůnková, Staněk, and Tvrdý [59]. The techniques used there are based on construction of certain sequences of regular problems that are assumed to be solvable and whose solutions can be regarded as approximations of the solution in question. Certain singular two-point problems with separated boundary conditions are studied in [59] by using the method of upper and lower functions.

Singular boundary value problems for systems (2.1) in various complicated cases are studied by Kiguradze [36] by using techniques based upon properties of differential inequalities.

The solvability of various boundary value problems by using fixed point theorems is studied, in particular, by Ntouyas [52]. The works cited also contain an extensive bibliography on the subject.

In this text, we investigate systems (2.1) subjected to the boundary conditions

$$g\big(x(0), x(T)\big) = 0, \tag{2.3}$$

which are, generally speaking, non-separated and non-linear. Conditions (2.3), obviously, cover the periodic conditions (2.2), the separated non-linear conditions

$$x(0) = b\big(x(T)\big), \tag{2.4}$$

and so on. We also study the three-point problems with the conditions

$$Ax(0) + Bx(\xi) + Cx(T) = d. \tag{2.5}$$

In order to show the existence of a solution and provide an efficient algorithm for its approximate finding, we construct a certain uniformly convergent sequence of functions that depends essentially on the character of concrete non-linearities in system (2.1) and on the form of the given boundary conditions. This approach may be regarded as a generalisation of the Picard successive approximation scheme.

In the original works of Samoilenko [82,83] and Samoilenko and Ronto [88–90], the approach used and described here had been referred to as the numerical-analytic method based upon successive approximations. The idea of the method, originally aimed at the investigation of periodic solutions only, had been later applied in studies of numerous types

of boundary value problems. The monograph of Farkas [17], in particular, refers to it as to a relatively new, powerful, and popular means for the investigation of periodic solutions of non-linear periodic systems of ordinary differential equations. The merit of this method is that it gives an opportunity to solve two main questions of the theory of periodic motions, namely, to establish existence theorems and to construct approximate solutions.

It should be noted that appropriate versions of the method considered can be applied in many situations for handling periodic or two-point non-linear boundary value problems in the case of systems of first or second order ordinary differential equations, integro-differential equations, equations with retarded argument and equations with more general argument deviations, systems containing unknown parameters, and countable systems of differential equations. A survey of the investigations on the subject can be found in the series of papers by Ronto, Samoilenko, and Trofimchuk [74,76,77,75,78–80] where the achievements and trends in the development of the method for the period of about 40 years are described. This text is devoted to applications of the method to the existence analysis and approximate finding of solutions of periodic and some other boundary value problems for ordinary differential equations.

In what follows, it is convenient to use the term “periodic boundary value problem” rather than “T-periodic solution of a T-periodic system of differential equations” and, instead of $\mathbb{R}$, consider the segment $[0, T]$ as the domain of definition of the equation.

The scheme of the method under consideration is rather simple. According to the basic idea of the method, the given periodic boundary value problem is replaced by the Cauchy problem for a “perturbed” differential equation containing some artificially introduced parameter, whose value is to be determined later. The solution of this family of systems of differential equations parametrised by a single vector parameter is sought for in the analytic form by the method of successive approximations. Thus, all the iterations depend upon the parameter mentioned.

As regards the way how the auxiliary problem is constructed, it is essential that the form of the “perturbation term” generates a certain system of (algebraic or transcendental) “determining equations,” which give the numerical values of the parameter corresponding to the solutions sought for. By studying these determining equations, we can establish existence results for the original problem.

Thus, the method consists of two main parts, namely, the “analytic” part, i.e., solving the auxiliary problem for the “perturbed” equation by the method of successive approximations, and then the “numerical ”part, when the determining equations are solved.

The text is organised as follows. Sections 1 and 2 contain the notation used in what follows and provide a short introduction.

In Section 3, the successive approximation techniques are treated for the investigation of periodic solutions of non-autonomous periodic systems.

In Section 4, we apply the method for the study the periodic solutions of autonomous systems by using the appropriate reduction to a non-autonomous system.

In Section 5, we establish conditions under which a system of non-linear non-autonomous ordinary differential equations has a family of solutions that are periodic with a common period and possess a certain generalised symmetry property.

Sections 6 and 7 deal with the investigation of non-linear boundary problems by using a parametrisation that leads one to a family of problems with linear two-point conditions considered together with certain additional algebraic or transcendent equations.

In Section 8, we use the parametrisation approach to study some three-point boundary value problems, which, as a result, can be investigated through auxiliary two-point problems.

Section 9 contains some historical remarks concerning the development and application of the method.

In Section 10, we pose several problems concerning the method of successive approximations.

3. Periodic successive approximations for non-autonomous systems

Let us consider the system of ordinary differential equations

$$x'(t) = f\big(t, x(t)\big), \quad t \in (-\infty, \infty), \tag{3.1}$$

where the right-hand side function f is such that $f \in C([0, T] \times D, \mathbb{R}^n)$ and

$$f(t, x) = f(t + T, x) \tag{3.2}$$

for all $x \in D$ and all real t. Here, T is a positive number and D is the closure of a bounded and connected domain in $\mathbb{R}^n$.

We are interested in continuously differentiable solutions of system (3.1) which are periodic with period T (i.e., belonging to C_T^n). It follows from Lemma 3.1 below that this problem is equivalent to finding a solution of the periodic boundary value problem (2.1), (2.2) on the bounded interval $[0, T]$:

$$x'(t) = f\big(t, x(t)\big), \quad t \in [0, T], \tag{3.3}$$

$$x(0) = x(T). \tag{3.4}$$

LEMMA 3.1. (See, e.g., [17, Lemma 2.2.1].) *If a function $x : (-\infty, \infty) \to D$ is a T-periodic solution of system* (3.1)*, then its restriction $x|_{[0,T]}$ to* $[0, T]$ *satisfies system* (3.3), (3.4). *Conversely, if x: $[0, T] \to D$ is a solution of the boundary value problem* (3.3), (3.4)*, then its continuous T-periodic extension to $(-\infty, \infty)$ is a T-periodic solution of system* (3.1).

The solution of the periodic boundary value problem (3.3), (3.4) can be considered as a restriction of the T-periodic solution of the T-periodic system (3.1) onto the segment $[0, T]$. Thus, below we consider the segment $[0, T]$ instead of $\mathbb{R}$ in the domain of definition (3.2): $f \in C([0, T] \times D, \mathbb{R}^n)$.

It is assumed that the following conditions hold for the periodic boundary value problem (3.3), (3.4):

(1) For any $t \in [0, T]$ fixed, the function $f(t, \cdot)$ for all $\{x_1, x_2\} \subset D$ satisfies the Lipschitz condition

$$\bigl| f(t, x_1) - f(t, x_2) \bigr| \leqslant K |x_1 - x_2| \tag{3.5}$$

with some non-negative matrix $K = \{K_{ij}\}_{i,j=1}^{n} \in \mathrm{GL}_n(\mathbb{R})$ and, furthermore, the estimate

$$\frac{1}{2}\Bigl[\max_{(t,x)\in[0,T]\times D} f(t, x) - \min_{(t,x)\in[0,T]\times D} f(t, x) \Bigr] \leqslant M \tag{3.6}$$

is true with some $M \in \mathbb{R}_+^n$.

(2) The maximal eigenvalue $r(K)$ of the matrix K in (3.5) is such that

$$r(K) < \frac{10}{3T}. \tag{3.7}$$

(3) The set $D_{\frac{T}{2}M}$ is non-empty:

$$D_{\frac{T}{2}M} \neq \varnothing \tag{3.8}$$

Recall that the set $D_{\frac{T}{2}M}$ is defined according to notation (13), Section 1. It is clear that the form of this set essentially depends on the properties of the function f and the structure of the domain D.

REMARK 3.2. Inequality (3.7) is equivalent to the condition that the greatest eigenvalue of the matrix

$$Q = \frac{T}{q} K, \tag{3.9}$$

where

$$q = \frac{10}{3}, \tag{3.10}$$

is less than 1.

REMARK 3.3. By virtue of the non-negativity of K, it follows from the Perron theorem (see, e.g., Krein and Rutman [38]) that the largest eigenvalue of this matrix is real, non-negative, and is estimated from above by the number

$$\min\Biggl\{ \max_{i=1,\dots,n} \sum_{j=1}^{n} K_{ij}, \max_{j=1,\dots,n} \sum_{i=1}^{n} K_{ij} \Biggr\}.$$

In (1.2), as well as in (3.6), (3.5) and similar relations below, the signs $|\cdot|$, max, min, and $\leqslant$ are understood componentwise. Note that the componentwise estimate of form (3.5), (3.6), etc., in general, are more accurate than the corresponding estimates in terms of the norm because they use the algebraic structure of the space $\mathbb{R}^n$ in a more subtle way.

REMARK 3.4. One can verify (see Remark 2 in Rontó and Mészáros [69]) that the inequality

$$\frac{1}{2}\Big[\max_{(t,x)\in[0,T]\times D} f(t,x) - \min_{(t,x)\in[0,T]\times D} f(t,x)\Big] \leqslant \max_{(t,x)\in[0,T]\times D}\big|f(t,x)\big|, \tag{3.11}$$

is true. Moreover, the equality

$$\frac{1}{2}\Big[\max_{(t,x)\in[0,T]\times D} f(t,x) - \min_{(t,x)\in[0,T]\times D} f(t,x)\Big] = \max_{(t,x)\in[0,T]\times D}\big|f(t,x)\big|$$

holds if, and only if

$$\max_{(t,x)\in[0,T]\times D} f(t,x) = -\min_{(t,x)\in[0,T]\times D} f(t,x) = \max_{(t,x)\in[0,T]\times D}\big|f(t,x)\big|.$$

EXAMPLE 3.5. For the function

$$f(t,x) = \frac{1}{2}\ln x \sin t + 12$$

defined for (t,x) from the domain $[0,T]\times D = [0,2\pi]\times[\mathrm{e},\mathrm{e}^2] =: G$, estimate (3.6) is sharper than the inequality $|f(t,x)| \leqslant M$. Indeed, it easy to see that

$$\max_{(t,x)\in[0,T]\times D}\big|f(t,x)\big| = 13$$

and

$$\frac{1}{2}\Big[\max_{(t,x)\in[0,T]\times D} f(t,x) - \min_{(t,x)\in[0,T]\times D} f(t,x)\Big] = \frac{1}{2}[13-11] = 1.$$

Taking into account that the vector M and the matrix K in estimates (3.6), (3.5) depend on the domain D, it may happen that conditions (3.6)–(3.8) for system (3.1) are satisfied in some domain D (or some subdomains of D) and does not hold in another one. Let us consider some examples to illustrate this fact.

EXAMPLE 3.6. Assume that the function f in (3.1) satisfies conditions (3.6), (3.5) for $(t,x)\in\mathbb{R}\times\mathbb{R}^n$.

If inequality (3.7) is satisfied with some Lipschitz matrix K, then condition (3.8) also holds in any closed bounded domain D containing an n-dimensional rectangle with the side length componentwise greater than $\frac{T}{2}M$.

EXAMPLE 3.7. Consider the case where (3.1) is a linear system,

$$x'(t) = P(t)x + q(t), \tag{3.12}$$

where $q \in C_T^n$ and $P = \{P_{ij}\}_{i,j=1}^n$, P_{ij}, $i, j = 1, 2, \dots, n$, are continuous T-periodic functions.

Let us put $\hat{P} := \{\max_{t\in[0,T]} |P_{ij}(t)|\}_{i,j=1}^n$,

$$\hat{q} := \operatorname{col}\Bigl(\max_{t\in[0,T]}\bigl|q_1(t)\bigr|, \dots, \max_{t\in[0,T]}\bigl|q_n(t)\bigr|\Bigr)$$

and consider system (3.12) in the domain

$$D := \bigl\{x \mid |x| \leqslant b\bigr\}, \tag{3.13}$$

where $b = \operatorname{col}(b_1, \dots, b_n)$ is such that conditions (3.6)–(3.8) hold for system (3.12).

We can put $K = \hat{P}$ in (3.5). According to (3.11), condition (3.6) takes the form

$$\bigl|f(t,x)\bigr| \leqslant \bigl|P(t)x + q(t)\bigr| \leqslant \hat{P}b + \hat{q} =: M.$$

Condition (3.8) holds in domain (3.13) if the inequality

$$\hat{q} \leqslant \frac{2}{T}\left(\mathbb{1}_n - \frac{T}{2}\hat{P}\right)b.$$

is true. In particular, when $r(\hat{P}) < 2T^{-1}$, we can put

$$b = \frac{T}{2}\left(\mathbb{1}_n - \frac{T}{2}\hat{P}\right)\hat{q}.$$

Condition (3.7) holds whenever the greatest eigenvalue of the matrix $\hat{P}$ does not exceed $\frac{10}{3T}$.

EXAMPLE 3.8. Let us consider the system of equations

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= -x_1(t) - \sin(20t)\bigl(x_2(t)\bigr)^s + \cos(20t) \end{aligned} \tag{3.14}$$

for $(t, x) \in [0, \frac{\pi}{10}] \times D$, where

$$D := \bigl\{(x_1, x_2) \mid |x_1| \leqslant a, |x_2| \leqslant 1\bigr\}. \tag{3.15}$$

Then, for $s \geqslant 0$, the function

$$f(t, x_1, x_2) := \begin{pmatrix} x_2 \\ -x_1 - (\sin 20t)x_2^s + \cos(20t) \end{pmatrix} \tag{3.16}$$

defined on the set $[0, T] \times D$ with $T = \frac{\pi}{10}$ satisfies the relation

$$\left| f(t, x) \right| \leqslant \operatorname{col}(1, a+2) =: M$$

for all $(t, x) \in [0, \frac{\pi}{10}] \times D$. Furthermore, the Lipschitz condition (3.5) holds with $K = \left[\begin{smallmatrix} 0 & 1 \\ 1 & s \end{smallmatrix}\right]$. The set (3.8) corresponding to system (3.14) is defined as follows:

$$D_{\frac{\pi}{20}M} = \left\{ (x_1, x_2) \;\middle|\; |x_1| \leqslant a - \frac{\pi}{20},\, |x_2| \leqslant 1 - \frac{\pi}{20}(a+2) \right\}.$$

This subdomain is non-empty if

$$\begin{aligned} & a - \frac{\pi}{20} \geqslant 0, \\ & 1 - \frac{\pi}{20}(a+2) \geqslant 0, \end{aligned}$$

whence we see that $D_{\frac{\pi}{20}M} \neq \varnothing$ whenever a satisfies the relations

$$\frac{\pi}{20} \leqslant a \leqslant \frac{20}{\pi} - 2. \tag{3.17}$$

It is easy to see that λ is an eigenvalue of the matrix K if, and only if $\lambda^2 - \lambda s - 1 = 0$, whence

$$r(K) = \frac{s + \sqrt{s^2 + 4}}{2}.$$

Therefore, condition (3.7) holds whenever $\frac{3\pi}{100} \frac{s+\sqrt{s^2+4}}{2} < 1$, $\sqrt{s^2+4} < \frac{200}{3\pi} - s$, or, finally, when

$$s < \frac{100}{3\pi} - \frac{3\pi}{100}. \tag{3.18}$$

Thus, the conditions (3.6)–(3.8) for the system (3.14) are fulfilled in the domain (3.15) if the values a and s satisfy the inequalities (3.17), (3.18).

EXAMPLE 3.9. Let us consider the damped Duffing equation with periodic forcing term (see (3.3.8) and (4.3.15) in [17])

$$x''(t) + dx' + ax(t) + b\big(x(t)\big)^3 = H \sin\left(\frac{2\pi}{T} t\right), \tag{3.19}$$

where $a > 0$, $d > 0$, $b \in \mathbb{R}$ (“soft” or “hard spring”), $H \in \mathbb{R}$. Equation (3.19) is equivalent to the Cauchy normal form system

$$
\begin{aligned}
&x_1'(t) = x_2(t),\\
&x_2'(t) = -dx_2(t) - ax_1(t) - bx_1^3(t) + H\sin\left(\frac{2\pi}{T}t\right).
\end{aligned} \tag{3.20}
$$

For the domain of definition of this system, we take

$$
(t, x) \in [0, T] \times D, \quad D := \left\{(x_1, x_2)\colon |x_1| \leqslant m, |x_2| \leqslant 1\right\}. \tag{3.21}
$$

In this case, the right-hand side function, f, in (3.20) satisfies the relation

$$
\left|f(t, x)\right| \leqslant \operatorname{col}\left(1, d + am + |b|m^3 + |H|\right) =: M
$$

and the Lipschitz condition (3.5) with

$$
K = \begin{bmatrix} 0 & 1 \\ a + 3|b|m & d \end{bmatrix}.
$$

The set (3.8) for system (3.20) is determined by the formula

$$
D_{\frac{T}{2}M} = \left\{(x_1, x_2) \mid |x_1| \leqslant m - \frac{T}{2}, |x_2| \leqslant 1 - \frac{T}{2}\left(d + am + |b|m^3 + |H|\right)\right\}.
$$

This subdomain is non-empty if

$$
\begin{aligned}
&m - \frac{T}{2} \geqslant 0,\\
&1 - \frac{T}{2}\left(d + am + b|m^3 + |H|\right) \geqslant 0.
\end{aligned}
$$

Thus, condition (3.8) holds whenever

$$
m \geqslant \frac{T}{2}, \qquad am + |b|m^3 \leqslant \frac{2}{T} - d. \tag{3.22}
$$

Since the eigenvalues λ of the matrix K are determined by the equation

$$
\lambda^2 - d\lambda - \left(a + 3|b|m^2\right) = 0,
$$

it is obvious that

$$
r(K) = \frac{d + \sqrt{d^2 + 4(a + 3|b|m^2)}}{2}
$$

and, therefore, (3.21) holds if

$$d + \sqrt{d^2 + 4\big(a + 3|b|m^2\big)} < \frac{20}{3T}. \tag{3.23}$$

Conditions (3.6)–(3.8) for system (3.20) are satisfied in domain (3.21) when the values of the parameters a, b, d, H, and m satisfy inequalities (3.22), (3.23).

We can now return to the investigation of T-periodic solution of system (3.1), which, as we have mentioned above, is equivalent to the periodic boundary value problem (3.3), (3.4).

Let us put

$$(Lx)(t) := \int_0^t x(s)\,\mathrm{d}s - \frac{t}{T}\int_0^T x(s)\,\mathrm{d}s, \quad t \in [0, T]. \tag{3.24}$$

Obviously, formula (3.24) determines a continuous linear operator in the space $C([0, T], \mathbb{R}^n)$. Moreover, one can easily verify that the following lemma holds.

LEMMA 3.10. *For an arbitrary x from $C([0, T], \mathbb{R}^n)$,*

$$(Lx)(0) = (Lx)(T).$$

Consider the parametrised integral equation

$$x(t) = z + \int_0^t \left(f\big(s, x(s)\big) - \frac{1}{T}\int_0^T f\big(\tau, x(\tau)\big)\,\mathrm{d}\tau \right)\mathrm{d}s$$

or, which is the same,

$$x(\cdot) = z + Lf\big(\cdot, x(\cdot)\big) \tag{3.25}$$

where $z \in \mathbb{R}^n$ is a vector parameter.

REMARK 3.11. It is clear from Lemma 3.10 that every solution x of Eq. (3.25) satisfies the T-periodic boundary condition (3.4); see also the discussion of Eq. (4.1.19) in [17, Section 4.1].

We have the following simple

PROPOSITION 3.12. *If $x(\cdot, z)$ is a solution of the integro-differential equation*

$$x'(t) = f\big(t, x(t)\big) - \frac{1}{T}\int_0^T f\big(s, x(s)\big)\,\mathrm{d}s, \quad t \in [0, T], \tag{3.26}$$

with the initial value condition

$$x(0) = z \tag{3.27}$$

and, moreover,

$$\int_0^T f\big(s, x(s,z)\big)\,\mathrm{d}s = 0, \tag{3.28}$$

then $x(\cdot, z)$ *is also a solution of the* T*-periodic boundary value problem* (3.3), (3.4).

Conversely, if a function x *satisfies* (3.3), (3.4), *then it is also a solution of the initial value problem* (3.26), (3.27) *with* $z = x(0)$.

PROOF. It is obvious that a continuously differentiable function x satisfying the initial value problem (3.26), (3.27) is a continuous solution of Eq. (3.25) and *vice versa*. Therefore, by virtue of Remark 3.11, a solution x of (3.26), (3.27) satisfies the T-periodic boundary condition (3.4) for an arbitrary z. Thus, it follows that (3.26), (3.27), and (3.28) imply (3.3), (3.4).

Conversely, if x is a solution of the T-periodic problem (3.3), (3.4), then

$$\int_0^T f\big(s, x(s)\big)\,\mathrm{d}s = 0$$

and, hence, (3.26), (3.27) hold with $z = x(0)$. □

Equation (3.26) can be considered as a perturbed version of Eq. (3.1) with the perturbation term determined by the expression in the left-hand side of (3.28).

We shall show that, under conditions (3.6)–(3.8), the integral equation (3.25) can be solved by the method of successive approximations. This is the “analytic” part of our method.

With the integral equation (3.25), we associate the following sequence of T-periodic functions $\{x_m(\cdot, z)\}_{m=0}^{+\infty}$ depending on the parameter $z \in \mathbb{R}^n$:

$$x_m(t,z) = z + \int_0^t \left[f\big(s, x_{m-1}(s,z)\big) - \frac{1}{T}\int_0^T f\big(\tau, x_{m-1}(\tau,z)\big)\,\mathrm{d}\tau \right] \mathrm{d}s, \tag{3.29}$$

where $m = 1, 2, \ldots$ and $x_0(t,z) = z$.

It is obvious from Lemma 3.10 that

$$x_m(0,z) = x_m(T,z), \qquad x_m(0,z) = z$$

for every natural m.

Let us establish two subsidiary statements needed below for the convergence analysis of sequence (3.29).

LEMMA 3.13. *Let $f : [0, T] \to \mathbb{R}^n$ be a continuous function. Then, for an arbitrary $t \in [0, T]$, the inequality*

$$\big|(Lf)(t)\big| \leqslant \frac{1}{2}\alpha_{1,T}(t)\Big[\max_{t\in[0,T]} f(t) - \min_{t\in[0,T]} f(t)\Big], \tag{3.30}$$

holds, where the operator L is given by formula (3.24) *and*

$$\alpha_{1,T}(t) := 2t\left(1 - \frac{t}{T}\right), \quad t \in [0, T]. \tag{3.31}$$

REMARK 3.14. It is easy to see that the estimate

$$0 \leqslant \alpha_{1,T}(t) \leqslant \frac{T}{2}, \quad t \in [0, T] \tag{3.32}$$

holds for function (3.31).

PROOF OF LEMMA 3.13. It is obvious that

$$\begin{aligned}
&\int_0^t \left[f(\tau) - \frac{1}{T}\int_0^T f(s)\,\mathrm{d}s\right]\mathrm{d}\tau \\
&\quad = \frac{1}{T}\int_0^t\int_0^T \big[f(\tau) - f(s)\big]\,\mathrm{d}s\,\mathrm{d}\tau \\
&\quad = \frac{1}{T}\int_0^t\int_0^t \big[f(\tau) - f(s)\big]\,\mathrm{d}s\,\mathrm{d}\tau + \frac{1}{T}\int_0^t\int_t^T \big[f(\tau) - f(s)\big]\,\mathrm{d}s\,\mathrm{d}\tau.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\int_0^t\int_0^t \big[f(\tau) - f(s)\big]\,\mathrm{d}s\,\mathrm{d}\tau &= \int_0^t \left[tf(\tau) - \int_0^t f(s)\,\mathrm{d}s\right]\mathrm{d}\tau \\
&= t\int_0^t f(\tau)\,\mathrm{d}\tau - t\int_0^t f(s)\,\mathrm{d}s = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\int_0^t \left[f(\tau) - \frac{1}{T}\int_0^T f(s)\,\mathrm{d}s\right]\mathrm{d}\tau\right| &\leqslant \frac{1}{T}\int_0^t\int_t^T \big|f(\tau) - f(s)\big|\,\mathrm{d}s\,\mathrm{d}\tau \\
&\leqslant \frac{1}{T}\max_{\substack{\tau\in[0,t]\\ s\in[t,T]}} \big|f(\tau) - f(s)\big|\,t(T-t)
\end{aligned}$$

$$= \frac{1}{2}\alpha_{1,T}(t) \max_{\substack{\tau\in[0,t]\\ s\in[t,T]}} \bigl|f(\tau) - f(s)\bigr|$$

$$\leqslant \frac{1}{2}\alpha_{1,T}(t)\Bigl[\max_{t\in[0,T]} f(t) - \min_{t\in[0,T]} f(t)\Bigr].$$

The last estimate leads us to the required inequality (3.30). □

REMARK 3.15. Estimate (3.30) from Lemma 3.13 improves the corresponding estimate

$$\left|\int_0^t \left[f(\tau) - \frac{1}{T}\int_0^T f(s)\,\mathrm{d}s\right]\mathrm{d}\tau\right| \leqslant \alpha_{1,T}(t) \max_{t\in[0,T]}\bigl|f(t)\bigr| \tag{3.33}$$

of [70, Lemma 2.1].

Indeed, let us consider the function f from Example 3.5. Taking into account the fact that

$$\max_{(t,x)\in[0,T]\times D}\bigl|f(t,x)\bigr| = \max_{(t,x)\in[0,2\pi]\times[e,e^2]}\left|\frac{1}{2}\ln x \sin t + 12\right| = 13,$$

according to (3.33) and estimate (3.32) from Remark 3.14, we have

$$\left|\int_0^t \left[f(\tau,x) - \frac{1}{T}\int_0^T f(s,x)\,\mathrm{d}s\right]\mathrm{d}\tau\right| \leqslant \alpha_{1,T}(t)\cdot 13 \leqslant 13\pi \approx 40.82.$$

In view of estimate (3.30),

$$\left|\int_0^t \left[f(\tau,x) - \frac{1}{T}\int_0^T f(s,x)\,\mathrm{d}s\right]\mathrm{d}\tau\right| \leqslant \frac{\pi}{2}\cdot 1 \approx 1.57.$$

On the other hand, by direct computation, we obtain that

$$\begin{aligned}
&\int_0^t \left[\left(\frac{1}{2}\ln x \sin\tau + 12\right) - \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2}\ln x \sin s + 12\right)\mathrm{d}s\right]\mathrm{d}\tau\\
&\quad = \int_0^t \left[\left(\frac{1}{2}\ln x \sin\tau + 12\right) - \frac{1}{2\pi}24\pi\right]\mathrm{d}\tau\\
&\quad = \int_0^t \frac{1}{2}\ln x \sin\tau\,\mathrm{d}\tau = \frac{1}{2}\ln x(1-\cos t)
\end{aligned}$$

and

$$\max_{(t,x)\in[0,2\pi]\times[e,e^2]}\left|\frac{1}{2}\ln x(1-\cos t)\right| = 1.$$

LEMMA 3.16. *Let the sequence of continuous functions*

$$\alpha_{0,T}(t) = 1, \quad \alpha_{1,T}(t) = 2t\left(1 - \frac{t}{T}\right), \quad \ldots, \quad \alpha_{m,T}(t), \ldots \tag{3.34}$$

be defined by the recurrence relation

$$\begin{aligned} \alpha_{m+1,T}(t) = {} & \left(1 - \frac{t}{T}\right) \int_0^t \alpha_{m,T}(s)\,\mathrm{d}s \\ & + \frac{t}{T} \int_t^T \alpha_{m,T}(s)\,\mathrm{d}s, \quad m = 0, 1, 2, \ldots. \end{aligned} \tag{3.35}$$

Then, for all $m = 0, 1, 2, \ldots$, *the estimate*

$$\alpha_{m+1,T}(t) \leqslant \left(\frac{3T}{10}\right)^m \bar{\alpha}_{1,T}(t), \quad t \in [0, T] \tag{3.36}$$

holds, where

$$\bar{\alpha}_{1,T}(t) := \frac{10}{9}\alpha_{1,T}(t). \tag{3.37}$$

PROOF. By direct computation, from the iterative formula (3.35) we obtain

$$\alpha_{2,T}(t) = \alpha_{1,T}(t)\left[\frac{T}{6} + \frac{1}{3}\alpha_{1,T}(t)\right]$$

and

$$\alpha_{3,T}(t) = \alpha_{1,T}(t)\left[\frac{T^2}{20} + \frac{T}{15}\alpha_{1,T}(t) + \frac{1}{15}\alpha_{1,T}^2(t)\right]. \tag{3.38}$$

Taking estimate (3.32) into account, we get

$$\alpha_{2,T}(t) \leqslant \frac{T}{3}\alpha_{1,T}(t) = \frac{3}{10}T\left(\frac{10}{9}\alpha_{1,T}(t)\right) = \left(\frac{3}{10}T\right)\bar{\alpha}_{1,T}(t).$$

It follows from (3.38) that

$$\begin{aligned} \alpha_{3,T}(t) &= \frac{3}{10}T\alpha_{1,T}(t)\left[\frac{T}{6} + \frac{1}{3}\alpha_{1,T}(t)\right] - \frac{1}{30}T\alpha_{1,T}^2(t) + \frac{1}{15}\alpha_{1,T}^3(t) \\ &= \left(\frac{3}{10}T\right)\alpha_{2,T}(t) - \frac{1}{30}\alpha_{1,T}^2(t)\left[T - 2\alpha_{1,T}(t)\right] \\ &\leqslant \left(\frac{3}{10}T\right)\alpha_{2,T}(t) \leqslant \left(\frac{3}{10}T\right)^2 \bar{\alpha}_{1,T}(t). \end{aligned}$$

By induction, the last inequality yields

$$\begin{aligned}\alpha_{m+1,T}(t) &\leqslant \left(1-\frac{t}{T}\right)\int_0^t \left(\frac{3}{10}T\right)\alpha_{m-1,T}(s)\,\mathrm{d}s \\ &\quad + \frac{t}{T}\int_t^T \left(\frac{3}{10}T\right)\alpha_{m-1,T}(s)\,\mathrm{d}s \\ &\leqslant \left(\frac{3}{10}T\right)\alpha_{m,T}(t) \leqslant \left(\frac{3}{10}T\right)^m \bar{\alpha}_{1,T}(t).\end{aligned}$$

Thus, we have obtained the required estimate (3.36). □

The last lemma allows us to establish a statement on the convergence of sequence (3.29).

THEOREM 3.17. *Let the T-periodic boundary-value problem* (3.3), (3.4) *be such that conditions* (3.5), (3.6), *and* (3.8) *are satisfied. Then, for an arbitrary value of the parameter $z \in D_\beta$, the following assertions are true*:

(i) *The sequence $\{x_m(\cdot, z)\}_{m\geqslant 0}$ of the T-periodic functions* (3.29) *converges uniformly in (t, z) from*

$$[0, T] \times D_{\frac{T}{2}M}; \tag{3.39}$$

(ii) *The limit function*

$$x^*(t, z) := \lim_{m\to\infty} x_m(t, z) \tag{3.40}$$

of sequence (3.29) *is the unique T-periodic solution of the integral equation* (3.25)

$$x^*(t, z) = z + \int_0^t f\big(s, x^*(s, z)\big)\,\mathrm{d}s - \frac{t}{T}\int_0^T f\big(s, x^*(s, z)\big)\,\mathrm{d}s \tag{3.41}$$

in the domain D or, which is the same, it is the unique T-periodic solution of the integro-differential equation (3.26)

$$\frac{\partial x(t, z)}{\partial t} = f\big(t, x(t, z)\big) - \Delta(z), \quad t \in [0, T], \tag{3.42}$$

where the "perturbation term" has the form

$$\Delta(z) := \frac{1}{T}\int_0^T f\big(t, x^*(t, z)\big)\,\mathrm{d}t; \tag{3.43}$$

(iii) *The estimate*

$$\big|x^*(t, z) - x_m(t, z)\big| \leqslant \bar{\alpha}_{1,T}(t) Q^m (I - Q)^{-1} M, \tag{3.44}$$

holds for all $m = 1, 2, \ldots$ *and* $t \in [0, T]$, *where the function* $\bar{\alpha}_{1,T}$ *is given by* (3.37),

$$Q = \frac{3T}{10} K, \tag{3.45}$$

and K *is the matrix from the Lipschitz condition* (3.5).

REMARK 3.18. It is obvious from (3.37) and Remark 3.14 that

$$\bar{\alpha}_{1,T}(t) \leqslant \frac{5}{9} T, \quad t \in [0, T]. \tag{3.46}$$

PROOF OF THEOREM 3.17. We shall show that $\{x_m(\cdot, z)\}_{m \geqslant 0}$ given by (3.29) is a Cauchy sequence in the space $C([0, T], \mathbb{R}^n)$ and, therefore, this sequence is uniformly convergent in domain (3.39).

First we prove that if z belongs to the set $D_{\frac{T}{2}M} \subset D$, then $x_m(\cdot, z) \in D$ for all t and m. Indeed, by virtue of estimate (3.30) of Lemma 3.13 and relation (3.6), equality (3.29) implies

$$\begin{aligned}
\left|x_1(t, z) - z\right| &= \left|x_1(t, z) - x_0(t, z)\right| \\
&\leqslant \left| \int_0^t \left[f(s, z)\, \mathrm{d}s - \frac{1}{T} \int_0^T f(\tau, z)\, \mathrm{d}\tau \right] \mathrm{d}s \right| \mathrm{d}s \\
&\leqslant \frac{1}{2} \alpha_{1,T}(t) \left[\max_{(t,z) \in [0,T] \times D_{\frac{T}{2}M}} f(t, z) - \min_{(t,x) \in [0,T] \times D_{\frac{T}{2}M}} f(t, z) \right] \\
&\leqslant \alpha_{1,T}(t) M \leqslant \frac{T}{2} M,
\end{aligned} \tag{3.47}$$

where $\alpha_{1,T}$ is given by (3.31). Therefore, $x_1(t, z) \in D$ for $t \in [0, T]$ and $z \in D_{\frac{T}{2}M}$. Arguing by induction, one can show that

$$\left|x_m(t, z) - z\right| \leqslant \frac{T}{2} M, \tag{3.48}$$

i.e., the values of all the members $x_m(\cdot, z)$ of sequence (3.29) are contained in the domain D for all $t \in [0, T]$ and arbitrary $z \in D_{\frac{T}{2}M}$.

Let us estimate the differences between neighbouring members of sequence (3.29). We have

$$\begin{aligned}
x_{m+1}(t, z) - x_m(t, z) = \int_0^t &\Big[\big(f\big(s, x_m(s, z)\big) - f\big(s, x_{m-1}(s, z)\big) \big) \\
&- \frac{1}{T} \int_0^T \big(f\big(\tau, x_m(\tau, z)\big) - f\big(s, x_{m-1}(\tau, z)\big) \big)\, \mathrm{d}\tau \Big]\, \mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \left[f\big(s, x_m(s,z)\big) - f\big(s, x_{m-1}(s,z)\big)\right] \mathrm{d}s \\
&\quad - \frac{t}{T}\int_0^T \left[f\big(s, x_m(s,z)\big) - f\big(s, x_{m-1}(s,z)\big)\right] \mathrm{d}s \\
&= \left(1 - \frac{t}{T}\right)\int_0^t \left[f\big(s, x_m(s,z)\big) - f\big(s, x_{m-1}(s,z)\big)\right] \mathrm{d}s \\
&\quad - \frac{t}{T}\int_t^T \left[f\big(s, x_m(s,z)\big) - f\big(s, x_{m-1}(s,z)\big)\right] \mathrm{d}s.
\end{aligned} \tag{3.49}
$$

Putting

$$
r_m(t,z) := \left|x_{m+1}(t,z) - x_m(t,z)\right|, \quad t \in [0,T],
$$

and taking the Lipschitz condition (3.5) into account, we see that (3.49) yields

$$
r_{m+1}(t,z) \leqslant K\left[\left(1 - \frac{t}{T}\right)\int_0^t r_m(s,z)\,\mathrm{d}s + \frac{t}{T}\int_t^T r_m(s,z)\,\mathrm{d}s\right]. \tag{3.50}
$$

According to (3.47), we have

$$
r_1(t,z) \leqslant \alpha_{1,T}(t) M \leqslant \frac{T}{2} M.
$$

Therefore, for $m = 1$, on base of the recurrence formula (3.35), relation (3.50) implies

$$
\begin{aligned}
r_2(t,z) &\leqslant K\left[\left(1 - \frac{t}{T}\right)\int_0^t r_1(s,z)\,\mathrm{d}s + \frac{t}{T}\int_t^T r_1(s,z)\,\mathrm{d}s\right] \\
&\leqslant KM\left[\left(1 - \frac{t}{T}\right)\int_0^t \alpha_{1,T}(s)\,\mathrm{d}s + \frac{t}{T}\int_t^T \alpha_{1,T}(s)\,\mathrm{d}s\right] \\
&\leqslant KM\alpha_{2,T}(t).
\end{aligned}
$$

Arguing by induction, we easily obtain

$$
r_{m+1}(t,z) \leqslant K^m M \alpha_{m+1,T}(t), \quad m = 0, 1 \ldots,
$$

where $\alpha_{m+1,T}(t)$ is given by formula (3.33). By virtue of estimate (3.36) from Lemma 3.16, we have

$$
r_{m+1}(t,z) \leqslant \bar{\alpha}_{1,T}(t) K^m \left(\frac{3T}{10}\right)^m M = \bar{\alpha}_{1,T}(t) Q^m M \tag{3.51}
$$

for all $m = 1, 2 \ldots$, where Q is the matrix given by (3.45).

Furthermore, in view of (3.51), the relation

$$\begin{aligned}x_{m+j}(t,z)-x_m(t,z)&=\big(x_{m+j}(t,z)-x_{m+j-1}(t,z)\big)\\&\quad+\big(x_{m+j}(t,z)-x_{m+j-2}(t,z)\big)\\&\quad+\cdots+\big(x_{m+1}(t,z)-x_m(t,z)\big)\end{aligned}$$

yields

$$\begin{aligned}\big|x_{m+j}(t,z)-x_m(t,z)\big|&\leqslant\sum_{i=1}^{j}r_{m+i}\leqslant\bar{\alpha}_{1,T}(t)\sum_{i=1}^{j}Q^{m+i-1}M\\&\leqslant\bar{\alpha}_{1,T}(t)\sum_{i=0}^{j-1}Q^{m+i-1}M=\bar{\alpha}_{1,T}(t)Q^m\sum_{i=0}^{j-1}Q^iM.\end{aligned}\tag{3.52}$$

Taking into account the fact that, according to assumption (3.7), the maximal eigenvalue of matrix (3.45) is less than 1, we see that

$$\sum_{i=0}^{j-1}Q^{m+i}\leqslant\sum_{i=0}^{\infty}Q^{m+i}=Q^m(I-Q)^{-1}\tag{3.53}$$

and

$$\lim_{m\to\infty}Q^m=0.\tag{3.54}$$

Consequently, it follows from (3.52), (3.53), and (3.54) that, for an arbitrary $t\in[0,T]$,

$$\big|x_{m+j}(t,z)-x_m(t,z)\big|\leqslant\bar{\alpha}_{1,T}(t)Q^m(I-Q)^{-1}M.\tag{3.55}$$

Inequality (3.29) implies immediately that (3.29) is a Cauchy sequence in the space $C([0,T],\mathbb{R}^n)$ and, therefore, it uniformly converges to a continuous limit function $x^*(\cdot,z)$, i.e., (3.40) holds.

Since all the functions $x_m(\cdot,z)$ of the sequence (3.29) are T-periodic, it follows its uniform limit $x^*(\cdot,z)$ is also a T-periodic function. Passing to the limit as $m\to+\infty$ in equality (3.29), we conclude that the limit function $x^*(\cdot,z)$ is indeed a unique T-periodic solution of the integral equation (3.25) with the initial value

$$x^*(0,z)=z.\tag{3.56}$$

Thus, equalities (3.41), (3.42) hold. When $j\to+\infty$ and $m=1,2,\ldots,$ relation (3.55) yields the required estimate (3.43) for the difference between the exact solution $x^*(\cdot,z)$ of Eq. (3.41) and its mth approximation $x_m(\cdot,z)$. □

Equality (3.56) shows that, in fact, every point z from the subset $D_{\frac{T}{2}M}$ is the initial value of the unique T-periodic solution of Eqs. (3.25) and (3.26).

3.1. *Properties of the limit function*

Along with system (3.1), we shall also consider the system of equations

$$x'(t) = f\big(t, x(t)\big) - \mu, \quad t \in [0, T], \tag{3.57}$$

where $\mu = \mathrm{col}(\mu_1, \ldots, \mu_n)$ is a control parameter. Let us show that the parameter μ can always be chosen so that the solution of Eq. (3.57) having the initial value (3.27) with

$$z \in D_{\frac{T}{2}M} \tag{3.58}$$

also satisfies the T-periodic boundary condition (3.4).

In other words, this means that we can always introduce an additive perturbation of the right-hand side of the T-periodic system (3.1) such that, for an arbitrary $z \in D_{\frac{T}{2}M}$, the solution $x = x(t, 0, z, \mu)$ of the resulting initial value problem (3.57), (3.27) is, at the same time, a solution of the T-periodic problem (3.57), (3.4).

THEOREM 3.19. *Assume that the T-periodic system of differential Eqs.* (3.57) *satisfies the conditions of Theorem* 3.17.

Then, for an arbitrary $z \in D_\beta$, there exists a unique value of the control parameter μ,

$$\mu = \frac{1}{T}\int_0^T f\big(t, x^*(t, z)\big)\,\mathrm{d}t, \tag{3.59}$$

such that the solution $x = x(t, 0, z, \mu)$ of the initial value problem (3.57), (3.58), *with μ given by* (3.59), *is a T-periodic function and*

$$x(t, 0, z, \mu) = x^*(t, z) = \lim_{m\to\infty} x_m(t, z), \tag{3.60}$$

where $\{x_m(\cdot, z)\}_{m\geqslant 0}$ is the sequence of functions defined by formula (3.29).

PROOF. Since (3.1) satisfies conditions (3.5), (3.6), (3.61), and (3.62), the sequence of T-periodic functions given by (3.29) converges to a T-periodic function $x^*(\cdot, z)$ uniformly in domain (3.39). It follows from Theorem 3.17 that, according (3.42), (3.43), if (3.59) holds, then the T-periodic solution $x = x^*(t, z)$ of system (3.57) also satisfies the initial value condition (3.56). Thus, we have found the value of the parameter μ of the form (3.59) for which (3.60) holds.

It can be shown that this parameter value is unique because, for any other value $\mu = \nu$, the solution $x = x(t, 0, z, \nu)$ of the initial value problem (3.27) for the differential system

$$x'(t) = f\big(t, x(t)\big) - \nu, \quad t \in [0, T], \tag{3.61}$$

where $\nu \neq \Delta(z)$ and z satisfies (3.58), does not satisfy the T-periodic boundary condition (3.4). Indeed, assume the contrary. Then there exist at least two values μ, ν ($\mu \neq \nu$) such

that the solutions $x = x(t, 0, z, \mu)$ and $x = x(t, 0, z, \nu)$ of the Cauchy problems (3.57), (3.58) and (3.61) respectively also satisfy the T-periodic boundary condition (3.4). Then, by using (3.25), we obtain the following identity for the difference of these functions:

$$\begin{aligned}
&\big|x(t,0,z,\mu) - x(t,0,z,\nu)\big| \\
&\quad = \bigg|\int_0^t \big[f\big(s, x(s,0,z,\mu)\big) - f\big(s, x(s,0,z,\nu)\big)\big]\,\mathrm{d}s \\
&\qquad - \frac{t}{T}\int_0^T \big[f\big(s, x(s,0,z,\mu)\big) - f\big(s, x(s,0,z,\nu)\big)\big]\,\mathrm{d}s\bigg| \\
&\quad = \bigg|\left(1 - \frac{t}{T}\right)\int_0^t \big[f\big(s, x(s,0,z,\mu)\big) - f\big(s, x(s,0,z,\nu)\big)\big]\,\mathrm{d}s \\
&\qquad - \frac{t}{T}\int_t^T \big[f\big(s, x(s,0,z,\mu)\big) - f\big(s, x(s,0,z,\nu)\big)\big]\,\mathrm{d}s\bigg| \\
&\quad \leqslant \left(1 - \frac{t}{T}\right)\int_0^t \big|f\big(s, x(s,0,z,\mu)\big) - f\big(s, x(s,0,z,\nu)\big)\big|\,\mathrm{d}s \\
&\qquad + \frac{t}{T}\int_t^T \big|f\big(s, x(s,0,z,\mu)\big) - f\big(s, x(s,0,z,\nu)\big)\big|\,\mathrm{d}s. \qquad (3.62)
\end{aligned}$$

Putting

$$r(t) := \big|x(t,0,z,\mu) - x(t,0,z,\nu)\big|, \quad t \in [0,T], \tag{3.63}$$

and $\hat{r} := \operatorname{col}(\max_{t\in[0,T]} r_1(t), \ldots, \max_{t\in[0,T]} r_n(t))$, and using the Lipschitz condition (3.5), from (3.62) we find

$$r(t) \leqslant K\left[\left(1 - \frac{t}{T}\right)\int_0^t r(s)\,\mathrm{d}s + \frac{t}{T}\int_t^T r(s)\,\mathrm{d}s\right]. \tag{3.64}$$

In view of (3.64), application of Lemma 3.16 yields

$$r(t) \leqslant K\left[\left(1 - \frac{t}{T}\right)\int_0^t \hat{r}\,\mathrm{d}s + \frac{t}{T}\int_t^T \hat{r}\,\mathrm{d}s\right] \equiv \alpha_{1,T}(t)K\hat{r}. \tag{3.65}$$

Taking into account inequality (3.65) and Lemma 3.16 and arguing by induction, from (3.64) we obtain the estimate

$$r(t) \leqslant \alpha_{m+1,T}(t)K^{m+1}\hat{r}, \quad m = 0, 1, 2, \ldots, \tag{3.66}$$

where $\alpha_{m,T}(\cdot)$ are positive functions defined by formula (3.35). By virtue of inequality (3.36), relation (3.66) implies

$$r(t) \leqslant \left(\frac{3T}{10}K\right)^m K\bar{\alpha}_{1,T}(t)r_0 = Q^m K\bar{\alpha}_{1,T}(t)\hat{r}, \quad t \in [0,T],$$

where $\bar{\alpha}_{1,T}$ is given by (3.37). Hence, by virtue of estimate (3.46) from Remark 3.18, we obtain

$$\hat{r} \leqslant \frac{5T}{9} Q^m K \hat{r}, \quad m = 1, 2, \ldots. \tag{3.67}$$

On the assumption that the eigenvalues of matrix (3.45) are less than 1, inequality (3.67) is possible only when $\hat{r} = 0$ and, hence, we have $\mu = \nu$ in (3.63). This contradiction proves that (3.59) is the only suitable value of the parameter μ. □

EXAMPLE 3.20. Let us consider the differential system (3.14) from Example 3.8 in domain (3.15) and try to determine approximately the control parameter $\mu = \mu(z)$, for which the solution of the system

$$\begin{aligned} x_1'(t) &= x_2(t) - \mu_1, \\ x_2'(t) &= -x_1(t) - \big(x_2(t)\big)^s \sin 20t + \cos 20t - \mu_2, \quad t \in [0, T], \end{aligned} \tag{3.68}$$

such that

$$x_1(0) = z_1, \qquad x_2(0) = z_2, \tag{3.69}$$

where $(z_1, z_2) \in D_{\frac{T}{2}M}$, is periodic with period $T = \frac{\pi}{10}$.

Assume that s in (3.14) and a in (3.15) satisfy inequalities (3.17) and (3.18). Then we can seek for the control

$$\mu(z) = \operatorname{col}\big(\mu_1(z), \mu_2(z)\big)$$

as the limit

$$\mu(z) = \lim_{m\to\infty} \mu_m(z) \tag{3.70}$$

of the successive approximations

$$\mu_m(z) = \operatorname{col}\big(\mu_{1m}(z), \mu_{2m}(z)\big), \quad m = 0, 1, 2, \ldots,$$

defined by the equality

$$\mu_m(z) = \frac{1}{T}\int_0^T f\big(t, x_m(t, z)\big)\,\mathrm{d}t$$

with f given by formula (3.16). Thus, we have

$$\begin{aligned} \mu_1(z) &= \lim_{m\to\infty} \frac{10}{\pi}\int_0^{\frac{\pi}{10}} x_{2m}(t, z)\,\mathrm{d}t, \\ \mu_2(z) &= \lim_{m\to\infty} \frac{10}{\pi}\int_0^{\frac{\pi}{10}} \big[-x_{1m}(t, z) - \sin 20t\big(x_{2m}(t, z)\big)^s + \cos 20t\big]\,\mathrm{d}t, \end{aligned}$$

where $x_m(t,z) = \operatorname{col}(x_{1m}(t,z), x_{2m}(t,z))$,

$$x_{10}(t,z) = z_1, \qquad x_{20}(t,z) = z_2,$$

and, for all $m = 1, 2, \ldots$ and $t \in [0, T]$,

$$x_{1m}(t,z) = z_1 + \int_0^t x_{2,m-1}(t,z)\,\mathrm{d}t - \frac{10t}{\pi}\int_0^{\frac{\pi}{10}} x_{2,m-1}(t,z)\,\mathrm{d}t, \tag{3.71}$$

$$\begin{aligned} x_{2m}(t,z) = z_2 &+ \int_0^t \bigl[-x_{1,m-1}(t,z) - \sin 20t\bigl(x_{2,m-1}(t,z)\bigr)^s + \cos 20t\bigr]\,\mathrm{d}t, \\ &- \frac{10t}{\pi}\int_0^{\frac{\pi}{10}} \bigl[-x_{1,m-1}(t,z) - \sin 20t\bigl(x_{2,m-1}(t,z)\bigr)^s + \cos 20t\bigr]\,\mathrm{d}t. \end{aligned} \tag{3.72}$$

Let us compute the value of the control parameter for system (3.14) with $s = 1$ (i.e., when (3.14) is a linear system with periodic coefficients) and $a = 2$. In this case (see Example 3.8), we have

$$\begin{aligned} &D = \bigl\{(x_1, x_2) \mid |x_1| \leqslant 2, |x_2| \leqslant 1\bigr\}, \\ &D_{\frac{\pi}{20}M} = \left\{(x_1, x_2) \;\middle|\; |x_1| \leqslant 2 - \frac{\pi}{20}, |x_2| \leqslant 1 - \frac{\pi}{5}\right\}. \end{aligned} \tag{3.73}$$

Let us put, e.g.,

$$z_1 = \frac{3}{2}, \qquad z_2 = \frac{1}{4} \tag{3.74}$$

in the initial value condition (3.69). It is easy to see that the vector (z_1, z_2) defined by (3.74) belongs to set (3.73).

According to (3.71) with $m = 1$, the first coordinate of the first approximation $x_1(\cdot, z)$ has the form

$$x_{11}(t,z) = \frac{3}{2} + \int_0^t \frac{1}{4}\,\mathrm{d}t - \frac{10t}{\pi}\int_0^{\frac{\pi}{10}} \frac{1}{4}\,\mathrm{d}t = \frac{3}{2}.$$

Quite similarly, (3.72) yields

$$\begin{aligned} x_{21}(t,z) &= \frac{1}{4} + \int_0^t \left[-\frac{3}{2} - \frac{1}{4}\sin 20t + \cos 20t\right]\mathrm{d}t \\ &\quad - \frac{10t}{\pi}\int_0^{\frac{\pi}{10}} \left[-\frac{3}{2} - \frac{1}{4}\sin 20t + \cos 20t\right]\mathrm{d}t \\ &= \frac{19}{80} + \frac{1}{80}\cos 20t + \frac{1}{20}\sin 20t. \end{aligned} \tag{3.75}$$

Therefore, in view of (3.70), the control parameters μ_1, μ_2, in the first approximation, have the following values:

$$\mu_{11} = \frac{10}{\pi}\int_0^{\frac{\pi}{10}}\left[\frac{19}{80} + \frac{1}{80}\cos 20t + \frac{1}{20}\sin 20t\right]\mathrm{d}t = \frac{19}{80}$$
$$\approx 0.2125,$$
$$\mu_{21} = \frac{10}{\pi}\int_0^{\frac{\pi}{10}}\left[-\frac{3}{2} - \sin 20t\left(\frac{19}{80} + \frac{1}{80}\cos 20t + \frac{1}{20}\sin 20t\right)\right]\mathrm{d}t$$
$$= -\frac{3}{2} - \frac{\pi}{40}.$$

Analogously, by using (3.71), we can easily compute the second approximations to the limit function ($m = 2$):

$$\begin{aligned} x_{12}(t,z) &= \frac{3}{2} + \int_0^t\left[\frac{19}{80} + \frac{1}{80}\cos 20t + \frac{1}{20}\sin 20t\right]\mathrm{d}t \\ &\quad - \frac{10t}{\pi}\int_0^{\frac{\pi}{10}}\left[\frac{19}{80} + \frac{1}{80}\cos 20t + \frac{1}{20}\sin 20t\right]\mathrm{d}t \\ &= \frac{601}{400} + \frac{1}{1600}\sin 20t - \frac{1}{400}\cos 20t \end{aligned} \tag{3.76}$$

and

$$\begin{aligned} x_{22}(t,z) &= \frac{1}{4} + \int_0^t\left[-\frac{3}{2} - \sin 20t\left(\frac{19}{80} + \frac{1}{80}\cos 20t\right.\right. \\ &\quad \left.\left. + \frac{1}{20}\sin 20t + \cos 20t\right) + \cos 20t\right]\mathrm{d}t \\ &\quad - \frac{10t}{\pi}\int_0^{\frac{\pi}{10}}\left[-\frac{3}{2} - \sin 20t\left(\frac{19}{80} + \frac{1}{80}\cos 20t\right.\right. \\ &\quad \left.\left. + \frac{1}{20}\sin 20t + \cos 20t\right) + \cos 20t\right]\mathrm{d}t \\ &= \frac{1523}{6400} + \frac{19}{1600}\cos 20t + \frac{1}{20}\sin 20t + \frac{1}{6400}\cos 40t + \frac{1}{1600}\sin 40t. \end{aligned}$$

The control parameters μ_1, μ_2, in the second approximation, are given by the formulae

$$\begin{aligned} \mu_{12} &= \frac{10}{\pi}\int_0^{\frac{\pi}{10}}\left[\frac{1523}{6400} + \frac{19}{1600}\cos 20t + \frac{1}{20}\sin 20t\right. \\ &\quad \left. + \frac{1}{6400}\cos 40t + \frac{1}{1600}\sin 40t\right] = \frac{1523}{6400} \approx 0.2379, \end{aligned}$$

and

$$\mu_{22} = \frac{10}{\pi} \int_0^{\frac{\pi}{10}} \Big[-\frac{3}{2} - \frac{1}{1600} \sin 20t + \frac{1}{400} \cos 20t - \sin 20t \left(\frac{1523}{6400} + \frac{19}{1600} \cos 20t + \frac{1}{20} \sin 20t + \frac{1}{6400} \cos 40t + \frac{1}{1600} \sin 40t \right) \Big] \mathrm{d}t = -\frac{3}{2} - \frac{\pi}{400}.$$

Note that (3.75) and (3.76) are the first and second approximations to the exact solution of the initial value problem (3.68), (3.74) with z_1, z_2 given by (3.74). One can verify that an exact solution of this problem exists for control parameter

$$\mu = \mathrm{col}(\mu_1, \mu_2) = \mathrm{col}\left(\frac{1}{4}, -\frac{3}{2} \right).$$

This solution, at the same time, is a $\frac{\pi}{10}$-periodic function as well as $x^*(t, z)$ and satisfies the initial value conditions (3.69) with z_1, z_2 given by (3.74). Higher order approximations can be found in a similar way by using formulae (3.70), (3.71).

Let us find necessary and sufficient conditions for the limit function $x^*(\cdot, z)$ of sequence (3.29) to be a solution of the original T-periodic boundary value problem n (3.3), (3.4).

THEOREM 3.21. *For the T-periodic boundary value problem* (3.3), (3.4), *let conditions* (3.5)–(3.7), *and* (3.8) *hold. Then the solution $x = x(\cdot, 0, z)$ of the Cauchy problem* (3.3), (3.27) *with $z \in D_{\frac{T}{2}M}$ is a solution of the T-periodic problem* (3.3), (3.4) *if, and only if the equality*

$$\int_0^T f\big(t, x^*(t, z)\big)\, \mathrm{d}t = 0, \tag{3.77}$$

holds, where $x^*(\cdot, z)$ is the limit function of sequence (3.29). *Moreover, in this case*

$$x(t, 0, z) \equiv x^*(t, z) \tag{3.78}$$

and for all $m = 1, 2, \ldots$ inequality (3.44) *holds for the deviation of the exact solution* (3.78) *of the T-periodic boundary value problem* (3.3), (3.4) *from its approximate T-periodic solution $x = x_m(\cdot, z)$ given by* (3.29) .

PROOF. The sufficiency of condition (3.77) follows immediately from the fact that the limit function $x^*(\cdot, z)$ satisfying the periodic boundary condition (3.4). and the initial value condition (3.27), is a solution of the integral equation (3.41) or, which is the same, of the integro-differential equations (3.42), (3.43). Obviously, if (3.43) is true, this condition is

sufficient for $x^*(\cdot, z)$ to be a solution both of the periodic boundary value problem (3.3), (3.4) and the initial value problem (3.3), (3.27). In this case, (3.78) holds.

The necessity of condition (3.77) follows from the fact that if the solution $x = x(\cdot, 0, x_0)$ of (3.3), (3.27) is a solution of the T-periodic boundary value problem (3.3), (3.4), then the solution $x = x(\cdot, 0, z, \Delta)$ of system (3.42) with the same initial value $x(0, 0, z, \Delta) = z$ is T-periodic provided that $\Delta(z) = 0$ because, in this case,

$$x(t, 0, z, 0) \equiv x(t, 0, z) \equiv x^*(t, z) \tag{3.79}$$

and estimate (3.44) holds.

According to Theorem 3.19, a solution $x = x(t, 0, z, \mu)$ of the Cauchy problem (3.57), (3.58) is, at the same time, a T-periodic function only for the parameter value μ of the form (3.59). However, we already have this value of the parameter when$\mu = \Delta(z) = 0$. The theorem is proved. $\square$

Let us show that the function $x^*(\cdot, z)$ is Lipschitzian with respect to z.

THEOREM 3.22. *Under the assumptions of Theorem* 3.17, *the function* $x^*(\cdot, z)$ *satisfies the following Lipschitz condition for all* $\{y, z\} \subset D_{\frac{T}{2}M}$ *and* $t \in [0, T]$:

$$\left|x^*(t, z) - x^*(t, y)\right| \leqslant \left[I + \bar{\alpha}_{1,T}(t) K (I - Q)^{-1}\right] |z - y|, \tag{3.80}$$

where $\bar{\alpha}_{1,T}$ *is the function* (3.37) *and the matrix* Q *is given by* (3.45).

PROOF. It follows immediately from (3.29) that

$$\begin{aligned} x_1(t, z) - x_1(t, y) = {}& (z - y) + \int_0^t \left[f(s, z) - f(s, y)\right] \mathrm{d}s \\ & - \frac{t}{T} \int_0^T \left[f(s, z) - f(s, y)\right] \mathrm{d}s \\ = {}& (z - y) + \left(1 - \frac{t}{T}\right) \int_0^t \left[f(s, z) - f(s, y)\right] \mathrm{d}s \\ & - \frac{t}{T} \int_t^T \left[f(s, z) - f(s, y)\right] \mathrm{d}s. \end{aligned} \tag{3.81}$$

Taking into account identity (3.81), the Lipschitz condition (3.5), and formula (3.35) of Lemma 3.16, we obtain

$$\begin{aligned} \left|x_1(t, z) - x_1(t, y)\right| \leqslant {}& |z - y| + \left[\left(1 - \frac{t}{T}\right) \int_0^t \mathrm{d}s + \frac{t}{T} \int_t^T \mathrm{d}s\right] K |z - y| \\ = {}& \left[I + \alpha_{1,T}(t) K\right] |z - y|. \end{aligned} \tag{3.82}$$

Similarly,

$$\begin{aligned}\left|x_2(t,z)-x_2(t,y)\right| \leqslant |z-y| + K\Bigg[\left(1-\frac{t}{T}\right)\int_0^t \left|x_1(s,z)-x_1(s,y)\right|\mathrm{d}s \\ + \frac{t}{T}\int_t^T \left|x_1(s,z)-x_1(s,y)\right|\mathrm{d}s\Bigg].\end{aligned}$$

Taking into account (3.82) and the recurrence relation (3.35), we get

$$\begin{aligned}\left|x_2(t,z)-x_2(t,y)\right| &\leqslant |z-y| + K\Bigg[\left(1-\frac{t}{T}\right)\int_0^t \left|\big(I+\alpha_{1,T}(s)K\big)\,\mathrm{d}s\right|\mathrm{d}s \\ &\quad + \frac{t}{T}\int_t^T \big(I+\alpha_{1,T}(s)K\big)\,\mathrm{d}s\Bigg] \\ &\leqslant \big[I+\alpha_{1,T}(t)K+\alpha_{2,T}(t)K^2\big]|z-y|.\end{aligned}$$

From these relations we conclude by induction that

$$\begin{aligned}\left|x_m(t,z)-x_m(t,y)\right| \leqslant \big(I+\alpha_{1,T}(t)K+\alpha_{2,T}(t)K^2 \\ + \cdots + \alpha_{m,T}(t)K^m\big)|z-y|.\end{aligned} \tag{3.83}$$

Condition (3.7) means that the spectral radius $r(Q)$ of the matrix Q defined by (3.45) is less than 1. Passing to the limit as $m \to +\infty$ in (3.83) and using condition (3.7) together with estimate (3.36) of Lemma 3.16, we obtain

$$\begin{aligned}\left|x^*(t,z)-x^*(t,y)\right| &\leqslant \left(I+\bar{\alpha}_{1,T}(t)\sum_{i=0}^{\infty} K\left(\frac{3T}{10}K\right)^i\right)|z-y| \\ &= \big[I+\bar{\alpha}_{1,T}(t)K(I-Q)^{-1}\big]|z-y|,\end{aligned}$$

i.e., inequality (3.80) holds. □

When the set $D_{\frac{T}{2}M}$ given by (3.8) is not a singleton, to our T-periodic problem (3.3), (3.4) one can associate the "determining function"

$$\Delta : D_{\frac{T}{2}M} \to \mathbb{R}^n \tag{3.84}$$

defined by formula (3.43).

Assuming that $D_{\frac{T}{2}M}$ contains more than one point, we now establish some properties of the determining function (3.84).

THEOREM 3.23. *Under the conditions of Theorem* 3.17, *the function*

$$D_{\frac{T}{2}M} \ni z \longmapsto \Delta(z) := \frac{1}{T}\int_0^T f\big(t, x^*(t,z)\big)\,\mathrm{d}t, \tag{3.85}$$

where $x^*(\cdot, z)$ *is the limit function of sequence* (3.29), *is well-defined, continuous, and bounded inside the set* $D_{\frac{T}{2}M}$. *Furthermore, for all* $\{z, y\} \subset D_{\frac{T}{2}M}$, *it satisfies the Lipschitz condition*

$$\left|\Delta(z) - \Delta(y)\right| \leqslant K\left[I + \frac{10}{27}TK(I-Q)^{-1}\right]|z-y|. \tag{3.86}$$

PROOF. It follows from Theorem 3.17 that, for every $z \in D_{\frac{T}{2}M}$, the limit function $x^*(\cdot, z)$ of the uniformly convergent sequence (3.29) exists and is continuous. Therefore, when z varies over $D_{\frac{T}{2}M}$, the function Δ is also continuous and bounded,

$$\left|\Delta(z)\right| = \frac{1}{T}\left|\int_0^T f\big(t, x^*(t,z)\big)\,\mathrm{d}t\right| \leqslant \max_{(t,x)\in[0,T]\times D}\left|f(t,x)\right|.$$

Equality (3.85) implies

$$\begin{aligned}\left|\Delta(z) - \Delta(y)\right| &= \frac{1}{T}\left|\int_0^T \left[f\big(t, x^*(t,z)\big) - f\big(t, x^*(t,y)\big)\right]\mathrm{d}t\right| \\ &\leqslant \frac{1}{T}K\int_0^T \left|x^*(t,z) - x^*(t,y)\right|. \end{aligned}\tag{3.87}$$

Substituting (3.80) into (3.87), we get

$$\left|\Delta(z) - \Delta(y)\right| \leqslant \frac{1}{T}K\int_0^T \left[I + \bar{\alpha}_{1,T}(t)K(I-Q)^{-1}\right]\mathrm{d}t\,|z-y|. \tag{3.88}$$

Since

$$\int_0^T \bar{\alpha}_{1,T}(t)\,\mathrm{d}t = \frac{10}{9}\int_0^T 2t\left(1 - \frac{t}{T}\right)\mathrm{d}t = \frac{10}{27}T^2, \tag{3.89}$$

inequality (3.88) yields

$$\begin{aligned}\left|\Delta(z) - \Delta(y)\right| &\leqslant K|z-y| + \frac{10T}{27}K^2(I-Q)^{-1}|z-y| \\ &\leqslant K\left[I + \frac{10}{27}TK(I-Q)^{-1}\right]|z-y|,\end{aligned}$$

which completes the proof of our theorem. □

REMARK 3.24. Theorem 3.21 gives the following numerical-analytic algorithm for the construction of the T-periodic solutions of the non-autonomous T-periodic system (3.1).

(1) For a $z \in D_{\frac{T}{2}M}$, according to (3.29), we analytically construct the sequence of T-periodic functions $x_m(\cdot, z)$ depending on the parameter z.

(2) We find the limit $x^*(\cdot, z)$ of the sequence $x_m(\cdot, z)$.
(3) We construct the determining function Δ of the form (3.43).
(4) Using a suitable method for the numerical solution of non-linear algebraic or transcendental equations, we (approximately) find the solution(s) $z = z^*$ of the determining equation

$$\Delta(z) = 0 \tag{3.90}$$

lying in the set $D_{\frac{T}{2}M}$.
(5) After the substitution of the obtained value(s) $z = z^*$ into $x^*(\cdot, z)$, we get the (approximate) T-periodic solution(s) in the form $x^*(\cdot, z^*)$. This solution can also be obtained by solving the Cauchy problem $x(0) = z^*$ for the given differential system (3.3).

REMARK 3.25. Under our assumptions, the method under consideration allows one to find all the T-periodic solutions of (3.1) with values in the domain D because, by virtue of Theorem 3.21, the roots of the determining equation (3.90) "detect" the initial values of all the T-periodic solutions.

The principal difficulty in the realisation of the method is related with the analytic construction of the limit function $x^*(\cdot, z)$. However, in a number of cases, this problem can be avoided because, as can be shown, it is possible to prove the existence of a T-periodic solution on base of the properties of a certain approximation $x_m(\cdot, z)$ known in the analytic form.

3.2. *Existence theorems for periodic solutions*

In this section, we establish certain sufficient conditions for the solvability of periodic boundary value problem (3.3), (3.4) using the properties of successive approximations $\{x_m(\cdot, z)\}_{m\geqslant 0}$ of the form (3.29). Then, we indicate some necessary conditions for the existence of T-periodic solutions, i.e., conditions necessary for a certain subset of the set $D_{\frac{T}{2}M}$ to contain a point z^* determining the initial value $x^*(0) = z^*$ of the solution $x^*(\cdot)$ of the T-periodic problem (3.3), (3.4). We describe also the numerical algorithm for selecting these points.

To investigate the solvability of the periodic boundary value problem (3.3), (3.4), in addition to the *determining function* (3.85), we introduce the mth *approximate determining function*

$$D_{\frac{T}{2}M} \ni z \longmapsto \Delta_m(z) := \frac{1}{T} \int_0^T f\big(t, x_m(t, z)\big)\, \mathrm{d}t, \tag{3.91}$$

replacing $x^*(\cdot, z)$ in (3.85) by the function $x_m(\cdot, z)$ given by (3.29).

It is natural to expect that, under suitable conditions, the functions Δ and Δ_m are "close enough" to one another for m sufficiently large.

LEMMA 3.26. *Let us suppose that the function* f *satisfies assumptions* (3.5)–(3.7), *and* (3.8). *Then, for arbitrary* $z \in D_{\frac{T}{2}M}$ *and* $m \geqslant 0$,

$$\left|\Delta(z) - \Delta_m(z)\right| \leqslant \frac{10}{27} T K Q^m (I - Q)^{-1} M, \tag{3.92}$$

where the matrix Q *and vector* M *are given by* (3.45) *and* (3.6), *respectively.*

PROOF. By virtue of estimate (3.44) and relations (3.85), (3.91), we have

$$\begin{aligned}\left|\Delta(z) - \Delta_m(z)\right| &\leqslant \left|\frac{1}{T}\int_0^T \left[f\bigl(t, x^*(t,z)\bigr) - \int_0^T f\bigl(t, x_m(t,z)\bigr)\right] \mathrm{d}t\right| \\ &\leqslant \frac{1}{T} K \int_0^T \left|x^*(t,z) - x_m(t,z)\right| \mathrm{d}t \\ &\leqslant \frac{1}{T} K Q^m (I - Q)^{-1} M \int_0^T \bar{\alpha}_{1,T}(t)\, \mathrm{d}t.\end{aligned}$$

Taking (3.89) into account, we arrive at (3.92). □

REMARK 3.27. It is easy to see from Theorem 3.23, Lemma 3.26, and inequality (3.44) that the mappings (3.85), (3.91) are completely continuous.

Let us formulate a statement that establishes sufficient conditions for the solvability of the of the periodic boundary value problem (3.3), (3.4).

THEOREM 3.28. *Let us suppose that, in addition to assumptions* (3.5)–(3.7), *and* (3.8), *there exist a bounded closed subdomain* $\Omega \subset D_{\frac{T}{2}M}$ *and an number* $m \in \mathbb{N}$ *such that the approximate determining function* (3.91) *satisfies the following conditions*:

(a) *The relation*

$$\inf_{z \in \partial\Omega} \left|\Delta_m(z)\right| > \frac{10T}{27} K Q^m (I - Q)^{-1} M, \tag{3.93}$$

holds, where Q *is the matrix given by* (3.45);

(b) *The Brouwer degree*[1] *of* Δ_m *over* Ω *with respect to* 0 *satisfies the inequality*

$$\deg(\Delta_m, \Omega, 0) \neq 0. \tag{3.94}$$

Then the periodic boundary value problem (3.3), (3.4) *has a* T*-periodic solution* $x = x^*(\cdot)$ *with the initial value* $x^*(0)$ *belonging to* Ω.

[1] See, e.g., [17, Definition A2.5, Definition A2.9] and [81, Section 5].

PROOF. Recall that [17, Theorem A2.5] establishes the invariance of the topological degree by homotopy. Thus, if we show that the vector fields (3.85) and (3.91) are homotopic, then, in view of assumption (3.94),

$$\deg(\Delta, \Omega, 0) = \deg(\Delta_m, \Omega, 0), \tag{3.95}$$

whence we see that

$$\deg(\Delta, \Omega, 0) \neq 0. \tag{3.96}$$

Let us prove that the fields Δ and Δ_m are homotopic. For this purpose, we consider the "linear deformation" determined by the family of mappings

$$P(\theta, z) := \Delta_m(z) + \theta\big[\Delta(z) - \Delta_m(z)\big], \quad z \in \partial\Omega, \tag{3.97}$$

where $\theta \in [0, 1]$. Obviously, $P(\theta, \cdot)$ is a continuous mapping on $\partial\Omega$ for every θ from $[0, 1]$ and, furthermore,

$$P(0, z) = \Delta_m(z), \qquad P(1, z) = \Delta(z) \tag{3.98}$$

for all $z \in \partial\Omega$.

For an arbitrary z from $\partial\Omega$, in view of (3.98), (3.92), and (3.93), we have

$$\begin{aligned} |P(\theta, z)| &= \big|\Delta_m(z) + \theta\big[\Delta(z) - \Delta_m(z)\big]\big| \\ &\geqslant \big|\Delta_m(z)\big| - \big|\Delta(z) - \Delta_m(z)\big| > 0, \end{aligned} \tag{3.99}$$

i.e., $P(\theta, \cdot)$ does not vanish on $\partial\Omega$. We see that, being a non-degenerate deformation, (3.97) guarantees that (3.85) and (3.91) are homotopic. Therefore, (3.96) is true.

By [17, Theorem A2.4], inequality (3.96) guarantees the existence of a point $z^* \in \Omega$ such that $\Delta(z^*) = 0$. Consequently, in view of Theorem 3.21, the solution $x = x^*(\cdot)$ of the Cauchy problem

$$x(0) = z^*$$

for Eq. (3.3) is also a solution of the T-periodic boundary value problem (3.3), (3.4). Moreover,

$$x^*(t) = x^*\big(t, z^*\big), \quad t \in [0, T],$$

where $x^*(\cdot, z^*)$ is the limit function of the T-periodic sequence of functions defined by formula (3.29). □

REMARK 3.29. To verify condition (a) of Theorem 3.28 in concrete cases, one has to use the recurrence formula (3.29) to compute the function $x_m(\cdot, z)$ depending on $z \in \Omega$ as a

parameter, construct the corresponding mapping (3.91), and verify whether its topological degree is non-zero.

This is a rather difficult problem in general. However, in a number of important cases, there are relatively simple criteria to study this question. In particular, when Δ_m is an odd mapping, i.e.,

$$\Delta_m(-z) = -\Delta_m(z),$$

for any z, then, according to the Borsuk theorem (see, e.g., [81, p. 174] or [17, Theorem A2.12]), the Brouwer degree $\deg(\Delta_m, \Omega, 0)$ is an odd number and, therefore, is not equal to zero.

Alternatively, it follows directly from [17, Definition A2.1] of the topological degree that if the Jacobian matrix of the function (3.91) is non-degenerate at its isolated zero $z_{m0} \in \Omega$, i.e., if

$$\det \frac{\partial \Delta_m}{\partial z}(z_{m0}) \neq 0,$$

then inequality (3.96) holds.

REMARK 3.30. The assertion of Theorem 3.28 is valid, in particular, for $m = 0$. In this case, condition (3.93) is replaced by the relation

$$\inf_{z\in\partial\Omega} \left| \int_0^T f(t,z)\,\mathrm{d}t \right| > \frac{10T^2}{27} K(I-Q)^{-1} M. \tag{3.100}$$

Here, we do not have to compute any approximations at all. Due to its simplicity, inequality (3.100) may prove convenient for the investigation of periodic solutions with "small" periods.

To seek for points z^* through which the T-periodic solutions of (3.3) pass at $t = 0$, i.e., to search for zeroes of the determining function (3.85), is by no means an easy problem, because the analytic form of this mapping is, in general, unknown. For this reason, one has to use approximate determining functions and apply statements similar to Theorem 3.28. However, there are particular cases where the existence of zeroes of function (3.85) is clear from some other considerations.

Let us consider one of such cases, where the right-hand side function of Eq. (3.1) has a certain symmetric property.

We recall the classical Arzelà–Ascoli theorem (see, e.g., Arzelà [5], Petrovski [58]).

THEOREM 3.31. (See [5].) *Let $\{x_m(\cdot)\}_{m=1}^{\infty}$ be a uniformly bounded and equicontinuous sequence of functions on the interval $[\alpha, \beta]$. Then it contains a subsequence $\{x_{m_k}(\cdot)\}_{k=1}^{\infty}$ uniformly convergent on $[\alpha, \beta]$.*

The uniform boundedness of the sequence means that there exists a positive constant K for which $\max_{t\in[\alpha,\beta]} |x_m(t)| \leqslant K$ independently of $m = 1, 2, \ldots$, whereas its equicon-

tinuity means that, for an arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\{t_1, t_2\} \subset [\alpha, \beta]$ satisfy the inequality $|t_1 - t_2| < \delta$, then

$$\sup_{m \geqslant 1} \left| x_m(t_1) - x_m(t_2) \right| < \varepsilon. \quad m = 1, 2, \ldots.$$

The following theorem is a modified version of [89, Theorem 8.1].

THEOREM 3.32. *Let the right-hand side function* $f : \mathbb{R} \times D \to \mathbb{R}^n$ *of the differential system* (3.1) *be defined on* $\mathbb{R} \times D$, *where* D *is the closure of a bounded domain in* $\mathbb{R}^n$, *and let* f *satisfy the following conditions*:

(1) *The function* f *is continuous and periodic of period* T *in the first argument and, with a certain vector* $M \in \mathbb{R}^n_+$, *satisfies condition* (3.6);
(2) *The equality*

$$f(-t, x) = -f(t, x) \tag{3.101}$$

holds for all $t \in (-\infty, \infty)$ *and* $x \in D$;
(3) *Condition* (3.8) *is satisfied*: $D_{\frac{T}{2}M} \neq \varnothing$.

Then, for every $z \in D_{\frac{T}{2}M}$, *system* (3.1) *possesses an even* T*-periodic solution satisfying the initial condition*

$$x(0) = z. \tag{3.102}$$

If, moreover, there exists a non-negative matrix $K \in \mathrm{GL}_n(\mathbb{R})$ *such that* f *satisfies the Lipschitz* (3.5) *for all* $t \in \mathbb{R}$ *and* $\{x_1, x_2\}$ *from* D, *then, for any* $z \in D_{\frac{T}{2}M}$, *system* (3.1) *has no other solutions with property* (3.102) *and graph lying in* D.

In other words, under conditions of Theorem 3.32, the set $D_{\frac{T}{2}M}$ consists of the initial values of T-periodic solutions of system (3.1).

PROOF OF THEOREM 3.32. Let us fix a vector $z \in D_{\frac{T}{2}M}$ and consider the Picard successive approximations

$$x_{m+1}(t) = z + \int_0^t f\big(s, x_m(s)\big)\, \mathrm{d}s, \quad m = 0, 1, 2, \ldots, \tag{3.103}$$

where $x_0(t) = z$ for all t from $(-\infty, \infty)$.

By assumption, $t \mapsto f(t, z)$ is an odd function for any fixed z and, hence,

$$\int_0^T f(t, z)\, \mathrm{d}t = 0.$$

Therefore, formula (3.103) and the T-periodicity of sequence (3.29) yield

$$x_1(t) = z + \int_0^t \left(f(s, z) - \frac{1}{T} \int_0^T f(\tau, z)\, \mathrm{d}\tau \right) \mathrm{d}s = x_1(t + T) \tag{3.104}$$

for any $t \in (-\infty, \infty)$, i.e., the function x_1 is T-periodic. Let us argue by analogy with the proof of Theorem 3.17. Taking assumption (3.6) into account, we arrive at estimate (3.47), whence, in view of (3.104), it follows that, for all t,

$$\left|x_1(t) - z\right| \leqslant \frac{T}{2} M,$$

i.e., the function x_1 takes values in D. Furthermore, being the integral of an odd function, x_1 is even, i.e., $x_1(t) = x_1(-t)$ for any t from $(-\infty, \infty)$.

Taking assumption (3.6) into account and arguing by induction, one can show that, for an arbitrary $m \geqslant 1$, the function $x_m(\cdot)$ given by formula (3.103) is well-defined on $\mathbb{R}$, periodic with period T, and satisfies the relations

$$\int_0^T f\bigl(t, x_m(t)\bigr)\,\mathrm{d}t = 0, \tag{3.105}$$

$$x_m(t) = x_m(-t), \tag{3.106}$$

and (3.48) for all t from $(-\infty, \infty)$.

Inequality (3.48) implies the uniform boundedness of sequence (3.103). The form of sequence (3.103) and the continuity of f on the bounded set $[0, T] \times D$ guarantees the equicontinuity of sequence (3.103). Thus, in view of the Arzelà–Ascoli theorem (Theorem 3.31), we can select a uniformly convergent subsequence $\{x_{m_k} \mid k \geqslant 0\}$ of the sequence of T-periodic functions $\{x_m \mid m \geqslant 0\}$. Hence, the function

$$\lim_{k\to\infty} x_{m_k} =: x \tag{3.107}$$

is well-defined on the entire $\mathbb{R}$. Setting $m = m_k$, $k = 1, 2, \ldots$, in (3.103), (3.106) and passing to the limit as $k \to +\infty$, we obtain that x is an even function satisfying the differential system (3.1). Moreover, x is periodic with period T because each of the functions x_{m_k}, $k = 1, 2, \ldots$, has this property and the limit in (3.107) is uniform. Thus, x is an even T-periodic solution of (3.1) satisfying condition (3.102).

Let now f satisfy the Lipschitz condition (3.5) for all $t \in \mathbb{R}$ and $\{x_1, x_2\}$ from D. According to assumption (3.5), the conditions of the Picard–Lindelöf existence and uniqueness theorem (see, for instance [17, Theorem 1.1.1]) hold, whence the uniqueness of the solution in question follows. □

EXAMPLE 3.33. Let us apply the results above to study the 1-periodic solutions of the system

$$\begin{aligned} x_1'(t) &= a x_2(t) \sin 4\pi t, \\ x_2'(t) &= b\bigl(x_1(t)\bigr)^2 \sin 2\pi t + \sin 4\pi t, \quad t \in [0, T], \end{aligned} \tag{3.108}$$

where $a \neq 0$, $b \in \mathbb{R}$, $T = 1$.

It is obvious that property (3.101) holds for the right-hand side of (3.108). Let the system be defined in the domain

$$(t, x) \in [0, T] \times D, \quad D: |x_1| \leqslant A, |x_2| \leqslant B. \tag{3.109}$$

Then

$$\left| f(t, x) \right| \leqslant \operatorname{col}\bigl(|a|B, 1 + |b|A^2\bigr). \tag{3.110}$$

The set (3.8) for system (3.108) is defined as follows:

$$D_{\frac{T}{2}M} = \left\{ (x_1, x_2) \,\middle|\, |x_1| \leqslant A - \frac{1}{2}|a|B, |x_2| \leqslant B - \frac{1}{2}\bigl[1 + |b|A^2\bigr] \right\}. \tag{3.111}$$

This set is non-empty if

$$A - \frac{1}{2}|a|B \geqslant 0,$$
$$B - \frac{1}{2}\bigl[1 + |b|A^2\bigr] \geqslant 0,$$

i.e., when

$$\frac{1}{2} + \frac{1}{2}|b|A^2 \leqslant B \leqslant \frac{2A}{|a|}. \tag{3.112}$$

The Lipschitz condition (3.5) holds with the matrix

$$K = \begin{bmatrix} 0 & |a| \\ 2b|A| & 0 \end{bmatrix}.$$

Consequently, if our domain satisfies condition (3.112), by virtue of Theorem 3.32, every solution $x(t)$ of system (3.108) with the initial value $x(0)$ lying in domain (3.111) is periodic with period 1. For instance, if

$$a = \frac{1}{2}, \quad b = -1, \quad A = 1,$$

and, according to (3.112),

$$1 \leqslant B \leqslant 4,$$

then the set

$$D_{\frac{T}{2}M} = \left\{ (x_1, x_2) \,\middle|\, |x_1| \leqslant 1 - \frac{B}{4}, |x_2| \leqslant B - 1 \right\}$$

consists of the initial values of 1-periodic solutions of system (3.108). In particular, for $A = 1$ and $B = 3$, all points from the set

$$D_{\frac{T}{2}M} = \left\{ (x_1, x_2) \,\middle|\, |x_1| \leqslant \frac{1}{4}, |x_2| \leqslant 2 \right\}$$

can serve as the initial values of 1-periodic solutions of system (3.108).

Let us find the first approximation of the 1-periodic solutions of the given system (3.108), supposing that the parameters a, b and the domain (3.109) such that the inequalities (3.112) are true and the vector of initial values $z = \operatorname{col}(z_1, z_2)$ belongs to subset $D_{\frac{T}{2}M}$ of the form (3.111). According to formula (3.103) for $m = 0$ we get:

$$\begin{aligned} x_{11}(t) &= z_1 + \int_0^t a z_2 \sin 4\pi s \,\mathrm{d}s = \left(z_1 + \frac{a z_2}{4\pi} \right) - \frac{a z_2}{4\pi} \cos 4\pi t, \\ x_{21}(t) &= z_2 + \int_0^t \left[b z_1^2 \sin 2\pi s + \sin 4\pi s \right] \mathrm{d}s \\ &= \left(z_2 + \frac{b z_1^2}{2\pi} + \frac{1}{4\pi} \right) - \frac{b z_1^2}{2\pi} \cos 2\pi t - \frac{\cos 4\pi t}{4\pi}. \end{aligned} \tag{3.113}$$

Similarly to (3.113) one can easily get the second approximation.

EXAMPLE 3.34. Consider the damped Duffing equation with periodic forcing term (see (4.3.15) in [17])

$$x''(t) + d x'(t) + a x + b\big(x(t)\big)^3 = H \sin \frac{2\pi t}{T}, \quad t \in [0, T], \tag{3.114}$$

for some $T > 0$, $d \in \mathbb{R}$, $a \in \mathbb{R}$, $H \in \mathbb{R}$, $b \in \mathbb{R}$ (if $b > 0$ this is the "hard spring case," and if $b < 0$ this is the "soft spring case"). Rewriting the differential equation (3.114) in the Cauchy normal form, we get

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= H \sin \frac{2\pi t}{T} - d x_2(t) - a x_1(t) - b\big(x_1(t)\big)^3. \end{aligned} \tag{3.115}$$

Let the system (3.115) be defined for (t, x) from $[0, T] \times D$, where

$$D = \big\{ (x_1, x_2) \mid |x_1| \leqslant A, |x_2| \leqslant B \big\}. \tag{3.116}$$

Then

$$\big| f(t, x) \big| \leqslant \operatorname{col}\big(B, |H| + |d| B + |a| A + |b| A^3 \big).$$

The subset (3.8) for system (3.115) is given by the formula

$$D_{\frac{T}{2}M} = \Big\{(x_1, x_2) \;\Big|\; |x_1| \leqslant A - \frac{T}{2}B,$$
$$|x_2| \leqslant B - \frac{T}{2}\big[|H| + |d|B + |a|A + |b|A^3\big]\Big\}, \tag{3.117}$$

which is non-empty if

$$A - \frac{T}{2}B \geqslant 0,$$
$$B - \frac{T}{2}\big[|H| + |d|B + |a|A + |b|A^3\big] \geqslant 0,$$

i.e., when

$$\begin{aligned} &\frac{T}{2 - T|d|}\big[|H| + |a|A + |b|A^3\big] \leqslant B \leqslant \frac{2}{T}A, \\ &T|d| < 2. \end{aligned} \tag{3.118}$$

The Lipschitz condition (3.5) is fulfilled for system (3.115) with the matrix

$$K = \begin{bmatrix} 0 & 1 \\ |a| + 3|b|A^2 & |d| \end{bmatrix}, \tag{3.119}$$

whose greatest eigenvalue is equal to

$$r(K) = \frac{|d|}{2} + \sqrt{\frac{d^2}{4} + |a| + 3|b|A^2}.$$

Therefore, the maximal eigen-value of the matrix Q of the form (3.45) is less that 1 if

$$\frac{|d|}{2} + \sqrt{\frac{d^2}{4} + |a| + 3|b|A^2} < \frac{10}{3T}. \tag{3.120}$$

Thus, for system (3.115), conditions (3.5)–(3.7), and (3.8) hold in domain (3.116) if the parameters in Eq. (3.114) satisfy inequalities (3.117)–(3.120). If, for instance,

$$T = \frac{1}{2}, \quad b = 2, \quad d = 1, \quad a = 1, \quad H = 1, \quad A = 1 \tag{3.121}$$

(the “hard spring case”), the domain D is

$$D = \big\{(x_1, x_2) \mid |x_1| \leqslant 1, |x_2| \leqslant B\big\} \tag{3.122}$$

and, according to (3.118),

$$\frac{4}{3} \leqslant B \leqslant 4.$$

In particular, for $B = 2$ the subset $D_{\frac{T}{2}M}$ has the form

$$D_{\frac{T}{2}M} = \left\{ (x_1, x_2) \;\middle|\; |x_1| \leqslant \frac{1}{2}, |x_2| \leqslant \frac{1}{2} \right\} \tag{3.123}$$

and, thus, the initial value of the $\frac{1}{2}$-periodic solution must belong to the set (3.123). The successive approximations (3.29) are well-defined. So, the first approximation has the form:

$$\begin{aligned}
x_{11}(t, z) &= z_1 + \int_0^t \left[z_2 - \frac{1}{T} \int_0^T z_2 \, \mathrm{d}\tau \right] \mathrm{d}s = z_1, \\
x_{21}(t, z) &= z_2 + \int_0^t \Bigg[H \sin \frac{2\pi}{T} s - d z_2 - a z_1 - b z_1^3 \\
&\qquad - \frac{1}{T} \int_0^T \left(H \sin \frac{2\pi}{T} \tau - d z_2 - a z_1 - b z_1^3 \right) \mathrm{d}\tau \Bigg] \mathrm{d}s \\
&= z_2 + \frac{HT}{2\pi} - \frac{HT}{2\pi} \cos \frac{2\pi}{T} t = z_2 + \frac{1}{4\pi} - \frac{1}{4\pi} \cos 4\pi t.
\end{aligned} \tag{3.124}$$

The first approximation to the determining equation of the form (3.91) is

$$\begin{aligned}
\Delta_{11}(z) &= \frac{1}{T} \int_0^T \left[z_2 + \frac{1}{4\pi} - \frac{1}{4\pi} \cos 4\pi t \right] \mathrm{d}t \\
&= z_2 + \frac{1}{4\pi} = 0, \\
\Delta_{21}(z) &= \frac{1}{T} \int_0^T \left[\sin 4\pi t - \left(-\frac{1}{4\pi} + \frac{1}{4\pi} - \frac{1}{4\pi} \cos 4\pi t \right) - z_1 - 2 z_1^3 \right] \mathrm{d}t \\
&= -z_1 - 2 z_1^3 = 0,
\end{aligned} \tag{3.125}$$

whence

$$z_2 = -\frac{1}{4\pi}, \qquad z_1 = 0.$$

Thus, the initial value of the $\frac{1}{2}$-periodic solution of system (3.115), in the first approximation, is

$$\operatorname{col}(z_1, z_2) = \operatorname{col}\left(0, -\frac{1}{4\pi} \right).$$

Let us take, instead of (3.121), the following values:

$$T = \frac{1}{2}, \quad b = -1, \quad d = 1, \quad a = \frac{1}{9}, \quad H = 1, \quad A = 1 \tag{3.126}$$

(the "soft spring case"). Then, in view of (3.118), the domain D is given by (3.122), where

$$\frac{19}{27} < B \leqslant 4. \tag{3.127}$$

In particular, for $A = 1$ and $B = 2$ the subset $D_{\frac{T}{2}M}$ is the following:

$$D_{\frac{T}{2}M} = \left\{ (x_1, x_2) \;\middle|\; |x_1| \leqslant \frac{1}{2}, |x_2| \leqslant \frac{19}{28} \right\}. \tag{3.128}$$

Taking into account (3.29) and performing the computation, we obtain that the first approximation is also given by formula (3.124). However, instead of (3.125), we have

$$\begin{aligned}\Delta_{21}(z) &= \frac{1}{T}\int_0^T \left[\sin 4\pi t + \frac{1}{4\pi}\cos 4\pi t - \frac{1}{9}z_1 + z_1^3\right] \mathrm{d}t \\ &= -\frac{1}{9}z_1 + z_1^3 = 0,\end{aligned}$$

whence

$$z_1\left(-\frac{1}{9} + z_1^2\right) = 0, \quad z_1^{(1)} = 0, \quad z_1^{(2)} = \frac{1}{3}, \quad z_1^{(3)} = -\frac{1}{3}.$$

Thus, for the Duffing equation (3.114), (3.126) (the "soft spring case"), in the domain (3.128), on base of the first approximation, we find three initial values for the $\frac{1}{2}$-periodic solutions:

$$\begin{aligned}&\operatorname{col}(z_1, z_2) = \operatorname{col}\left(0, -\frac{1}{4\pi}\right), \\ &\operatorname{col}(z_1, z_2) = \operatorname{col}\left(\frac{1}{3}, -\frac{1}{4\pi}\right), \qquad \operatorname{col}(z_1, z_2) = \operatorname{col}\left(-\frac{1}{3}, -\frac{1}{4\pi}\right).\end{aligned} \tag{3.129}$$

Note that it is impossible to apply the results of Rouche and Mawhin [81] (see also Theorem 4.3.4 in Farkas [17]) to study the Duffing differential equation (3.114) with parameters (3.121), (3.126) in the domain D: $|x_1| \leqslant 1$, $|x_2| \leqslant 2$, because we assume that the "hard spring case" ($b > 0$) may take place and, moreover, the inequalities

$$aA + bA^3 + H\sin\omega t < 0, \quad -aA - bA^3 + H\sin\omega t > 0$$

of the mentioned [17, Theorem 4.3.4] (see also Corollary 5.6 from [81, p. 195]) do not hold in the domain D under consideration. These inequalities may hold in a "smaller"

domain. So, this approach adds to the possibilities of investigation of the Duffing type equation (3.114).

EXAMPLE 3.35. Let us investigate the T-periodic solution of the non-autonomous Van der Pol equation with a periodic forcing term of the form

$$x''(t) + a\big(\big(x(t)\big)^2 - 1\big)x'(t) + x(t) = H\sin\omega t, \quad t \in [0, T], \tag{3.130}$$

where $T = 2\pi\omega^{-1}$, $a \in \mathbb{R}$, $H \in \mathbb{R}$. Equation (3.130) can be rewritten as the system

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= H\sin\omega t - a\big(\big(x_1(t)\big)^2 - 1\big)x_2(t) - x_1(t), \quad t \in [0, T], \end{aligned} \tag{3.131}$$

which is considered in $[0, T] \times D$ with

$$D = \big\{(x_1, x_2) \mid |x_1| \leqslant A, |x_2| \leqslant B\big\}. \tag{3.132}$$

It is easy to see that f, the right-hand side function in (3.131), satisfies the estimate

$$\big|f(t, x)\big| \leqslant \operatorname{col}\big(B, |H| + |a|\big(A^2 + 1\big)B + A\big) \tag{3.133}$$

for any $(t, x) \in [0, T] \times D$. For M, we can thus take the term from the right-hand side of (3.133). Then

$$\begin{aligned} D_{\frac{T}{2}M} = \bigg\{(x_1, x_2) \,\bigg|\, &|x_1| \leqslant A - \frac{T}{2}B, \\ &|x_2| \leqslant B - \frac{T}{2}\big[|H| + |a|(A^2 + 1)B + A\big]\bigg\}. \end{aligned} \tag{3.134}$$

The set $D_{\frac{T}{2}M}$ is non-empty if

$$A - \frac{T}{2}B \geqslant 0, \tag{3.135}$$

$$B - \frac{T}{2}\big[|H| + |a|(A^2 + 1)B + A\big] \geqslant 0. \tag{3.136}$$

It is easy to see that inequalities (3.135) and (3.136) hold whenever

$$\frac{T[|H| + A]}{2 - T|a|(A^2 + 1)} \leqslant B \leqslant \frac{2}{T}A, \tag{3.137}$$

for a and A such that $T|a|(A^2 + 1) < 2$.

For the system (3.131) the Lipschitz condition (3.5) holds with the matrix

$$K = \begin{bmatrix} 0 & 1 \\ 1 + 2|a|AB & |a|\bigl(A^2 + 1\bigr) \end{bmatrix},$$

whose eigenvalues λ are given by the equalities

$$\lambda = \frac{|a|(A^2+1)}{2} \pm \sqrt{\frac{a^2(A^2+1)^2}{4} + 1 + 2|a|AB}.$$

The greatest eigenvalue of the matrix Q of the form (3.45) is, therefore, less then 1 whenever

$$|a|\bigl(A^2+1\bigr) + \sqrt{a^2\bigl(A^2+1\bigr)^2 + 4 + 8|a|AB} < \frac{20}{3T}. \tag{3.138}$$

Thus, conditions (3.5), (3.6), (3.7), and (3.8) are fulfilled for system (3.131) in domain (3.132) if (3.134), (3.137), and (3.138) are true. For instance, if

$$T = 1, \quad \omega = 2\pi, \quad a = -\frac{1}{2}, \quad A = 1, \quad H = \frac{1}{2}, \tag{3.139}$$

then (3.137) means that $\frac{3}{2} \leqslant B \leqslant 2$. Specifically, for the domain

$$D = \left\{ (x_1, x_2) \;\middle|\; |x_1| \leqslant 1 = A, |x_2| \leqslant \frac{7}{4} \right\} \tag{3.140}$$

corresponding to $B = \frac{7}{4}$, set (3.134) is given by the formula

$$D_{\frac{T}{2}M} = \left\{ (x_1, x_2) \;\middle|\; |x_1| \leqslant \frac{1}{8}, |x_2| \leqslant \frac{1}{8} \right\}. \tag{3.141}$$

Thus, for the Van der Pol system (3.131) conditions (3.5)–(3.140) with parameters (3.139), (3.141) and, hence, the successive approximations (3.29) are well-defined.

According to (3.29), the first approximation is

$$\begin{aligned}
x_{11}(t,z) &= z_1 + \int_0^t \left[z_2 - \int_0^1 z_2 \,\mathrm{d}\tau \right] \mathrm{d}s = z_1, \\
x_{21}(t,z) &= z_2 + \int_0^t \left[\frac{1}{2} \sin 2\pi t + \frac{1}{2}\bigl(z_1^2 - 1\bigr) z_2 - z_1 \right. \\
&\qquad \left. - \int_0^1 \left(\frac{1}{2} \sin 2\pi\tau + \frac{1}{2}\bigl(z_1^2 - 1\bigr) z_2 - z_1 \right) \mathrm{d}\tau \right] \mathrm{d}s \\
&= z_2 + \frac{1}{4\pi} - \frac{1}{4\pi} \cos 2\pi t.
\end{aligned}$$

The first approximation to determining equation (3.91) has the form

$$\Delta_{11}(z) = \int_0^1 \left[z_2 + \frac{1}{4\pi} - \frac{1}{4\pi}\cos 4\pi t \right] \mathrm{d}t = z_2 + \frac{1}{4\pi} = 0,$$

$$\Delta_{21}(z) = \int_0^1 \left[\frac{1}{2}\sin 2\pi t + \frac{1}{2}\big(z_1^2 - 1\big)\left(z_2 + \frac{1}{4\pi} - \frac{1}{4\pi}\cos 2\pi t \right) - z_1 \right] \mathrm{d}t$$

$$= \int_0^1 \left[-\frac{1}{8\pi} z_1^2 + \frac{1}{8\pi} + \frac{1}{8\pi} z_1^2 - \frac{1}{8\pi} - z_1 \right] \mathrm{d}t = 0,$$

whence we find that

$$z_2 = -\frac{1}{4\pi}, \qquad z_1 = 0.$$

So, the approximate initial value of the 1-periodic solution of the Van der Pol system (3.131), (3.139) is given by the formula

$$\mathrm{col}(z_1, z_2) = \mathrm{col}\left(0, -\frac{1}{4\pi} \right).$$

The components of the second approximation are:

$$x_{12}(t, z) = z_1 + \int_0^t \left[z_2 + \frac{1}{4\pi} - \frac{1}{4\pi}\cos 2\pi t - \int_0^1 \left(z_2 + \frac{1}{4\pi} - \frac{1}{4\pi}\cos 2\pi s \right) \right] \mathrm{d}t = z_1 - \frac{1}{8\pi^2}\sin 2\pi t,$$

$$x_{22}(t, z) = z_2 + \int_0^t \left[\frac{1}{2}\sin 2\pi t + \frac{1}{2}\big(z_1^2 - 1\big)\left(z_2 + \frac{1}{4\pi} - \frac{1}{4\pi}\cos 2\pi t \right) - z_1 - \int_0^1 \left[\frac{1}{2}\sin 2\pi s + \frac{1}{2}\big(z_1^2 - 1\big)\left(z_2 + \frac{1}{4\pi} - \frac{1}{4\pi}\cos 2\pi s \right) - z_1 \right] \mathrm{d}s \right] \mathrm{d}t$$

$$= z_2 + \frac{1}{4\pi} - \frac{1}{4\pi}\cos 2\pi t - \frac{1}{16\pi^2}\big(z_1^2 - 1\big)\sin 2\pi t,$$

whereas the second approximate determining system

$$\Delta_{12}(z) = \int_0^1 \left[z_2 + \frac{1}{4\pi} - \frac{1}{4\pi}\cos 2\pi t - \frac{1}{16\pi^2}(z_1^2 - 1)\sin 2\pi t \right] \mathrm{d}t$$

$$= z_2 + \frac{1}{4\pi} = 0,$$

$$\begin{aligned}\Delta_{22}(z) &= \int_0^1 \left[\frac{1}{2}\sin 2\pi t + \frac{1}{2}\left(z_1^2 - \frac{1}{4\pi^2}\sin 2\pi t + \frac{1}{64\pi^4}\sin^2 2\pi t - 1\right)\right.\\ &\quad \times \left(z_2 + \frac{1}{4\pi} - \frac{1}{4\pi}\cos 2\pi t - \frac{1}{16\pi^2}\left(z_1^2 - 1\right)\sin 2\pi t\right)\\ &\quad \left. - z_1 + \frac{1}{8\pi^2}\sin 2\pi t\right]\mathrm{d}t\\ &= \frac{1}{2}\int_0^1 \left[\frac{1}{64\pi^4}\left(z_1^2 - 1\right)\sin^2 2\pi t - z_1\right]\mathrm{d}t\\ &= \frac{1}{128\pi^4}\left(z_1^2 - 1\right) - 2z_1 = 0,\end{aligned}$$

has the solution

$$z_2 = -\frac{1}{4\pi}, \qquad z_1 = 128\pi^4 \pm \sqrt{\left(128\pi^4\right)^2 + 1},$$

because z_1 satisfies the equality $z_1^2 - 256\pi^4 z_1 - 1 = 0$.

Taking into account the fact that $\operatorname{col}(z_1, z_2)$ must belong to domain (3.141), we obtain that, in the second approximation, the initial value of the 1-periodic solution of (3.131), (3.139) has the form

$$\operatorname{col}(z_1, z_2) = \operatorname{col}\left(128\pi^4 - \sqrt{\left(128\pi^4\right)^2 + 1}, -\frac{1}{4\pi}\right).$$

Again we note that Corollary 5.6 from [81, p. 195] cannot be applied to Van der Pol's system (3.131), (3.139) in domain (3.140), because the inequalities

$$-1 + \frac{1}{2}\sin\omega t > 0, \qquad 1 + \frac{1}{2}\sin\omega t < 0$$

from the assumptions of the corollary mentioned do not hold for all $t \in [0, T]$.

The following theorem gives a condition necessary for $G \subset D_{\frac{T}{2}M}$ to contain the initial value z^* of a T-periodic solution $x^*(t) = x^*(t, 0, z^*)$ of system (3.3).

THEOREM 3.36. *Suppose that the T-periodic boundary value problem* (3.3), (3.4) *satisfies conditions* (3.5), (3.6), (3.61), *and* (3.62). *Then in order that a subset $G \subset D_{\frac{T}{2}M}$ contain the initial value $z^* = x^*(0)$ of the T-periodic solution $x = x^*(t)$ of* (3.1) *at $t = 0$, it is necessary that, for arbitrary $w \in G$ and integer m, the following inequality be satisfied*:

$$\begin{aligned}\Delta_m(w) &\leqslant \sup_{z\in G} K\left[I + \frac{10}{27}TK(I-Q)^{-1}\right]|z - w|\\ &\quad + \frac{10}{27}TKQ^m(I-Q)^{-1}M,\end{aligned} \tag{3.142}$$

where Δ_m *is the* m*-th approximation* (3.91) *of the determining function* (3.85) *and* Q *is the matrix given by* (3.45).

PROOF. Assume that $z = z^*$ is a solution of the determining equation (3.85), i.e., $\Delta(z^*) = 0$. Consequently, according to Theorem 3.21, $z^* = x^*(0)$ is the initial value at $t = 0$ of the periodic solution $x^*(t) = x^*(t, z^*)$, $t \in [0, T]$, where $x^*(\cdot, z)$ is the limit function of the sequence (3.29).

Let apply estimate (3.86) from Theorem 3.23 with $z = z^*$, $y = w$. Then it follows from (3.86) that

$$\left|\Delta(w)\right| = K\left[I + \frac{10}{27}TK(I-Q)^{-1}\right]|z^* - w|. \tag{3.143}$$

In view of inequality (3.92) of Lemma 3.26, we obtain

$$\begin{aligned}
&\left|\Delta(w) - \Delta_m(w)\right| \leqslant \frac{10}{27}TKQ^m(I-Q)^{-1}M, \\
&\left|\Delta_m(w)\right| \leqslant \left|\Delta(w)\right| + \frac{10}{27}TKQ^m(I-Q)^{-1}M. \qquad (3.144)
\end{aligned}$$

Combining (3.144) with (3.143), we prove the validity of inequality (3.142):

$$\begin{aligned}
\left|\Delta_m(w)\right| &\leqslant K\left[I + \frac{10}{27}TK(I-Q)^{-1}\right]|z^* - w| + \frac{10}{27}TKQ^m(I-Q)^{-1}M \\
&\leqslant \sup_{z\in G} K\left[I + \frac{10}{27}TK(I-Q)^{-1}\right]|z - w| + \frac{10}{27}TKQ^m(I-Q)^{-1}M.
\end{aligned}$$

The theorem is proved. □

REMARK 3.37. According to Theorem 3.36, we can indicate an algorithm for approximate determination of the initial value z^* of a solution of the T-periodic boundary value problem (3.3), (3.4). For this purpose, we represent the subset $D_{\frac{T}{2}M}$ defined in (3.8) as the union of finitely many subsets Ω_i, i.e.,

$$D_{\frac{T}{2}M} = \bigcup_{i=1}^{N} \Omega_i.$$

In every Ω_i, we choose an arbitrary point $w^i \in \Omega_i$ and, for a certain m, we compute the value of the mth successive approximation $x_m(t, w^i)$ according to formula (3.29) and the value of the corresponding approximate determining function, $\Delta_m(w^i)$ according to (3.91). Then, by using (3.142), we exclude the subsets Ω_i for an arbitrary point w^i of which the inequality inverse to (3.142), i.e.,

$$\left|\Delta_m(w^i)\right| > K\left[I + \frac{10}{27}TK(I-Q)^{-1}\right]\sup_{z\in G}|z - w^i| + \frac{10}{27}TKQ^m(I-Q)^{-1}M$$

holds. The reason for this is provided by Theorem 3.36, which guarantees that such subsets cannot contain initial values of solution of the periodic boundary value problem (3.3), (3.4).

The remaining subsets $\Omega_{i_1}, \Omega_{i_2}, \ldots, \Omega_{i_s}$ form a certain set $\Omega_{m,N}$,

$$\Omega_{m,N} := \bigcup_{\nu=1}^{s} \Omega_{i_\nu},$$

such that only through its points the T-periodic solutions of system (3.1) can pass. When N and m tend to ∞, the set $\Omega_{m,N}$ "tends" (see Problem 10.1) to the set Ω^* of *all* the initial values of the T-periodic solutions with values in $D_{\frac{T}{2}M}$. Every point u from $\Omega_{m,N}$ can be regarded as an approximate value of an exact initial value $z^* \in D^*$. In this case, obviously,

$$\left|u - z^*\right| \leqslant \sup_{u \in \Omega_{m,N}} |u - z|, \tag{3.145}$$

and the function $x_m(\cdot, u)$ given by the recurrence formula (3.29) can naturally be taken as an approximate T-periodic solution of the T-periodic boundary value problem (3.3), (3.4).

Let us estimate the absolute error of the approximate T-periodic solution, i.e., the deviation of the limit function $x^*(\cdot, z^*)$ from the approximate T-periodic solution $x_m(\cdot, u)$.

THEOREM 3.38. *Let conditions* (3.5), (3.6), (3.61), *and* (3.62) *be satisfied for the periodic boundary value problem* (3.3), (3.4); z^* *be a solution of the exact determining equation* (3.85), *and* u *an arbitrary point from the subset* $\Omega_{m,N}$ *satisfying condition* (3.145). *Then for the deviation of the exact* T*-periodic solution* $x^*(\cdot, z^*)$ *from its approximation* $x_m(\cdot, u)$ *given by* (3.29), *the following inequality is true*:

$$\begin{aligned} \left|x^*(t, z^*) - x_m(t, u)\right| \leqslant{}& \bar{\alpha}_{1,T}(t) Q^m (I - Q)^{-1} M \\ &+ \left(I + \bar{\alpha}_{1,T}(t) K (I - Q)^{-1}\right) \sup_{z \in \Omega_{m,N}} |z - u|, \end{aligned} \tag{3.146}$$

where Q *is given by* (3.45) *and* $\bar{\alpha}_{1,T}$ *is the function* (3.37).

PROOF. Let us use the obvious inequality

$$\begin{aligned} \left|x^*(t, z^*) - x_m(t, u)\right| \leqslant{}& \left|x^*(t, z^*) - x_m(t, z^*)\right| \\ &+ \left|x_m(t, z^*) - x_m(t, u)\right|. \end{aligned} \tag{3.147}$$

We estimate the first term on the right-hand side of (3.147) by using (3.44):

$$\left|x^*(t, z^*) - x_m(t, z^*)\right| \leqslant \bar{\alpha}_{1,T}(t) Q^m (I - Q)^{-1} M, \tag{3.148}$$

whereas the second term is estimated according to (3.83):

$$\begin{aligned}
&\bigl|x_m(t,z^*) - x_m(t,u)\bigr| \\
&\quad\leqslant \bigl[I + \alpha_{1,T}(t)K + \alpha_{2,T}(t)K^2 + \cdots + \alpha_{m,T}(t)K^m\bigr]|z^* - u| \\
&\quad\leqslant \left[I + \bar{\alpha}_{1,T}(t)\sum_{i=0}^{m-1} K\left(\frac{3T}{10}K\right)^i\right]|z^* - u| \\
&\quad\leqslant \bigl[I + \bar{\alpha}_{1,T}(t)K(I-Q)^{-1}\bigr]|z^* - u|.
\end{aligned} \tag{3.149}$$

Consequently, in view of (3.145), (3.146), (3.149) inequality (3.147) yields

$$\begin{aligned}
\bigl|x^*(t,z^*) - x_m(t,u)\bigr| &\leqslant \bar{\alpha}_{1,T}(t)Q^m(I-Q)^{-1}M \\
&\quad + \bigl[I + \bar{\alpha}_{1,T}(t)K(I-Q)^{-1}\bigr]|z^* - u|,
\end{aligned}$$

or

$$\begin{aligned}
\bigl|x^*(t,z^*) - x_m(t,u)\bigr| &\leqslant \bar{\alpha}_{1,T}(t)Q^m(I-Q)^{-1}M \\
&\quad + \bigl[I + \bar{\alpha}_{1,T}(t)K(I-Q)^{-1}\bigr]\sup_{z\in\Omega_{m,N}}|z - u|,
\end{aligned} \tag{3.150}$$

i.e., estimate (3.146) is true. □

4. Successive approximations for autonomous systems

In this section, we present some results concerning the application of the successive approximation techniques to autonomous systems of differential equations. Consider the case where the right-hand side of system (3.1) is independent of t:

$$x'(t) = f\bigl(x(t)\bigr), \quad -\infty < t < \infty. \tag{4.1}$$

Here, $f : D \to \mathbb{R}^n$ is a continuous function and $D \subset \mathbb{R}^n$ is a closed bounded set. We are interested in T-periodic solutions of system (4.1).

Unlike the non-autonomous systems, where the period T of the solution is known, in the autonomous case, besides the initial values of the periodic solution of (4.1), one should also find its period. Moreover, the T-periodic solution $x = x^*(t)$ of autonomous system (4.1) is not isolated in the extended phase space, which means that every member of the one-parameter family of functions

$$[0,T] \ni t \longmapsto x^*(t+\varphi), \quad \varphi \in [0,T], \tag{4.2}$$

is also a T-periodic solution. The latter circumstance, in particular, excludes every possibility of studying this problem via topological degree methods.

In the autonomous case (4.1), the direct application of the successive approximation techniques with the main equations (3.29), (3.41), (3.85), described above for the non-autonomous case, yields

$$\begin{aligned} x_m(t,z) &= z + \int_0^t f\big(x_{m-1}(s,z)\big)\,\mathrm{d}s - \frac{t}{T}\int_0^T f\big(x_{m-1}(s,z)\big)\,\mathrm{d}s, \\ \Delta(z) &= \frac{1}{T}\int_0^T f\big(x_{m-1}(t,z)\big)\,\mathrm{d}t, \end{aligned} \tag{4.3}$$

which implies that

$$x_m(t,z) = x_m^*(t,z) = z, \qquad \Delta(z) \equiv f(z) = 0.$$

Therefore, the successive approximation scheme determined by (4.3) “detects” only the stationary (i.e., constant) periodic solutions of the autonomous system (4.1). In order to find the non-constant T-periodic solutions of (4.1), the following modification can be used [90,91].

The change of variables

$$t = \frac{T}{\tilde{T}}\theta \tag{4.4}$$

allows one to study the periodic solution of (4.1) with a certain fixed period $\tilde{T} \in (0, +\infty)$, i.e., to investigate the following parametrised $\tilde{T}$-periodic boundary value problem

$$x'(\theta) = \mu f\big(x(\theta)\big), \tag{4.5}$$

$$x(0) = x(\tilde{T}), \tag{4.6}$$

where $x : [0, \tilde{T}] \to D$ and $\mu = T\tilde{T}^{-1}$.

Let us suppose that, for some fixed

$$\mu = \mu_0 > 0, \tag{4.7}$$

the $\tilde{T}$-periodic boundary value problem (4.5), (4.6) has a non-constant solution $x = \bar{x}(\theta)$. As in (4.2), the members of the one-parameter family

$$[0, T] \ni \theta \longmapsto \bar{x}(\theta + \varphi), \quad \varphi \in [0, \tilde{T}] \tag{4.8}$$

are also $\tilde{T}$-periodic solutions. Due to the periodicity and continuity of functions (4.8), the components of the solution

$$\bar{x}(\theta) = \mathrm{col}\big(\bar{x}_1(\theta), \bar{x}_2(\theta), \ldots, \bar{x}_n(\theta)\big) \tag{4.9}$$

assume their extremal values over the interval $[0, \tilde{T}]$ at some points θ_{i1}, θ_{i2}:

$$\max_{\theta \in [0,\tilde{T}]} \bar{x}_i(\theta) = \bar{x}_i(\theta_{i1}), \qquad \min_{\theta \in [0,\tilde{T}]} \bar{x}_i(\theta) = \bar{x}_i(\theta_{i2})$$

for $i = 1, 2, \dots, n$.

Since all the periodic solutions (4.8) represent one and the same trajectory, it follows that for all $i = 1, 2, \dots, n$

$$\max_{\theta \in [0,\tilde{T}]} \bar{x}_i(\theta + \varphi) = \bar{x}_i(\theta_{i1}), \qquad \min_{\theta \in [0,\tilde{T}]} \bar{x}_i(\theta + \varphi) = \bar{x}_i(\theta_{i2}).$$

Therefore, without loss of generality, having replaced x^0 by $x^0(\cdot + \varphi)$ with a suitable φ, we can assume in the subsequent consideration that a certain fixed, say jth, component of function (4.8) takes its extremal value over $[0, \tilde{T}]$ at the point $\theta = 0$:

$$\max_{\theta \in [0,\tilde{T}]} \bar{x}_j(\theta) = \bar{x}_j(0) \quad \text{or} \quad \min_{\theta \in [0,\tilde{T}]} \bar{x}_j(\theta) = \bar{x}_j(0).$$

Thus, for the boundary value problem (4.5), (4.6) to be solvable it is necessary that

$$\frac{\mathrm{d}\bar{x}_j}{\mathrm{d}\theta}(0) = 0,$$

i.e., the initial value $x_0 = \bar{x}(0)$ of the solution should lie on the hypersurface

$$\Gamma_j = \left\{x \in \mathbb{R}^n \mid f_j(x) = 0\right\},$$

where $f = \mathrm{col}(f_1, f_2, \dots, f_n)$.

Let us suppose that, in D, the equation

$$f_j(x) = 0 \tag{4.10}$$

has an $(n-1)$-parametric family of solutions

$$x = x_0(c), \tag{4.11}$$

where $c = \mathrm{col}(c_1, c_2, \dots, c_{n-1}) \in \mathbb{R}^{n-1}$ is a vector parameter. Therefore, according to our approach, the $\tilde{T}$-periodic solution of the boundary value problem (4.5), (4.6) will be determined on the set of solutions of the Cauchy problem of (4.5), (4.12):

$$x(0) = x_0(c). \tag{4.12}$$

4.1. *Non-linear substitution of variables*

The next step is to change the variable x according to the formula

$$x(\theta) = \Phi\big(\theta, y(\theta)\big), \quad \theta = \frac{\tilde{T}}{T}t, \ t \in [0, T], \tag{4.13}$$

which allows us to transform the autonomous system (4.5) into a non-autonomous one and then study problem (4.5), (4.6) near the hypothetical solution. The function

$$\Phi : \mathbb{R} \times \mathcal{B} \to D, \tag{4.14}$$

where $\mathcal{B} \subset \mathbb{R}^n$ is a bounded set, is $\tilde{T}$-periodic in the first variable and satisfies all conditions which are usually imposed on the change of variables, namely, $\Phi(\theta, y)$ is continuously differentiable with respect to the variables θ and y and satisfies the condition

$$\det\left[\frac{\partial \Phi(\theta, y)}{\partial y}\right] \neq 0, \quad (\theta, y) \in [0, \tilde{T}] \times \mathcal{B}. \tag{4.15}$$

As a result of the transformation of variables (4.13), Eq. (4.5) takes the form

$$y'(\theta) = F\big(\theta, y(\theta), \mu\big), \tag{4.16}$$

where

$$F(\theta, y, \mu) = \left[\frac{\partial \Phi(\theta, y)}{\partial y}\right]^{-1}\left[\mu f\big(\Phi(\theta, y)\big) - \frac{\partial \Phi(\theta, y)}{\partial \theta}\right]. \tag{4.17}$$

After the coordinate transformation (4.13), at $\theta = 0$, the hypersurface (4.9) turns into the hypersurface

$$\bar{\Gamma}_j = \big\{y_0(c) \mid c \in \mathbb{R}^n\big\},$$

where $y = y_0(c)$ is the solution of the equation

$$x_0(c) = \Phi(0, y). \tag{4.18}$$

This means that the initial value $x = x_0(c)$ that may determine a $\tilde{T}$-periodic solution of (4.12), turns into the initial value

$$y = y_0(c), \tag{4.19}$$

of the $\tilde{T}$-periodic solution of Eq. (4.16).

In order to use the successive approximation techniques with the main equations (3.29), (3.41), (3.85) for system (4.16), it will be assumed that for the values of parameter

$$\mu = \frac{T}{\tilde{T}} \tag{4.20}$$

from some interval

$$J = [\mu_1, \mu_2], \tag{4.21}$$

and for $(\theta, y) \in [0, \tilde{T}] \times \mathcal{B}$, conditions analogous to (3.5)–(3.7), and (3.8) are satisfied for system (4.16). More precisely, we suppose that:

(1) In the domain

$$G = [0, \tilde{T}] \times \mathcal{B} \times J, \tag{4.22}$$

the function F is bounded by the vector $M \in \mathbb{R}^n_+$ in the sense that

$$\frac{1}{2}\Big[\max_{(\theta,y,\mu)\in G} F(\theta, y, \mu) - \min_{(\theta,y,\mu)\in G} F(\theta, y, \mu)\Big] \leqslant M, \tag{4.23}$$

and, furthermore, satisfies the Lipschitz condition in the second argument with some matrix $K = \{K_{ij}\}_{i,j=1}^n \in \mathrm{GL}_n(\mathbb{R})$:

$$\big|F(\theta, u, \mu) - F(\theta, v, \mu)\big| \leqslant K|u - v| \tag{4.24}$$

for all $\theta \in [0, \tilde{T}]$, $u, v \in \mathcal{B}$, $\mu \in J$.

(2) The maximal eigenvalue $r(K)$ of the matrix K is such that

$$r(K) < \frac{10}{3\tilde{T}}. \tag{4.25}$$

(3) The set

$$\mathcal{B}_{\frac{\tilde{T}}{2}M} = \left\{ y \in \mathbb{R}^n \;\middle|\; B\left(y, \frac{\tilde{T}}{2}M\right) \subset \mathcal{B} \right\},$$

determined by the non-negative vector M in (4.23), is non-empty:

$$\mathcal{B}_{\frac{\tilde{T}}{2}M} \neq \varnothing. \tag{4.26}$$

REMARK 4.1. Similarly to (3.11), the following estimate holds:

$$\max_{(\theta,y,\mu)\in G} F(\theta, y, \mu) - \min_{(\theta,y,\mu)\in G} F(\theta, y, \mu) \leqslant 2\max\,(\theta, y, \mu) \in G\big|F(\theta, y, \mu)\big|.$$

We note also that if the relations (4.25) and (4.26) are true for some $\mu = \mu_0$, then they also hold for all $\mu > \mu_0$.

In connection with Eq. (4.16), we introduce the sequence of $\tilde{T}$-periodic functions $\{y_m(\theta, c, \mu)\}$ defined by the recurrence relation

$$y_m(\theta, c, \mu) = y_0(c) + \int_0^\theta \Bigl[F\bigl(s, y_{m-1}(s, c, \mu), \mu\bigr) - \frac{1}{\tilde{T}} \int_0^{\tilde{T}} F\bigl(\tau, y_{m-1}(\tau, c, \mu), \mu\bigr)\, \mathrm{d}\tau \Bigr] \mathrm{d}s, \quad m = 1, 2, \ldots. \tag{4.27}$$

These functions depend on the n real parameters

$$c = \operatorname{col}(c_1, c_2, \ldots, c_{n-1}) \in \mathbb{R}^{n-1}, \quad \mu \in J.$$

We define the starting member of sequence (4.27) as

$$y_0(\theta, c, \mu) = y_0(c)$$

with c such that $y_0(c) \in \mathcal{B}_{\frac{\tilde{T}}{2}M}$. The set of all such c from $\mathbb{R}^{n-1}$ will be denoted by H_f. Recall that $y_0(c)$ is given by formulae (4.18) and (4.19)

The following statement establishes the convergence of the sequence (4.27).

THEOREM 4.2. *Let the function f in the autonomous system* (4.5) *and the coordinate transformation* (4.13) *be such that for the non-autonomous $\tilde{T}$-periodic system* (4.16) *with F given by* (4.17), *conditions* (4.23)–(4.26) *hold. Then, for arbitrary values of the parameters*

$$c \in H_f, \qquad \mu \in J,$$

the following assertions are true:

(i) *The sequence $\{y_m(\theta, c, \mu)\}$ of $\tilde{T}$-periodic functions* (4.27) *is uniformly convergent,*

$$\lim_{m\to\infty} y_m(\theta, c, \mu) = y^*(\theta, c, \mu) \tag{4.28}$$

with respect to

$$(\theta, c, \mu) \in (-\infty, \infty) \times H_f \times J;$$

(ii) *The limit function $y^*(\theta, c, \mu)$ is the unique $\tilde{T}$-periodic solution of the integral equation*

$$y(\theta) = y_0(c) + \int_0^\theta F\bigl(s, y(s), \mu\bigr)\, \mathrm{d}s - \frac{\theta}{\tilde{T}} \int_0^{\tilde{T}} F\bigl(s, y(s), \mu\bigr)\, \mathrm{d}s, \tag{4.29}$$

with values in the domain $\mathcal{B}$; alternatively, it is the unique $\tilde{T}$-periodic solution of the Cauchy problem for the integro-differential equation

$$\frac{\partial y(\theta, c, \mu)}{\partial \theta} = F\big(\theta, y(s, c, \mu), \mu\big) - \Delta(c, \mu), \tag{4.30}$$

$$y(0, c, \mu) = y_0(c), \tag{4.31}$$

where

$$\Delta(c, \mu) = \frac{1}{\tilde{T}} \int_0^{\tilde{T}} F\big(\theta, y^*(\theta, c, \mu), \mu\big)\, d\theta; \tag{4.32}$$

(iii) *The estimate*

$$\big|y^*(\theta, c, \mu) - y_m(\theta, c, \mu)\big| \leqslant \bar{\alpha}_{1,\tilde{T}}(\theta)\left(\frac{3\tilde{T}}{10}K\right)^m \left(I - \frac{3\tilde{T}}{10}K\right)^{-1} M \tag{4.33}$$

holds for an arbitrary $\theta \in (-\infty, \infty)$, where the function $\bar{\alpha}_{1,\tilde{T}}$ is given by formulae (3.37), (3.31).

PROOF. It is easy to see that Theorem 4.2 is rather similar to Theorem 3.17 and, hence, its proof is omitted. □

Let us establish a general condition necessary and sufficient for the existence of a T-periodic solution of the autonomous system (4.1).

THEOREM 4.3. *Let us assume that, for the autonomous system* (4.5), *for the coordinate transformation* (4.13), *and for the non-autonomous $\tilde{T}$-periodic system* (4.16), (4.17), *conditions* (4.23)–(4.26) *hold. Then the solution $y = y(\theta, 0, y_0(c^*), \mu^*)$ of the Cauchy problem*

$$y'(\theta) = F\big(\theta, y(\theta), \mu^*\big), \tag{4.34}$$

$$y(0) = y_0(c^*), \tag{4.35}$$

where $c \in H_f$ and, hence $y_0(c^*) \in \mathcal{B}_{\frac{T}{2}M}$, is a $\tilde{T}$-periodic solution of (4.34) *if and only if $(c^*, \mu^*) \in H_f \times J$ is a zero of the determining function* (4.32), *i.e.*

$$\Delta(c^*, \mu^*) := \frac{1}{\tilde{T}} \int_0^{\tilde{T}} F\big(\theta, y^*(\theta, c^*, \mu^*), \mu^*\big)\, d\theta = 0, \tag{4.36}$$

and in this case $y = y(\theta, 0, y_0(c^*), \mu^*) = y^*(\theta, c^*, \mu^*)$, where

$$y^*(\theta, c^*, \mu^*) = \lim_{m \to \infty} y_m(\theta, c^*, \mu^*). \tag{4.37}$$

According to the change of variables (4.13), *the values* (c^*, μ^*) *and the* $\tilde{T}$*-periodic solution* $y^*(\theta, c^*, \mu^*)$ *of the non-autonomous system* (4.34) *determine the* T*-periodic solution of the autonomous system* (4.1)

$$x = x^*(t) = \Phi\left(\frac{t}{\mu^*}, y^*\left(\frac{t}{\mu^*}, c^*, \mu^*\right)\right) \tag{4.38}$$

and its period

$$T^* = \mu^* \tilde{T}. \tag{4.39}$$

PROOF. The sufficiency of condition (4.36) follows immediately from the fact that the function $y^*(\theta, c, \mu)$ is a $\tilde{T}$-periodic solution of the initial value problem (4.30), (4.31). It is obvious that if (4.36) holds, then this condition is sufficient for (4.37) to be a $\tilde{T}$-periodic solution of (4.16) and of the initial value problem (4.34), (4.35). Our change of variables,

$$\frac{t}{\mu} = \theta, \quad \tilde{T}\mu = T, \quad x = \Phi(\theta, y),$$

yields (4.38), (4.39).

The necessity of condition (4.36) can be established by analogy with the proof of Theorem 3.21. □

Now we establish some properties of the limit function (4.28) and the determining function (4.32).

THEOREM 4.4. *Let us suppose that, in domain* (4.22), *conditions* (4.23), (4.25), (4.26) *hold and, moreover, there exist some non-negative matrix* $K = \{K_{ij}\}_{i,j=1}^n \in \mathrm{GL}_n(\mathbb{R})$ *and vector* $\tilde{M} = \{\tilde{M}_i\}_{i=1}^n \in \mathbb{R}_+^n$ *such that the function* $(\theta, y, \mu) \mapsto F(\theta, y, \mu)$ *satisfies the following Lipschitz condition in* y *and* μ:

$$\left|F(\theta, u, \mu) - F(\theta, v, \xi)\right| \leqslant K|u - v| + |\mu - \xi|\tilde{M}. \tag{4.40}$$

Then the function $y^*(\theta, c, \mu)$ *given by* (4.28), (4.27) *and the determining function* $\Delta(c, \mu)$ *of the form* (4.32) *are Lipschitzian in the sense that, for all*

$$\{c, a\} \subset H_f, \quad \{\mu, \xi\} \subset J, \quad \theta \in [0, \tilde{T}], \tag{4.41}$$

we have

$$\begin{aligned} &\left|y^*(\theta, c, \mu) - y^*(\theta, a, \xi)\right| \\ &\quad \leqslant \left[I + \bar{\alpha}_{1,\tilde{T}}(\theta) K \left(I - \frac{3\tilde{T}}{10} K\right)^{-1}\right] \left|y_0(c) - y_0(a)\right| \\ &\qquad + \left[I + \left(I - \frac{3\tilde{T}}{10} K\right)^{-1}\right] \bar{\alpha}_{1,\tilde{T}}(\theta) |\mu - \xi| \tilde{M}, \end{aligned} \tag{4.42}$$

and the relations

$$\left|\Delta(c,\mu)-\Delta(a,\xi)\right| \leqslant K\left[I+\frac{10\tilde{T}}{27}K\left(I-\frac{3\tilde{T}}{10}K\right)^{-1}\right]\left|y_0(c)-y_0(a)\right| + K\left[I+\frac{10\tilde{T}}{27}K\left(I-\frac{3\tilde{T}}{10}K\right)^{-1}\right]|\mu-\xi|\tilde{M}, \tag{4.43}$$

$$\left|\Delta(c,\mu)-\Delta_m(c,\mu)\right| \leqslant \frac{10\tilde{T}}{27}K\left(\frac{3\tilde{T}}{10}K\right)^m\left(I-\frac{3\tilde{T}}{10}K\right)^{-1}M, \tag{4.44}$$

hold true, where

$$\Delta_m(c,\mu) := \frac{1}{\tilde{T}}\int_0^{\tilde{T}} F\bigl(\theta, y_m(\theta,c,\mu),\mu\bigr)\,\mathrm{d}\theta \tag{4.45}$$

is the mth approximation to function (4.32) *and* $y_0(c)$*, and* $y_0(a)$ *are determined by* (4.11), (4.18)*, and* (4.19).

PROOF. The uniform convergence of sequence (4.27) in domain (4.41) guarantees the continuity of the function $y^*(\theta,c,\mu)$ with respect to all the variables. Equality (4.32) ensures the continuity of the function $\Delta(c,\mu)$ in domain (4.41) and the fulfilment of the inequality

$$\left|\Delta(c,\mu)\right| \leqslant \max_{(\theta,y,\mu)\in[0,\tilde{T}]\times\mathcal{B}\times J}\left|F(\theta,y,\mu)\right|.$$

It follows directly from (4.27) that

$$\begin{aligned}
&y_1(\theta,c,\mu)-y_1(\theta,a,\xi)\\
&\quad= y_0(c)-y_0(a)+\int_0^{\theta}\bigl[F\bigl(s,y_0(c),\mu\bigr)-F\bigl(s,y_0(a),\xi\bigr)\bigr]\,\mathrm{d}s\\
&\qquad-\frac{\theta}{\tilde{T}}\int_0^{\tilde{T}}\bigl[F\bigl(s,y_0(c),\mu\bigr)-F\bigl(s,y_0(a),\xi\bigr)\bigr]\,\mathrm{d}s\\
&\quad= y_0(c)-y_0(a)+\left(1-\frac{\theta}{\tilde{T}}\right)\int_0^{\theta}\bigl[F\bigl(s,y_0(c),\mu\bigr)-F\bigl(s,y_0(a),\xi\bigr)\bigr]\,\mathrm{d}s\\
&\qquad-\frac{\theta}{\tilde{T}}\int_{\theta}^{\tilde{T}}\bigl[F\bigl(s,y_0(c),\mu\bigr)-F\bigl(s,y_0(a),\xi\bigr)\bigr]\,\mathrm{d}s.
\end{aligned} \tag{4.46}$$

Due to the Lipschitz condition (4.40) and equality (3.33) of Lemma 3.16, from (4.46) we get

$$\begin{aligned}
&\left|y_1(\theta,c,\mu)-y_1(\theta,a,\xi)\right|\\
&\quad\leqslant \left|y_0(c)-y_0(a)\right|+\left(1-\frac{\theta}{\tilde{T}}\right)\int_0^{\theta}\bigl[K\left|y_0(c)-y_0(a)\right|+|\mu-\xi|\tilde{M}\bigr]\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\theta}{\tilde{T}} \int_{\theta}^{\tilde{T}} \left[K \left| y_0(c) - y_0(a) \right| + |\mu - \xi| \tilde{M} \right] \mathrm{d}s \\
&= \left[I + \alpha_{1,\tilde{T}}(\theta) K \right] \left| y_0(c) - y_0(a) \right| + \alpha_{1,\tilde{T}}(\theta) |\mu - \xi| \tilde{M}.
\end{aligned} \tag{4.47}
$$

Similarly, taking into account (4.47), we obtain

$$
\begin{aligned}
&\left| y_2(\theta, c, \mu) - y_2(\theta, a, \xi) \right| \\
&\leqslant \left| y_0(c) - y_0(a) \right| \\
&\quad + \left(1 - \frac{\theta}{\tilde{T}} \right) \int_0^{\theta} \left[K \left| y_1(s, c, \mu) - y_1(s, a, \xi) \right| + |\mu - \xi| \tilde{M} \right] \mathrm{d}s \\
&\quad + \frac{\theta}{\tilde{T}} \int_{\theta}^{\tilde{T}} \left[K \left| y_1(s, c, \mu) - y_1(s, a, \xi) \right| + |\mu - \xi| \tilde{M} \right] \mathrm{d}s \\
&= \left[I + \alpha_{1,\tilde{T}}(\theta) K + \alpha_{2,\tilde{T}}(\theta) K^2 \right] \left| y_0(c) - y_0(a) \right| \\
&\quad + \left[\alpha_{1,\tilde{T}}(\theta) K + \alpha_{2,\tilde{T}}(\theta) K^2 \right] |\mu - \xi| \tilde{M}.
\end{aligned}
$$

Proceeding analogously and arguing by induction, we finally conclude that

$$
\begin{aligned}
&\left| y_m(\theta, c, \mu) - y_m(\theta, a, \xi) \right| \\
&\leqslant \left[I + \alpha_{1,\tilde{T}}(\theta) K + \alpha_{2,\tilde{T}}(\theta) K^2 + \cdots + \alpha_{m,\tilde{T}}(\theta) K^m \right] \left| y_0(c) - y_0(a) \right| \\
&\quad + \left[\alpha_{1,\tilde{T}}(\theta) K + \alpha_{2,\tilde{T}}(\theta) K^2 + \cdots + \alpha_{m,\tilde{T}}(\theta) K^{m-1} \right] |\mu - \xi| \tilde{M},
\end{aligned} \tag{4.48}
$$

where $\alpha_{m,\tilde{T}}$ is the function given by (3.33), i.e.,

$$
\alpha_{m+1,\tilde{T}}(\theta) = \left(1 - \frac{\theta}{\tilde{T}} \right) \int_0^{\theta} \alpha_{m,\tilde{T}}(s) \, \mathrm{d}s + \frac{\theta}{\tilde{T}} \int_{\theta}^{\tilde{T}} \alpha_{m,\tilde{T}}(s) \, \mathrm{d}s. \tag{4.49}
$$

In view of inequality (4.25) the maximal eigenvalue of the matrix $\frac{3\tilde{T}}{10} K$ is less than one and, therefore, by letting $m \to \infty$ in (4.48) and using estimates (3.36) of Lemma 3.16 and the inequality

$$
\alpha_{1,\tilde{T}}(\theta) \leqslant \frac{10}{9} \alpha_{1,\tilde{T}}(\theta) = \bar{\alpha}_{1,\tilde{T}}(t),
$$

we get

$$
\begin{aligned}
\left| y^*(\theta, c, \mu) - y^*(\theta, a, \xi) \right| \leqslant &\left[I + \bar{\alpha}_{1,\tilde{T}}(\theta) K \sum_{i=0}^{\infty} \left(\frac{3\tilde{T}}{10} K \right)^i \right] \left| y_0(c) - y_0(a) \right| \\
&+ \left[I + \sum_{i=0}^{\infty} \left(\frac{3\tilde{T}}{10} K \right)^i \right] \bar{\alpha}_{1,\tilde{T}}(\theta) |\mu - \xi| \tilde{M}.
\end{aligned}
$$

Thus, we have

$$\left|y^*(\theta,c,\mu)-y^*(\theta,a,\xi)\right| \leqslant \left[I+\bar{\alpha}_{1,\tilde{T}}(\theta)K\left(I-\frac{3\tilde{T}}{10}K\right)^{-1}\right]\left|y_0(c)-y_0(a)\right| + \left[I+\left(I-\frac{3\tilde{T}}{10}K\right)^{-1}\right]\bar{\alpha}_{1,\tilde{T}}(\theta)|\mu-\xi|\tilde{M}. \tag{4.50}$$

Equality (4.32), together with the Lipschitz condition (4.24), yields

$$\begin{aligned}\left|\Delta(c,\mu)-\Delta(a,\mu)\right| &= \frac{1}{\tilde{T}}\left|\int_0^{\tilde{T}}\left[F\big(\theta,y^*(\theta,c,\mu),\mu\big)-F\big(\theta,y^*(\theta,a,\xi),\xi\big)\right]\mathrm{d}\theta\right| \\ &\leqslant \frac{1}{\tilde{T}}K\int_0^{\tilde{T}}\left|y^*(\theta,c,\mu)-y^*((\theta,a,\xi),\xi)\right|\mathrm{d}\theta. \end{aligned} \tag{4.51}$$

Substituting (4.42) into (4.51) and taking into account the equality

$$\int_0^{\tilde{T}}\bar{\alpha}_{1,T}(\theta)\,\mathrm{d}\theta=\frac{10}{9}\int_0^{\tilde{T}}2\theta\left(1-\frac{\theta}{\tilde{T}}\right)\mathrm{d}\theta=\frac{10}{27}\tilde{T}^2, \tag{4.52}$$

we get

$$\begin{aligned}&\left|\Delta(c,\mu)-\Delta(a,\mu)\right| \\ &\quad\leqslant \frac{1}{\tilde{T}}K\int_0^{\tilde{T}}\left[I+\bar{\alpha}_{1,\tilde{T}}(\theta)K\left(I-\frac{3\tilde{T}}{10}K\right)^{-1}\right]\mathrm{d}\theta\left|y_0(c)-y_0(a)\right| \\ &\qquad+\frac{1}{\tilde{T}}K\int_0^{\tilde{T}}\left[I+\left(I-\frac{3\tilde{T}}{10}K\right)^{-1}\right]\bar{\alpha}_{1,\tilde{T}}(\theta)|\mu-\xi|\tilde{M} \\ &\quad\leqslant K\left[I+\frac{10\tilde{T}}{27}K\left(I-\frac{3\tilde{T}}{10}K\right)^{-1}\right]\left|y_0(c)-y_0(a)\right| \\ &\qquad+K\left[I+\frac{10\tilde{T}}{27}K\left(I-\frac{3\tilde{T}}{10}K\right)^{-1}\right]|\mu-\xi|\tilde{M}.\end{aligned}$$

In view of estimate (4.33), it follows directly from (4.32), (4.45), and (4.52) that

$$\begin{aligned}\left|\Delta(c,\mu)-\Delta_m(c,\mu)\right| = \Bigg|\frac{1}{\tilde{T}}\int_0^{\tilde{T}}&\left[F\big(\theta,y^*(\theta,c,\mu),\mu\big)\right. \\ &\left.-F\big(\theta,y_m(\theta,c,\mu),\mu\big)\right]\mathrm{d}\theta\Bigg|\end{aligned}$$

$$\leqslant \frac{1}{\tilde{T}} K \int_0^{\tilde{T}} \left| y^*(\theta, c, \mu) - y_m(\theta, c, \mu) \right| \mathrm{d}\theta$$
$$\leqslant \frac{1}{\tilde{T}} K \left(\frac{3\tilde{T}}{10} K \right)^m \left(I - \frac{3\tilde{T}}{10} K \right)^{-1} M \int_0^{\tilde{T}} \overline{\alpha}_{1,\tilde{T}}(\theta) \, \mathrm{d}\theta$$
$$= \frac{10\tilde{T}}{27} K \left(\frac{3\tilde{T}}{10} K \right)^m \left(I - \frac{3\tilde{T}}{10} K \right)^{-1} M.$$

The theorem is proved. □

Theorem 4.3 provides the following numerical-analytic algorithm for finding a T-periodic solution of the autonomous system (4.1) approximately and for the determination of its unknown period T.

(1) We choose a $\tilde{T}$-periodic coordinate transformation (4.13) with properties (4.14) and (4.15) such that the transformed equation satisfies conditions (4.23)–(4.26) in domain (4.22);

(2) We find a $\tilde{T}$-periodic limit function $y^*(\theta, c, \mu)$ of sequence (4.27) starting from $y_0(\theta, c, \mu) = y_0(c) \in \mathcal{B}_{\frac{\tilde{T}}{2}M}$ given by formulae (4.18) and (4.19);

(3) Given the function $y^*(\theta, c, \mu)$, we construct the determining equation

$$\Delta(c, \mu) \equiv \frac{1}{\tilde{T}} \int_0^{\tilde{T}} F\big(\theta, y^*(\theta, c, \mu), \mu\big) \, \mathrm{d}\theta = 0; \tag{4.53}$$

(4) By using certain numerical methods for solving non-linear algebraic or transcendental equations, we find approximately the solution(s) $c = c^*$ of the determining equation (4.53) belonging to the set $D_{\frac{T}{2}M}$ and the value $\mu = \mu^*$ from J;

(5) According to the coordinate transformations (4.4) and (4.13), the function

$$x^*(t) = \Phi\left(\frac{t}{\mu^*}, y^*\left(\frac{t}{\mu^*}, c^*, \mu^* \right) \right) \tag{4.54}$$

gives the T-periodic solution of the autonomous system (4.1) with the period

$$T^* = \mu^* \tilde{T}. \tag{4.55}$$

This solution, by virtue of (4.12), can also be obtained by solving the Cauchy problem with the values of parameters given by (4.53).

The functions

$$x_m(t) = \Phi\left(\frac{t}{\mu^*}, y_m\left(\frac{t}{\mu^*}, c^*, \mu^* \right) \right), \tag{4.56}$$

$$\tilde{x}_m(t) = \Phi\left(\frac{t}{\mu_m}, y_m\left(\frac{t}{\mu_m}, c_m, \mu_m \right) \right) \tag{4.57}$$

are natural to be taken for the mth approximation of the exact T-periodic solution (4.54), where the pair (c_m, μ_m) satisfies the mth approximate determining equation

$$\Delta_m(c,\mu) \equiv \frac{1}{\tilde{T}} \int_0^{\tilde{T}} F\big(\theta, y_m(\theta,c,\mu),\mu\big)\,\mathrm{d}\theta = 0. \tag{4.58}$$

Let us estimate the differences $|x^*(t) - x_m(t)|$ and $|x^*(t) - \tilde{x}_m(t)|$. Assume that, for all $\{\theta_1, \theta_2\} \subset \mathbb{R}$ and $\{y, z\} \subset \mathcal{B}$, function (4.13) satisfies the Lipschitz condition

$$\big|\Phi(\theta_1, y) - \Phi(\theta_2, z)\big| \leqslant |\theta_1 - \theta_2| M_\Phi + K_\Phi |y - z| \tag{4.59}$$

with some non-negative vector $M_\Phi \in \mathbb{R}^n_+$ and matrix $K_\Phi \in \mathrm{GL}_n(\mathbb{R})$. Then, taking into account (4.54), (4.56), and (4.33), we obtain the estimate

$$\begin{aligned}
\big|x^*(t) - x_m(t)\big| &= \left|\Phi\left(\frac{t}{\mu^*}, y^*\left(\frac{t}{\mu^*}, c^*, \mu^*\right)\right) - \Phi\left(\frac{t}{\mu^*}, y_m\left(\frac{t}{\mu^*}, c^a st, \mu^*\right)\right)\right| \\
&\leqslant K_\Phi \left| y^*\left(\frac{t}{\mu^*}, c^*, \mu^*\right) - y_m\left(\frac{t}{\mu^*}, c^*, \mu^*\right)\right| \\
&\leqslant \bar{\alpha}_{1,\tilde{T}}\left(\frac{t}{\mu^*}\right) K_\Phi \left(\frac{3\tilde{T}}{10}K\right)^m \left(I - \frac{3\tilde{T}}{10}K\right)^{-1} M.
\end{aligned} \tag{4.60}$$

Furthermore, due to (4.54), (4.57), and (4.59),

$$\begin{aligned}
\big|x^*(t) - \widetilde{x}_m(t)\big| &= \left|\Phi\left(\frac{t}{\mu^*}, y^*(\frac{t}{\mu^*}, c^*, \mu^*)\right) - \Phi\left(\frac{t}{\mu_m}, y_m\left(\frac{t}{\mu_m}, c_m, \mu_m\right)\right)\right| \\
&\leqslant \left|\frac{1}{\mu^*} - \frac{1}{\mu_m}\right| t M_\Phi + K_\Phi \left| y^*\left(\frac{t}{\mu^*}, c^*, \mu^*\right)\right. \\
&\qquad \left. - y_m\left(\frac{t}{\mu_m}, c_m, \mu_m\right)\right|.
\end{aligned} \tag{4.61}$$

Obviously,

$$\begin{aligned}
&\left| y^*\left(\frac{t}{\mu^*}, c^*, \mu^*\right) - y_m\left(\frac{t}{\mu_m}, c_m, \mu_m\right)\right| \\
&\quad \leqslant \left| y^*\left(\frac{t}{\mu^*}, c^*, \mu^*\right) - y_m\left(\frac{t}{\mu^*}, c_m, \mu_m\right)\right| \\
&\qquad + \left| y_m\left(\frac{t}{\mu^*}, c_m, \mu_m\right) - y_m\left(\frac{t}{\mu_m}, c_m, \mu_m\right)\right|.
\end{aligned} \tag{4.62}$$

The first term in the right-hand side of (4.62) can be estimated by using (4.42):

$$\begin{aligned}
&\left|y^*\left(\frac{t}{\mu^*}, c^*, \mu^*\right) - y_m\left(\frac{t}{\mu^*}, c_m, \mu_m\right)\right| \\
&\quad \leqslant \left[I + \bar{\alpha}_{1,\tilde{T}}(\theta) K \left(I - \frac{3\tilde{T}}{10} K\right)^{-1}\right] \left|y_0(c^*) - y_0(c_m)\right| \\
&\qquad + \left[I + \left(I - \frac{3\tilde{T}}{10} K\right)^{-1}\right] \bar{\alpha}_{1,\tilde{T}}(\theta) |\mu^* - \mu_m| \tilde{M}.
\end{aligned} \tag{4.63}$$

Let us put

$$\frac{1}{\mu^*} = \frac{1}{\mu_m} + \frac{1}{\gamma_m}. \tag{4.64}$$

By virtue of Lemma 3.13, it then follows from (4.27) that

$$\begin{aligned}
&\left|y_m\left(\frac{t}{\mu^*}, c_m, \mu_m\right) - y_m\left(\frac{t}{\mu_m}, c_m, \mu_m\right)\right| \\
&\quad = \Bigg| y_0(c_m) + \int_0^{\frac{t}{\mu^*}} \Bigg[F\big(s, y_{m-1}(s, c_m, \mu_m), \mu_m\big) \\
&\qquad - \frac{1}{\tilde{T}} \int_0^{\tilde{T}} F\big(\tau, y_{m-1}(\tau, c_m, \mu_m), \mu_m\big)\,\mathrm{d}\tau \Bigg] \mathrm{d}s \\
&\qquad - y_0(c_m) - \int_0^{\frac{t}{\mu_m}} \Bigg[F\big(s, y_{m-1}(s, c_m, \mu_m), \mu_m\big) \\
&\qquad - \frac{1}{\tilde{T}} \int_0^{\tilde{T}} F\big(\tau, y_{m-1}(\tau, c_m, \mu_m), \mu_m\big)\,\mathrm{d}\tau \Bigg] \mathrm{d}s \Bigg| \\
&\quad = \Bigg| \int_{\frac{t}{\mu_m}}^{\frac{t}{\gamma_m}} \Bigg[F\big(s, y_{m-1}(s, c_m, \mu_m), \mu_m\big) \\
&\qquad - \frac{1}{\tilde{T}} \int_0^{\tilde{T}} F\big(\tau, y_{m-1}(\tau, c_m, \mu_m), \mu_m\big)\,\mathrm{d}\tau \Bigg] \mathrm{d}s \Bigg| \\
&\quad \leqslant \alpha_{1,\tilde{T}}(t) M, \quad t \in [0, \tilde{T}].
\end{aligned} \tag{4.65}$$

Combining (4.63) and (4.65), we see that (4.62), (4.61) yield the following estimate:

$$\begin{aligned}
\left|x^*(t) - \tilde{x}_m(t)\right| \leqslant{}& \left|\frac{1}{\mu^*} - \frac{1}{\mu_m}\right| t M_\Phi \\
&+ K_\Phi \Bigg\{ \left[I + \bar{\alpha}_{1,\tilde{T}}\left(\frac{t}{\mu_m}\right) K \left(I - \frac{3\tilde{T}}{10} K\right)^{-1}\right] \\
&\times \left|y_0(c^*) - y_0(c_m)\right|
\end{aligned}$$

$$+\left[I+\left(I-\frac{3\tilde{T}}{10}K\right)^{-1}\right]\bar{\alpha}_{1,\tilde{T}}\left(\frac{t}{\mu_m}\right)|\mu^*-\mu_m|\tilde{M}\Bigg\}$$
$$+\alpha_{1,\tilde{T}}(t)M, \quad t\in[0,T]. \tag{4.66}$$

REMARK 4.5. By using inequality (4.44) of Theorem 4.4, one can obtain a condition sufficient for the absence of $\tilde{T}$-periodic solutions of system (4.16). Indeed, (4.44) implies

$$\left|\Delta(c,\mu)\right| \geqslant \frac{10\tilde{T}}{27}K\left(\frac{3\tilde{T}}{10}K\right)^m\left(I-\frac{3\tilde{T}}{10}K\right)^{-1}M-\left|\Delta_m(c,\mu)\right|.$$

Therefore, the determining equation (4.53) has no solutions (c,μ) in the set $H_f\times J$ whenever

$$\sup_{(c,\mu)\in H_f\times J}\left|\frac{1}{\tilde{T}}\int_0^{\tilde{T}}F\bigl(\theta,y_m(\theta,c_m,\mu_m),\mu_m\bigr)\,\mathrm{d}\theta\right|$$
$$>\frac{10\tilde{T}}{27}K\left(\frac{3\tilde{T}}{10}K\right)^m\left(I-\frac{3\tilde{T}}{10}K\right)^{-1}M \tag{4.67}$$

for some $m\in\{0,1,2,\ldots\}$. Consequently, it follows from Theorem 4.3 that, under condition (4.67), the $\tilde{T}$-periodic system (4.16) has no $\tilde{T}$-periodic solutions and, hence, the autonomous system (4.1) has no T-periodic solutions.

4.2. *Linear substitution of variables*

It is clear that the simplest case of substitution (4.13) is the linear change of variables

$$x(\theta)=A(\theta)y(\theta)+b(\theta), \quad \theta\in\mathbb{R}, \tag{4.68}$$

where A and b are $\tilde{T}$-periodic matrix and vector-valued functions, respectively. We assume that these functions are continuously differentiable and such that

$$\det A(\theta)\neq 0, \quad \theta\in[0,\tilde{T}]. \tag{4.69}$$

The change of variable (4.68) transforms the autonomous system (4.1) into the non-autonomous system

$$\frac{\mathrm{d}y}{\mathrm{d}\theta}=F(\theta,y,\mu), \tag{4.70}$$

where μ is given by (4.20) and

$$F(\theta,y,\mu):=A(\theta)^{-1}\bigl[\mu f\bigl(A(\theta)y+b(\theta)\bigr)-A'(\theta)y-b'(\theta)\bigr].$$

The use of the linear substitution (4.68) instead of (4.13) simplifies the transformation of the autonomous system (4.1) to a non-autonomous one and the verification of conditions (4.23)–(4.26). In this case, the initial data (4.11) and (4.19), for which there may exist $\tilde{T}$-periodic solutions of (4.5) and (4.70), are related by the formula

$$x_0(c) = A(0)y_0(c) + b(0), \tag{4.71}$$

whence, in view of (4.69),

$$y_0(c) = A(0)^{-1}\big[x_0(c) - b(0)\big]. \tag{4.72}$$

Therefore, we can start our iteration (4.27) from $y_0(c)$ given by (4.72).

The T-periodic solution of the autonomous system (4.1) is given by the formula

$$x^*(t) = A\left(\frac{t}{\mu^*}\right)y^*\left(\frac{t}{\mu^*}, c^*, \mu^*\right) + b\left(\frac{t}{\mu^*}\right), \tag{4.73}$$

where the pair (c^*, μ^*) is a solution of the determining equation (4.36). Similarly, the approximate solutions (4.56), (4.57) take the form

$$x_m(t) = A\left(\frac{t}{\mu^*}\right)y_m\left(\frac{t}{\mu^*}, c^*, \mu^*\right) + b\left(\frac{t}{\mu^*}\right), \tag{4.74}$$

and

$$\tilde{x}_m(t) = A\left(\frac{t}{\mu_m}\right)y_m\left(\frac{t}{\mu_m}, c_m, \mu_m\right) + b\left(\frac{t}{\mu_m}\right), \tag{4.75}$$

where the pair (c_m, μ_m) is a solution of the approximate determining equation (4.58).

For the linear change of variable (4.68), the vector M_Φ and the matrix K_Φ in the Lipschitz condition (4.59) as well as in the error estimates (4.60), (4.66) are given by the formulae $K_\Phi = \max_{\theta\in[0,\tilde{T}]} |A(\theta)|$,

$$M_\Phi = \max_{(\theta,y)\in[0,\tilde{T}]\times\Omega} \big|A'(\theta)y + f'(\theta)\big|.$$

EXAMPLE 4.6. Let us suppose that the system (4.1) has the form

$$\begin{aligned} x_1'(t) &= f_1\big(x_1(t), x_2(t)\big), \\ x_2'(t) &= f_2\big(x_1(t), x_2(t)\big). \end{aligned} \tag{4.76}$$

Consider (4.76), (4.77) together with the T-periodic boundary conditions

$$x_1(0) = x_1(T), \qquad x_2(0) = x_2(T), \tag{4.77}$$

where $f_1 : \mathbb{R}^n \to \mathbb{R}^p$, $f_2 : \mathbb{R}^n \to \mathbb{R}^{n-p}$, p is even, and T is an positive constant.

Let us use the special case of the linear transformation (4.68) where

$$A(\theta) = \operatorname{diag}\bigl(\exp(A\theta), I_{n-p}\bigr), \qquad b(\theta) \equiv 0, \tag{4.78}$$

where A is a real skew-symmetric matrix with purely imaginary simple eigenvalues.

In this case, substitution (4.68) is rewritten as follows:

$$x_1(t) = \exp(A\theta)y_1(\theta), \qquad x_2(\theta) = y_2(\theta), \tag{4.79}$$

and the transformed system (4.16) corresponding to (4.76) has the form

$$\begin{aligned} y_1'(\theta) &= \mu \exp[-\theta A] f_1\bigl(\exp(\theta A)y_1(\theta), y_2(\theta)\bigr) - Ay_1(\theta), \\ y_2'(\theta) &= \mu f_2\bigl(\exp(\theta A)y_1(\theta), y_2(\theta)\bigr), \end{aligned}$$

where μ is given by (4.20).

EXAMPLE 4.7. Consider the autonomous system of the form

$$\begin{aligned} x_1'(t) &= Ax_1(t) + f_1\bigl(x_1(t), x_2(t)\bigr), \\ x_2'(t) &= f_2\bigl(x_1(t), x_2(t)\bigr), \end{aligned} \tag{4.80}$$

where $f_1 : \mathbb{R}^n \to \mathbb{R}^p$, $f_2 : \mathbb{R}^n \to \mathbb{R}^{n-p}$, p is even, $T > 0$, and A is a square matrix of dimension p having the same properties as in Example 4.6. Let us assume that the functions f_1 and f_2 are continuous in the domain

$$G = \bigl\{(x_1, x_2) \mid \|x_1\| \leqslant d, x_2 \in D_2\bigr\}, \tag{4.81}$$

where $\|x_1\|^2 \equiv \sum_{k=1}^{p} x_{1k}^2$ and D_2 is a closed bounded domain.

Suppose that the functions f_1 and f_2 satisfy the following conditions in domain (4.81): there exist some non-negative constants k_1, m_1, a non-negative vector $M_2 \in \mathbb{R}^{n-p}$, and a non-negative $(n-p) \times n$ matrix K_2 such that

$$\bigl\|f_1(x)\bigr\| \leqslant m_1, \qquad \bigl|f_2(x)\bigr| \leqslant M_2 \tag{4.82}$$

and

$$\bigl\|f_1(x_1, x_2) - f_1(z_1, z_2)\bigr\| \leqslant k_1\|x - z\|, \tag{4.83}$$

$$\bigl|f_2(x_1, x_2) - f_2(z_1, z_2)\bigr| \leqslant K_2|x - z| \tag{4.84}$$

for all (x_1, x_2) and (z_1, z_2) from domain (4.81).

The change of variable (4.68) with A given by (4.78) transforms system (4.80) to the form

$$\begin{aligned}\frac{\mathrm{d}y_1}{\mathrm{d}\theta} &= (\mu-1)Ay_1 + \mu\exp[-\theta A]f_1\bigl(\exp(\theta A)y_1, y_2\bigr)\\ &=: \mu F_1(\theta, y_1, y_2),\\ \frac{\mathrm{d}y_2}{\mathrm{d}\theta} &= \mu f_2\bigl(\exp(\theta A)y_1, y_2\bigr) =: \mu F_2(\theta, y_1, y_2).\end{aligned} \tag{4.85}$$

Since A is skew-symmetric, it follows that the matrices $\exp(\pm\theta A)$ are orthogonal, i.e., $\mathrm{e}^{\pm\theta A}(\mathrm{e}^{\pm\theta A})^* = I_{p\times p}$ and, furthermore, x_1 and x_2 in (4.79) satisfy the relations

$$\|x_1\| = \|\mathrm{e}^{\theta A}y_1\| = \|y_1\|, \tag{4.86}$$

$$\bigl\|\mathrm{e}^{-\theta A}f_1(\mathrm{e}^{\theta A}y_1, y_2)\bigr\| = \bigl\|f_1(\mathrm{e}^{\theta A}y_1, y_2)\bigr\|. \tag{4.87}$$

It follows from (4.86), (4.87) that the right-hand side of system (4.85) is well defined and continuous in domain (4.81). Moreover, the functions $F_1(\theta, y_1, y_2)$ and $F_2(\theta, y_1, y_2)$ in (4.85) satisfy the inequalities

$$\begin{aligned}&\bigl|F_1(\theta, y_1, y_2)\bigr| \leqslant \bar{m}_1 e_p, \qquad \bigl|F_2(\theta, y_1, y_2)\bigr| \leqslant M_2,\\ &\bigl|F_1(\theta, y_1, y_2) - F_1(\theta, z_1, z_2)\bigr| \leqslant \bar{k}_1 J_{p\times n}|y-z|\end{aligned} \tag{4.88}$$

for all (y_1, y_2) and (z_1, z_2) from domain (4.81), where

$$\begin{aligned}\bar{m}_1 &= m_1 + (1-\mu^{-1})\max_{\|y_1\|\leqslant d}\|Ay_1\|,\\ \bar{k}_1 &= k_1 + (1-\mu^{-1})\|A\|,\end{aligned} \tag{4.89}$$

and m_1 and k_1 are the same as in (4.82), (4.83). Here, e_p and $J_{p\times n}$ are the vector and matrix of dimension $p \times 1$ and $p \times n$, respectively, with all elements equal to 1.

It follows from inequalities (4.88), (4.89) that the functions in the right-hand side of system (4.85) satisfy conditions (4.23) and (4.24) with the vector

$$M = \mu\,\mathrm{col}\,(\bar{m}_1 e_p, M_2) \tag{4.90}$$

and matrix

$$K = \mu\begin{pmatrix}\bar{k}_1 J_{p\times n}\\ K_2\end{pmatrix}, \tag{4.91}$$

where K_2 is the matrix from (4.84).

We thus see that, in some cases (in particular, for system (4.80)) one can verify conditions (4.23), (4.24) in terms of the constants m_1, M_2, k_1, and K_2 involved in conditions (4.82), (4.83), and (4.84) for the original system.

REMARK 4.8. Quite similarly, one can show that, for the autonomous system

$$x'(t) = Ax + f(x)$$

with A having the same properties as in Example 4.7, the linear change of variables

$$x(\theta) = \mathrm{e}^{\theta A} y(\theta) \tag{4.92}$$

brings the $T\mu^{-1}$-periodic problem (4.5), (4.6)

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \mu\big[Ax + f(x)\big], \tag{4.93}$$

$$x(0) = x(\tilde{T}) \tag{4.94}$$

to the non-autonomous $\tilde{T}$-periodic problem (4.94) for the equation

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = (\mu - 1)Ay + \mu \mathrm{e}^{-\theta A} f(\mathrm{e}^{\theta A} y). \tag{4.95}$$

EXAMPLE 4.9. Let us consider the second-order autonomous differential equation

$$\ddot{x} + \dot{x}^2 + (x - \varepsilon)^2 + x = \varepsilon^2 + \varepsilon, \tag{4.96}$$

where ε is a real parameter, and try do determine the period, T, which a periodic solution of (4.96) may have.

To Eq. (4.96), we associate the first-order system

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= (2\varepsilon - 1)x_1 - x_1^2 - x_2^2 + \varepsilon \end{aligned} \tag{4.97}$$

and the following T-periodic boundary conditions:

$$x_1(0) = x_1(T), \qquad x_2(0) = x_2(T). \tag{4.98}$$

System (4.97) will be considered for $x_1^2 + x_2^2 \leqslant \varrho^2$, where $\varrho \geqslant 2|\varepsilon|$.

It is clear that (4.97), (4.98) is equivalent to the problem on T-periodic C^2 solutions of (4.96).

Applying the coordinate transformation (4.4) with $\tilde{T} = 2\pi$, we transform (4.97), (4.98) into the system

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}\theta} &= \mu x_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}\theta} &= \mu\big((2\varepsilon - 1)x_1 - x_1^2 - x_2^2 + \varepsilon\big) \end{aligned} \tag{4.99}$$

with the 2π-periodic conditions

$$x_1(0) = x_1(2\pi), \qquad x_2(0) = x_2(2\pi). \tag{4.100}$$

In (4.99), according to (4.20), $\mu = T(2\pi)^{-1}$.

Rewriting (4.99), (4.100) as (4.93) by putting

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad f(x) = \begin{pmatrix} 0 \\ -x_1^2 - x_2^2 + 2\varepsilon x_1 + \varepsilon \end{pmatrix} \tag{4.101}$$

and using transformation (4.92), we arrive at system (4.95) of the form

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}\theta} = \mu\Bigg[&\left(1 - \frac{1}{\mu}\right)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y \\ &+ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} f\left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} y\right)\Bigg]. \end{aligned} \tag{4.102}$$

Here, we have used the equality

$$\exp(-A\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{4.103}$$

easily derived from the series expansion of $\exp(-A\theta)$ in a neighbourhood of 0.

It follows from (4.103) that, in our case, the variable transformation (4.92) is given by the equalities

$$x_1 = \cos\theta y_1 + \sin\theta y_2, \qquad x_2 = -\sin\theta y_1 + \cos\theta y_2 \tag{4.104}$$

and, hence, system (4.102) has the form

$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}\theta} &= (\mu - 1)y_2 + \mu\sin\theta\big[y_1^2 + y_2^2 - 2\varepsilon(\cos\theta y_1 + \sin\theta y_2) - \varepsilon\big], \\ \frac{\mathrm{d}y_2}{\mathrm{d}\theta} &= (1 - \mu)y_1 - \mu\cos\theta\big[y_1^2 + y_2^2 - 2\varepsilon(\cos\theta y_1 + \sin\theta y_2) - \varepsilon\big]. \end{aligned} \tag{4.105}$$

It is obvious from (4.104) that system (4.105) should be considered in the same ϱ-neighbourhood of zero as system (4.97).

We need to verify whether conditions (4.23)–(4.25) hold in our case.

As it has been already pointed out above, the vector M in (4.90) and matrix K in (4.91) are determined directly from the terms in the right-hand side of the original system (4.97), i.e., on base of the properties of function f given by (4.101).

It is easy to see that the function

$$W(x_1, x_2) = -x_1^2 - x_2^2 + 2\varepsilon x_1 + \varepsilon$$

takes its maximal value $W_{\max} = -|\varepsilon|(1 + |\varepsilon|)$ when $x_1 = \varepsilon$, $x_2 = 0$, whereas its minimal value $W_{\min} = -[\varrho^2 + 2\varepsilon^2 + (2\varrho + 1)|\varepsilon|]$ is achieved for $x_1 = \varrho \operatorname{sign} \varepsilon$, $x_2 = 0$. Therefore, according to (4.90), we may take the following vector M for system (4.105):

$$M = \mu_2\big[\varrho^2 + 2\varepsilon^2 + (2\varrho + 1)|\varepsilon| + |1 - \mu_2^{-1}|\varrho\big]e_2,$$

where $e_2 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$.

Furthermore, as can readily be verified, for the number k_1 appearing in (4.83) and (4.89), we may take

$$k_1 = \max_{(x_1,x_2)\in B(0,\varrho)} \sqrt{r\big(\Gamma(x_1, x_2)\big)}, \tag{4.106}$$

where

$$\Gamma(x_1, x_2) = 4\begin{bmatrix} 0 & 0 \\ 0 & (x_1 - \varepsilon)^2 + x_2^2 \end{bmatrix}. \tag{4.107}$$

Indeed, it is easy to see that

$$\Gamma(x) = \frac{\partial f(x)}{\partial x}\left[\frac{\partial f(x)}{\partial x}\right]^*.$$

The maximal eigenvalue of matrix (4.107) is, obviously, given by the formula

$$r\big(\Gamma(x_1, x_2)\big) = 4\big[(x_1 - \varepsilon)^2 + x_2^2\big],$$

whence it follows that, according to (4.106), we may put

$$k_1 = 2\big(\varrho + |\varepsilon|\big).$$

Hence, since $\|\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\| = 1$, we see that the constant $\bar{k}_1$ in (4.89) is given by the formula

$$\bar{k}_1 = 2\big(\varrho + |\varepsilon|\big) + \left|1 - \frac{1}{\mu_2}\right|.$$

Therefore, we have the following matrix K in (4.91):

$$K = \mu_2\left[2\big(\varrho + |\varepsilon|\big) + \left|1 - \frac{1}{\mu_2}\right|\right]J_{2\times 2},$$

where $J_{2\times 2} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$.

Since $r(J_{2\times 2}) = 2$, it is not difficult to verify that conditions (4.25) and (4.26) are satisfied in $B(0, \varrho)$ whenever

$$2(\varrho + |\varepsilon|) + \left|1 - \frac{1}{\mu_2}\right| < \frac{5}{6\pi\mu_2} \tag{4.108}$$

and

$$\varrho^2 + 2\varepsilon^2 + (2\varrho + 1)|\varepsilon| + \left|1 - \frac{1}{\mu_2}\right|\varrho < \frac{\varrho}{\pi\mu_2}. \tag{4.109}$$

Inequalities (4.108) and (4.109) hold, for example, when

$$\varepsilon = 0.02, \quad \varrho = 0.06, \quad 0.95 < \mu < 1.05. \tag{4.110}$$

Thus, the techniques described above are applicable to our system (4.97). For system (4.97), Eq. (4.10) with $j = 1$ has the form

$$x_2 = 0$$

and, hence, the one-parametric family of solutions

$$x_1 = c, \qquad x_2 = 0,$$

where $c \in \mathbb{R}$. According to the coordinate transformation (4.92), we have

$$x_1(0) = y_1(0), \qquad x_2(0) = y_2(0).$$

Therefore, we may start the successive approximations (4.27) from the constant function

$$y_0(\theta, c, \mu) = \begin{pmatrix} c \\ 0 \end{pmatrix}.$$

Consider the approximate determining equation (4.58) with $m = 0$ and $\tilde{T} = 2\pi$:

$$\mu \int_0^{2\pi} \sin\theta[c^2 - 2\varepsilon c \cos\theta - \varepsilon]\,\mathrm{d}\theta = 0,$$
$$\int_0^{2\pi} \big[(1-\mu)c - \mu\cos\theta[c^2 - 2\varepsilon c\cos\theta - \varepsilon]\big]\,\mathrm{d}\theta = 0.$$

The first of these two equations is, in fact, an identity for an arbitrary c, whereas the second one means that, for all real c,

$$2\pi(1-\mu)c + 2\varepsilon\mu c \int_0^{2\pi} \cos^2\theta\,\mathrm{d}\theta = 0$$

whence it follows that

$$\mu = \frac{1}{1+\varepsilon},$$

and therefore, in the zero approximation, the period, T, of a solution of Eq. (4.96) is given by the formula

$$T = \frac{2\pi}{1-\varepsilon}.$$

For instance, in the case where ε, ϱ, and μ are given by (4.110), we have $T = 2.04\pi$.

Note that the autonomous equation (4.96) has the periodic solution

$$x(t) = \varepsilon(1 + \sin t), \tag{4.111}$$

whose period is equal to 2π. Thus, for ε sufficiently small, even the zero approximation provides a reasonable degree of accuracy for the unknown value of the period.

5. Periodic solutions of differential systems with symmetries

In this section, we study a class of systems of non-linear non-autonomous ordinary differential equations that have "sufficiently many" periodic solutions possessing certain symmetry properties. The symmetry properties considered here (see Definition 5.1) are close to those studied by Hale [28] but, generally speaking, do not coincide with them; see Remark 5.43 in this relation. The theorem obtained in Section 5.7 contains, in particular, Theorem 6 from [70, p. 326] (see also [89]). Similar topics related to symmetries in differential equations were addressed to in the works of Fečkan [18,21–23], Mawhin and Walter [49] Vanderbauwhede [105].

We consider the problem on the T-periodic solutions of the system of n ordinary differential equations

$$x'(t) = f\bigl(t, x(t)\bigr), \quad -\infty < t < \infty, \tag{5.1}$$

where $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a certain function periodic in the first variable with the period T.

By a solution of the problem on the T-periodic solutions of system (5.1), we mean a continuously differentiable element of the Banach space C^n_T of continuous vector-functions $(-\infty, \infty) \to \mathbb{R}^n$ periodic with period T. The set of all continuous n-dimensional vector-functions $(-\infty, \infty)$ is denoted by the symbol C^n.

5.1. *Symmetry properties of periodic functions*

We are interested in those properties of the equations of the differential system (5.1) under which the T-periodic solutions of the system are "symmetric" in a certain sense. We restrict ourselves to the consideration of the following symmetry property.

DEFINITION 5.1. (See [63, Definition 3.1].) We say that a function $x \in C^n_T$ *possesses property* (τ, E) (or, alternatively, *is* (τ, E)*-proper*) for certain number τ and non-singular matrix $E \in \mathrm{GL}_n(\mathbb{R})$ if, for all real t, the equality

$$x(t) = Ex(-t - \tau) \tag{5.2}$$

is true.

The set of all functions from C^n_T possessing property (τ, E) will be denoted by $C^n_T(\tau, E)$:

$$C^n_T(\tau, E) := \left\{x \in C^n_T \mid (5.2) \text{ holds for all } t \in (-\infty, \infty)\right\}. \tag{5.3}$$

The following lemma is obvious.

LEMMA 5.2. *For arbitrary point $\tau \in (-\infty, \infty)$ and non-singular n-dimensional matrix E, the set $C^n_T(\tau, E)$ forms a closed linear subspace in C^n_T.*

REMARK 5.3. It is clear from (5.2) that, for $x \in C^n_T(\tau, E)$, the vectors $x(kT)$, where $k = 0, \pm 1, \pm 2, \ldots$, are fixed points of the linear operator determined by the matrix E.

The property dealt with in Definition 5.1 above can be regarded as a certain geometrical symmetry of the graph of the function under consideration. In particular, for $E \in \{-\mathbb{1}_n, \mathbb{1}_n\}$, where $\mathbb{1}_n$ stands for the unit matrix of dimension n, Definition 5.1 describes the very natural properties of τ-evenness and τ-oddness understood in the following sense.

DEFINITION 5.4. A function $x \in C^n_T$ will be called τ*-odd* (resp., τ*-even*) if, for an arbitrary t from $(-\infty, \infty)$, equality (5.2) holds with $E = -\mathbb{1}_n$ (resp., with $E = \mathbb{1}_n$).

Clearly, the usual notions of evenness and oddness of a vector-valued function of a scalar variable are obtained immediately from Definition 5.4 when $\tau = 0$.

REMARK 5.5. Properties similar to the τ-evenness and τ-oddness conditions arise in the theory of Fourier series [103]. Similar notions are used, e.g., in [62].

It is easy to construct examples of τ-even and τ-odd functions by using the following

PROPOSITION 5.6. *If a function $x \in C^n_T$ is even* (*resp., odd*), *then, for an arbitrary real τ, the corresponding function*

$$(-\infty, \infty) \ni t \longmapsto x\left(t + \frac{\tau}{2}\right) \tag{5.4}$$

is τ-even (*resp.*, *τ-odd*).

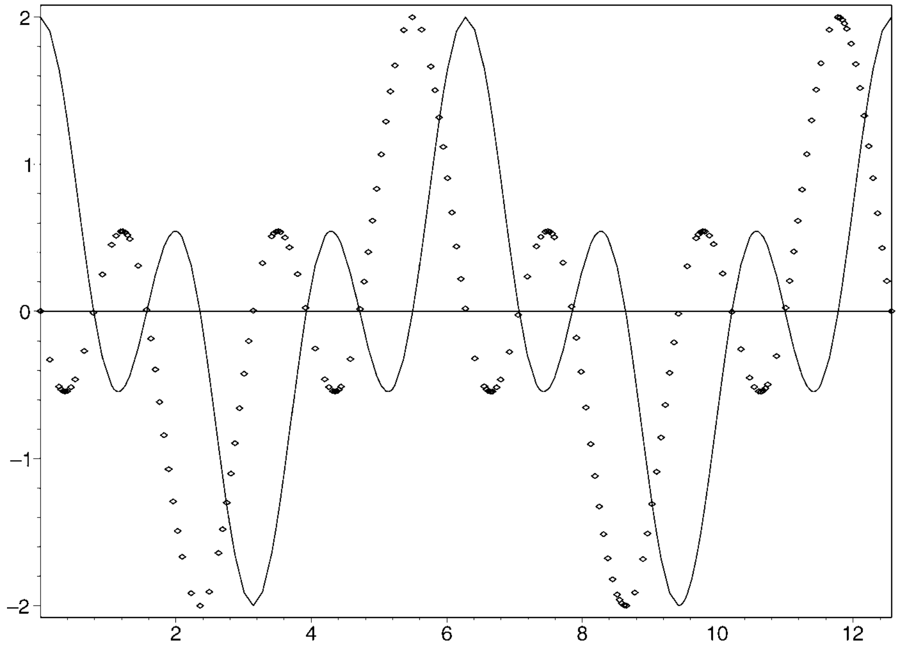

Fig. 1. Functions (5.5) and (5.6).

Proposition 5.6 is easily proved by using Corollary 5.11 of Proposition 5.10 below.

EXAMPLE 5.7. The function

$$u(t) := \cos\left(t + \frac{\pi}{4}\right) - \sin\left(3t + \frac{\pi}{4}\right), \quad t \in (-\infty, \infty), \tag{5.5}$$

is $\frac{\pi}{2}$-even.

Indeed, for the odd function x defined by the formula

$$x(t) := \cos t + \cos 3t, \quad t \in (-\infty, \infty), \tag{5.6}$$

the corresponding function (5.4) is given by equality (5.5). It remains to apply Proposition 5.6 for $T = 2\pi$ and $\tau = \frac{\pi}{2}$.

On Fig. 1, we draw with the “diamond” symbols the graph of the $\frac{\pi}{2}$-even function (5.5) obtained from the even function (5.6) through the shift of the argument by $\frac{\pi}{4}$.

EXAMPLE 5.8. The function

$$u(t) := \sin\left(t + \frac{\pi}{4}\right) - \cos\left(3t + \frac{\pi}{4}\right) - \sin\left(5t + \frac{\pi}{4}\right), \quad t \in (-\infty, \infty), \tag{5.7}$$

is $\frac{\pi}{2}$-odd.

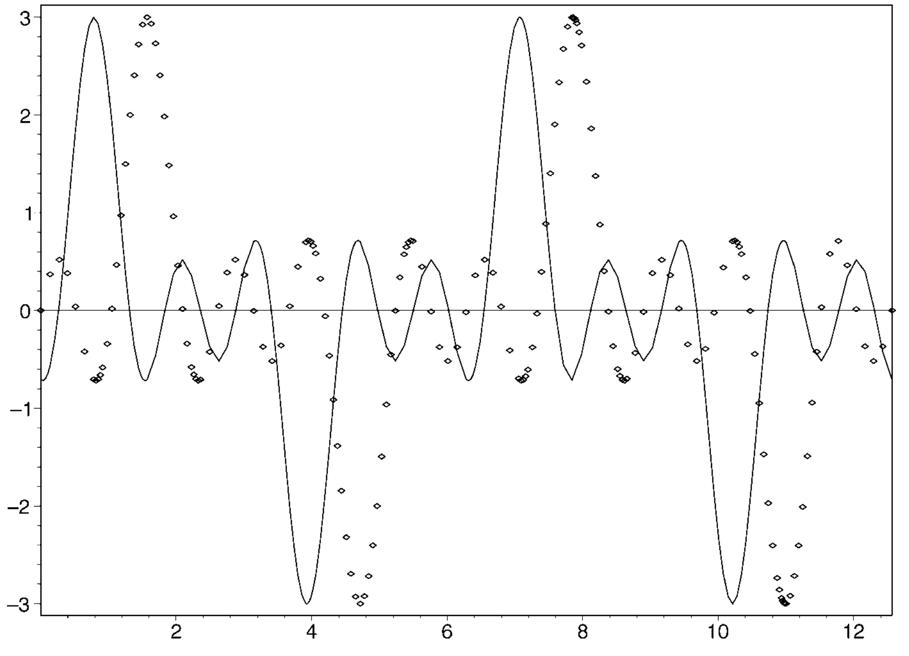

Fig. 2. Functions (5.7) and (5.8).

Indeed, for the even function x given by the formula

$$x(t) := \sin t - \sin 3t + \sin 5t, \quad t \in (-\infty, \infty), \tag{5.8}$$

the corresponding function (5.4) is defined by equality (5.7). Therefore, Proposition 5.6 with $T = 2\pi$ and $\tau = \frac{\pi}{2}$ implies the $\frac{\pi}{2}$-oddness of function (5.7).

On Fig. 2, the "diamond" symbols represent the graph of the $\frac{\pi}{2}$-odd function (5.7) obtained from the even function (5.8) through the shift of the argument by $\frac{\pi}{4}$.

REMARK 5.9. Proposition 5.6 is, in a sense, reversible (see Corollary 5.11 below).

It turns out that the functions with property (τ, E) can be relatively easily constructed from the functions possessing property $(0, E)$, and vice versa. This fact is a consequence of the following

PROPOSITION 5.10. *For arbitrary real σ and τ, the inclusion*

$$S_{-\frac{\tau}{2}} C_T^n(\sigma, E) \subset C_T^n(\sigma + \tau, E), \tag{5.9}$$

holds, where S_h stands for the operator of shift by $h \in [-T, T]$ acting in C_T^n, i.e.,

$$(S_h x)(t) := x(t - h), \quad t \in (-\infty, \infty). \tag{5.10}$$

PROOF. Assume that a certain function x belongs to $C_T^n(\sigma, E)$ and, hence, according to Definition 5.1, satisfies the condition

$$x(t) = Ex(-t-\sigma), \quad t \in (-\infty, \infty). \tag{5.11}$$

Let us set

$$x_\tau := S_{-\frac{\tau}{2}} x. \tag{5.12}$$

By virtue of the definition (5.10) of the operator $S_{-\frac{\tau}{2}}$, the function x_τ thus constructed satisfies the relations

$$\begin{aligned} x_\tau(-t-\sigma-\tau) &= x\left(-t-\sigma-\tau+\frac{\tau}{2}\right) = x\left(-t-\sigma-\frac{\tau}{2}\right) \\ &= x\left(-\left[t+\frac{\tau}{2}\right]-\sigma\right), \end{aligned}$$

whence, by (5.11), we have the relation

$$Ex_\tau(-t-\tau-\sigma) = x\left(t+\frac{\tau}{2}\right), \quad t \in (-\infty, \infty).$$

According to (5.12), the last equality can be rewritten as

$$Ex_\tau(-t-\tau-\sigma) = x_\tau(t), \quad t \in (-\infty, \infty),$$

whence, recalling Definition 5.4, we conclude that x_τ belongs to $C_T^n(\tau+\sigma, E)$. Since x is chosen arbitrarily from $C_T^n(\sigma, E)$, this proves the validity of the required inclusion (5.9). □

COROLLARY 5.11. *If a function $x \in C_T^n$ has property $(0, E)$, then, for an arbitrary real τ, the corresponding function*

$$(-\infty, \infty) \ni t \longmapsto x\left(t+\frac{\tau}{2}\right) \tag{5.13}$$

possesses property (τ, E). Conversely, if a certain function $x \in C_T^n$ has property (τ, E), then the corresponding function

$$(-\infty, \infty) \ni t \longmapsto x\left(t-\frac{\tau}{2}\right)$$

has property $(0, E)$.

PROOF. According to Proposition 5.10, inclusion (5.9) is true, which, for $\sigma = 0$, has the form

$$S_{-\frac{\tau}{2}} C_T^n(0, E) \subset C_T^n(\tau, E).$$

This means that, for an arbitrary function x from $C_T^n(0, E)$, the corresponding function (5.13) has property (τ, E), i.e., the first assertion is true.

On the other hand, for $\sigma = -\tau$, inclusion (5.9) takes the form

$$S_{-\frac{\tau}{2}} C_T^n(-\tau, E) \subset C_T^n(0, E),$$

whence it follows that, for $x \in C_T^n(\tau, E)$, function (5.13) belongs to $C_T^n(0, E)$, Replacing here τ by $-\tau$, we arrive at the second assertion of the corollary. $\square$

The assertions of Corollary 5.11 are illustrated by the diagram

$$C_T^n(\tau, E) \xrightarrow{S_{\frac{\tau}{2}}} C_T^n(0, E) \xrightarrow{S_{-\frac{\tau}{2}}} C_T^n(\tau, E).$$

A similar idea can sometimes be applied for the construction of (τ, E)-proper functions from even functions. Namely, the following statement holds.

PROPOSITION 5.12. *If $x \in C_T^n$ is an even function and E is a non-singular matrix of dimension n, then, for an arbitrary n-dimensional square matrix G whose columns belong to the kernel of the matrix $\mathbb{1}_n - E$, the function*

$$(-\infty, \infty) \ni t \longmapsto Gx\left(t + \frac{\tau}{2}\right), \tag{5.14}$$

has property (τ, E).

PROOF. For the function

$$u(t) := Gx\left(t + \frac{\tau}{2}\right)$$

given by formula (5.14), we have

$$Eu(-t-\tau) = EGx\left(-t + \frac{\tau}{2} - \tau\right) = EGx\left(-t - \frac{\tau}{2}\right). \tag{5.15}$$

Since, by assumption, the matrix G is such that

$$EG = G,$$

we see that, for the even functions x, relation (5.15) for an arbitrary $t \in (-\infty, \infty)$ implies the equality

$$Eu(-t-\tau) = u(t),$$

which means that the validity of the assertion required. $\square$

REMARK 5.13. In contrast to Proposition 5.10, the recipe described by Proposition 5.12 is only efficient when $\det(\mathbb{1}_n - E) = 0$ because in the contrary case formula (5.14) defines the function equal identically to zero.

Some further information on the structure of the functions with property (τ, E) is contained in the following

PROPOSITION 5.14. *For an arbitrary function $x \in C_T^n$ having property (τ, E), its value $x(t)$ at every point $t \in (-\infty, \infty)$ belongs to the kernel of the matrix $\mathbb{1}_n - E^2$.*

PROOF. Indeed, let x belong to $C_T^n(\tau, E)$. According to Definition 5.1, this means that, for all real t, equality (5.2) is true, whence

$$x(-t-\tau) = Ex\bigl(-(-t-\tau)-\tau\bigr) = Ex(t), \quad t \in (-\infty, \infty). \tag{5.16}$$

Substituting (5.16) into (5.2), we get the equality

$$x(t) = E^2 x(t), \quad t \in (-\infty, \infty), \tag{5.17}$$

i.e., $x(t) \in \ker(\mathbb{1}_n - E^2)$ for all $t \in (-\infty, \infty)$, as required. □

REMARK 5.15. The peculiar property (5.17) of the functions from $C_T^n(\tau, E)$ that is dealt with in Proposition 5.14, is, obviously, absent when E satisfies the equality

$$E^2 = \mathbb{1}_n. \tag{5.18}$$

Proposition 5.14 yields the following characterisation of the property (τ, E) for the scalar functions.

COROLLARY 5.16. *For an arbitrary point $\tau \in (-\infty, \infty)$, the set $C_T^1(\tau, E)$ consists of the zero function when $E \neq 1$ and coincides with the set of all τ-even functions when $E = 1$.*

PROOF. It follows from Proposition 5.14 that, for an arbitrary (τ, E)-proper scalar function x, identity (5.17) is true. Since, in our case, $E \in (-\infty, \infty)$, it follows that either $E \neq 1$ and x is equal identically to 0, or $E = 1$ and x is τ-even in the sense of Definition 5.4. □

Corollary 5.16 implies that property (τ, E) with E different of $\mathbb{1}_n$ has essentially many-dimensional nature and may be of interest in the cases where $n \geqslant 2$ only.

EXAMPLE 5.17. Assume that τ is a fixed real number and $\theta : (-\infty, \infty) \to (-\infty, \infty)$ is an arbitrary function such that, for a certain $\sigma \in \{-1, 1\}$,

$$\theta(t) = \sigma\theta(-t-\tau), \quad t \in (-\infty, \infty), \tag{5.19}$$

i.e., θ is either τ-even ($\sigma = 1$) or τ-odd ($\sigma = -1$). Let us fix some real α, β, and γ such that $\alpha\beta \neq 0$, and construct the matrix

$$E = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}. \tag{5.20}$$

This matrix is, obviously, non-singular.

The following assertions hold.

(i) For an arbitrary real μ, the function

$$u(t) = \mu\theta(t) \begin{pmatrix} \gamma(\sigma - \alpha)^{-1} \\ 1 \end{pmatrix}, \quad t \in (-\infty, \infty), \tag{5.21}$$

belongs to the class $C_T^2(\tau, E)$ with the matrix E given by

$$E = \begin{pmatrix} \alpha & \gamma \\ 0 & \sigma \end{pmatrix} \tag{5.22}$$

for $\alpha \neq \sigma$.

(ii) For all real λ and μ, the function

$$u(t) = \theta(t) \begin{pmatrix} \lambda \\ \mu \end{pmatrix}, \quad t \in (-\infty, \infty), \tag{5.23}$$

belongs to the class $C_T^2(\tau, E)$ if the matrix E is given by the formula

$$E = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}.$$

(iii) For arbitrary real λ, the function

$$u(t) = \theta(t) \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \quad t \in (-\infty, \infty),$$

belongs to the class $C_T^2(\tau, E)$ if matrix (5.20) has the form

$$E = \begin{pmatrix} \sigma & \gamma \\ 0 & \beta \end{pmatrix}$$

with $\beta \neq \sigma$.

It follows immediately from Proposition 5.14 that the functions u of the class $C_T^2(\tau, E)$ take the values in a subspace of a lower dimension. This dimension is either 0 or 1 in our case and, hence, every such function admits representation in form (5.23), where

$\theta : (-\infty, \infty) \to (-\infty, \infty)$ is continuous and T-periodic, and λ and μ are certain constants.

According to Definition 5.1, function (5.23) has property (τ, E) if, and only if, for all $t \in (-\infty, \infty)$,

$$\theta(t) \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \theta(-t - \tau) \begin{pmatrix} \alpha\lambda + \gamma\mu \\ \beta\mu \end{pmatrix}.$$

Therefore, analysing the system of linear algebraic equations

$$(\sigma - \alpha)\lambda = \gamma\mu, \qquad (\sigma - \beta)\mu = 0$$

and taking into account our assumption (5.19) on the function θ, we arrive at the assertions desired.

EXAMPLE 5.18. The function

$$u(t) = \begin{bmatrix} -\cos(t + \frac{\pi}{4}) + \sin(3t + \frac{\pi}{4}) \\ \frac{1}{3}\cos(t + \frac{\pi}{4}) - \frac{1}{3}\sin(3t + \frac{\pi}{4}) \end{bmatrix}, \quad t \in (-\infty, \infty), \tag{5.24}$$

has property $(\frac{\pi}{2}, E)$ for

$$E = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}. \tag{5.25}$$

Indeed, it is easy to see that function (5.21) from Example 5.17 has form (5.24) if, for θ, we take the $\frac{\pi}{2}$-even function (5.5) from Example 5.7:

$$\theta(t) = \cos\left(t + \frac{\pi}{4}\right) - \sin\left(3t + \frac{\pi}{4}\right), \quad t \in (-\infty, \infty). \tag{5.26}$$

Matrix (5.25) coincides with (5.22) for $\sigma = 1$, $\alpha = 2$ and $\gamma = 3$.

On Fig. 3, we present the graphs of the first (solid line) and second (“diamonds”) components of function (5.24). By the plus symbol, we draw the graph of the $\frac{\pi}{2}$-even function θ given by formula (5.26).

5.2. *Assumptions*

The implementation of the successive approximation method described below requires some technical assumptions on the function f in the right-hand side of Eq. (5.1). Namely, we assume the following conditions.

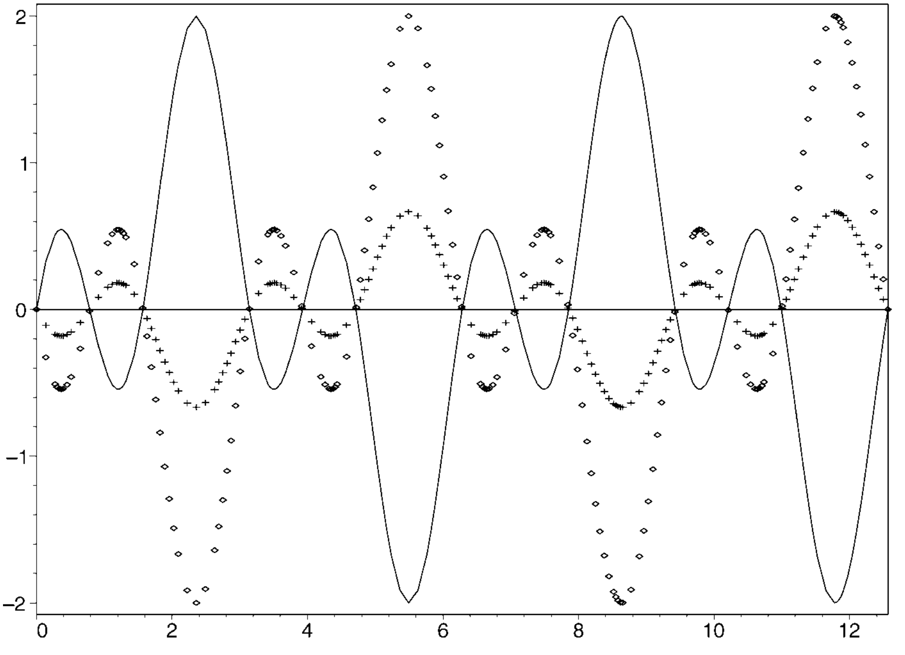

Fig. 3. The components of function (5.24), and function (5.26).

ASSUMPTION 5.19.

(1) The function $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is periodic with period T with respect to the first variable and is continuous on the set $\mathbb{R} \times D$, where D is the closure of a bounded domain in $\mathbb{R}^n$.

(2) There exists a matrix-valued function $K : \mathbb{R} \to \mathrm{GL}_n(\mathbb{R})$ with non-negative Riemann integrable elements such that

$$\left| f(t, x_1) - f(t, x_2) \right| \leqslant K(t)|x_1 - x_2| \tag{5.27}$$

for all $t \in (-\infty, \infty)$ and $\{x_1, x_2\} \subset D$.

(3) There exists an non-negative vector $M \in \mathbb{R}^n_+$ such that

$$\frac{1}{2}\Big[\max_{(t,x)\in(-\infty,\infty)\times D} f(t, x) - \min_{(t,x)\in(-\infty,\infty)\times D} f(t, x)\Big] \leqslant M.$$

(4) The set $D_{\frac{T}{2}M}$ that consists of all the vectors x from $\mathbb{R}^n$ lying in D together with their closed $\frac{T}{2}M$-neighbourhood is non-empty.

REMARK 5.20. Here and below, the inequality sign, the notion of a neighbourhood, and the symbols “max” and “min” are understood componentwise.

5.3. *Auxiliary statements. The operator H_σ*

Given some $\sigma \in (-\infty, \infty)$, for an arbitrary function u from C_T^n, we set

$$(H_\sigma u)(t) = \left(1 + k_\sigma(t) - \frac{t-\sigma}{T}\right) \int_\sigma^{t-Tk_\sigma(t)} u(s)\,\mathrm{d}s \\ + \left(\frac{t-\sigma}{T} - k_\sigma(t)\right) \int_{t-Tk_\sigma(t)}^{T+\sigma} u(s)\,\mathrm{d}s \tag{5.28}$$

at every point $t \in (-\infty, \infty)$. Here, by definition,

$$k_\sigma(t) := \lfloor (t-\sigma)T^{-1} \rfloor, \tag{5.29}$$

the integer part of the number $(t-\sigma)T^{-1}$.

LEMMA 5.21. *For any $u \in C_T^n$ and $t \in (-\infty, \infty)$, the equality $(H_\sigma u)(t+T) = (H_\sigma u)(t)$ holds.*

This statement is obtained by direct computation. In view of Lemma 5.21, the mapping H_σ given by (5.28) can be regarded as an operator from C_T^n to C_T^n.

Let us set

$$R_{K,T} := r(H_\sigma K), \tag{5.30}$$

where K in the right-hand side of the equality stands for the operator of multiplication by the continuous matrix-valued function $K : (-\infty, \infty) \to \mathrm{GL}_n(\mathbb{R})$ denoted by the same letter,

$$C_T^n \ni u(\cdot) \longmapsto K(\cdot)u(\cdot). \tag{5.31}$$

The symbol $r(A)$, as usual, denotes the spectral radius of a bounded linear operator A. It is clear from Lemma 5.21 that the composition mapping $H_\sigma K$ can be regarded as a linear operator transforming the space C_T^n into itself.

LEMMA 5.22. *For an arbitrary integrable matrix-valued function $K : \mathbb{R} \to \mathrm{GL}_n(\mathbb{R})$, the number $R_{K,T}$ is independent of the value of σ in* (5.30). *Moreover,*

$$R_{K,T} = 1 \Big/ \min\left\{ |\mu| \;\middle|\; \det\left(\frac{T}{\mu}\mathbb{1}_n - \Omega_{\mu K,0}(T) \int_0^T s\,\Omega_{\mu K,0}^{-1}(s)\,\mathrm{d}s\right) = 0 \right\}. \tag{5.32}$$

Here and in the proof below, for $P:[\sigma, T+\sigma] \to \mathrm{GL}_n(\mathbb{R})$, the symbol $\Omega_{P,\sigma}$ stands for the normalised, at the point σ, fundamental matrix of the homogeneous linear differential system

$$x'(t) = \left(1 - 2\frac{t-\sigma}{T}\right) P(t)x(t), \quad t \in [\sigma, T+\sigma].$$

PROOF OF LEMMA 5.22. Obviously, it is sufficient to consider the case where not all the elements of the matrix-valued function K are equal identically to zero.

Being compact, the linear integral operator $H_\sigma K : C_T^n \to C_T^n$ has discrete spectrum only. Let, therefore, λ be a non-zero eigenvalue of $H_\sigma K$ and $u \in C_T^n$ be an eigenfunction corresponding to it.

According to (5.29), for $t \in [\sigma, T+\sigma]$, we have $k_\sigma(t) = 0$ and, therefore, (5.28) yields

$$\left(1 - \frac{t-\sigma}{T}\right) \int_\sigma^t K(s)u(s)\,\mathrm{d}s + \frac{t-\sigma}{T} \int_t^{T+\sigma} K(s)u(s)\,\mathrm{d}s = \lambda u(t). \tag{5.33}$$

If we now set

$$y(t) = \int_\sigma^t K(s)u(s)\,\mathrm{d}s, \quad t \in [\sigma, T+\sigma], \tag{5.34}$$

then (5.33) implies that, for all t from $[\sigma, T+\sigma]$,

$$\left(1 - \frac{t-\sigma}{T}\right) K(t)y(t) + \frac{t-\sigma}{T} K(t)\big[y(T+\sigma) - y(t)\big] = \lambda y'(t). \tag{5.35}$$

Thus, if u is an eigenfunction of the operator $H_\sigma K$ corresponding to the eigenvalue λ, then function (5.34) necessarily satisfies the initial-value problem

$$\begin{aligned} &y'(t) = \frac{1}{\lambda}\left(1 - 2\frac{t-\sigma}{T}\right) K(t)y(t) + \frac{t-\sigma}{\lambda T} K(t)y(T+\sigma), \\ &\quad t \in [\sigma, T+\sigma], \\ &y(\sigma) = 0. \end{aligned} \tag{5.36}$$

The solution of this problem, as can be readily verified, satisfies the functional equation

$$\begin{aligned} y(t) = {}& \frac{1}{\lambda T} \Omega_{\frac{1}{\lambda}K,\sigma}(t) \int_\sigma^t \Omega^{-1}_{\frac{1}{\lambda}K,\sigma}(s)(s-\sigma)\,\mathrm{d}s \cdot y(T+\sigma), \\ & t \in [\sigma, T+\sigma]. \end{aligned} \tag{5.37}$$

Recall that the matrix-valued function $\Omega_{\frac{1}{\lambda}K,\sigma} : [\sigma, T+\sigma] \to \mathrm{GL}_n(\mathbb{R})$ have been defined immediately before the begin of present proof, whereas $\Omega^{-1}_{\frac{1}{\lambda}K,\sigma}$ denotes the matrix inverse for $\Omega_{\frac{1}{\lambda}K,\sigma}$.

Relation (5.37) implies, in particular, that the vector

$$c = y(T + \sigma) \tag{5.38}$$

is a solution of the system of homogeneous linear algebraic equations

$$\left[\lambda \mathbb{1}_n - \frac{1}{T}\Omega_{\frac{1}{\lambda}K,\sigma}(T+\sigma)\int_\sigma^{T+\sigma}\Omega^{-1}_{\frac{1}{\lambda}K,\sigma}(s)(s-\sigma)\,\mathrm{d}s\right]c = 0. \tag{5.39}$$

Solution (5.38) of system (5.39) is non-trivial because, in the contrary case, (5.36) and (5.34) imply the triviality of the eigenfunction u of the operator $H_\sigma K$. Thus, for the given λ, the matrix $B_{\frac{1}{\lambda}K,\sigma}$, where

$$B_{P,\sigma} := \lambda \mathbb{1}_n - \frac{1}{T}\Omega_{P,\sigma}(T+\sigma)\int_\sigma^{T+\sigma}\Omega^{-1}_{P,\sigma}(s)(s-\sigma)\,\mathrm{d}s, \tag{5.40}$$

is singular. Consequently,

$$R_{K,T} \leqslant \max\left\{|\lambda| \mid \det B_{\frac{1}{\lambda}K,\sigma} = 0\right\}. \tag{5.41}$$

On the other hand, if we assume that

$$\det B_{\frac{1}{\lambda}K,\sigma} = 0 \tag{5.42}$$

for a certain non-zero λ, then there exists a non-trivial solution c of the linear algebraic system (5.39). Then, as can be shown by using (5.42), the unique solution y of the initial-value problem

$$y'(t) = \frac{1}{\lambda}\left(1 - 2\frac{t-\sigma}{T}\right)K(t)y(t) + \frac{t-\sigma}{\lambda T}K(t)c, \quad t \in [\sigma, T+\sigma], \tag{5.43}$$

$$y(\sigma) = 0 \tag{5.44}$$

satisfies Eq. (5.37), whence the validity of equality (5.38) follows. Therefore, the function y is also a solution of problem (5.36).

Let us set

$$u(t) := \frac{1}{\lambda}\left(1 - 2\frac{t-\sigma}{T}\right)y(t) + \frac{t-\sigma}{\lambda T}c, \quad t \in [\sigma, T+\sigma]. \tag{5.45}$$

It is obvious from (5.43) and (5.45) that

$$y'(t) = K(t)u(t), \quad t \in [\sigma, T+\sigma],$$

and, hence, by virtue of (5.44), equality (5.34) is true.

We have thus established the existence of a non-trivial function $u \in C_T^n$ whose corresponding function (5.34) satisfies Eq. (5.35) for all t from $[\sigma, T+\sigma]$. Integrating the expressions on both sides of (5.35) from σ to $t \in [\sigma, T+\sigma]$ and taking (5.44) into account, we obtain that u satisfies relation (5.33), i.e., u is an eigenfunction of the operator $H_\sigma K$ with the eigenvalue λ. Consequently,

$$R_{K,T} \geqslant \max\left\{|\lambda| \mid \det B_{\frac{1}{\lambda}K,\sigma} = 0\right\}. \tag{5.46}$$

Finally, combining (5.41) and (5.46), we arrive at the equality

$$R_{K,T} = \max\left\{|\lambda| \mid \det B_{\frac{1}{\lambda}K,\sigma} = 0\right\}. \tag{5.47}$$

Let us now show that the value of $R_{K,T}$ is independent of σ, which fact will justify notation (5.30). Indeed, since, by the well-known property of the fundamental matrix, we have

$$\Omega_{P,\sigma}(t) = \Omega_{P,0}(t)\Omega_{P,\sigma}(0),$$

it follows that

$$\begin{aligned} B_{P,\sigma} &= \lambda \mathbb{1}_n - \frac{1}{T}\Omega_{P,\sigma}(T+\sigma)\int_\sigma^{T+\sigma} \Omega_{P,\sigma}^{-1}(s)(s-\sigma)\,\mathrm{d}s \\ &= \lambda \mathbb{1}_n - \frac{1}{T}\Omega_{P,0}(T+\sigma)\Omega_{P,\sigma}(0)\int_0^{T} s\Omega_{P,\sigma}^{-1}(0)\Omega_{P,0}^{-1}(s+\sigma)\,\mathrm{d}s \\ &= \lambda \mathbb{1}_n - \frac{1}{T}\Omega_{P,0}(T+\sigma)\int_0^{T} s\Omega_{P,0}^{-1}(s+\sigma)\,\mathrm{d}s. \end{aligned} \tag{5.48}$$

However,

$$\Omega_0(s+\sigma) = \Omega_0(s)\Omega_0(\sigma),$$

and, therefore, (5.48) yields

$$\begin{aligned} B_{P,\sigma} &= \lambda \mathbb{1}_n - \frac{1}{T}\Omega_{P,0}(T)\Omega_0(\sigma)\int_0^{T} s\Omega_0^{-1}(\sigma)\Omega_{P,0}^{-1}(s)\,\mathrm{d}s \\ &= \lambda \mathbb{1}_n - \frac{1}{T}\Omega_{P,0}(T)\int_0^{T} s\Omega_{P,0}^{-1}(s)\,\mathrm{d}s. \end{aligned}$$

The last equality implies, in particular, that the matrix $B_{K\lambda^{-1},\sigma}$, in fact, does not depend on σ. This, by virtue of equality (5.47), means that number (5.30) has the same property.

Finally, setting $\sigma = 0$ in (5.47), we arrive at equality (5.32), which completes the proof of our lemma. $\square$

COROLLARY 5.23. *Let the matrix-valued function* $K:(-\infty,\infty)\to \mathrm{GL}_n(\mathbb{R})$ *have the form*

$$K(t)=\operatorname{diag}\bigl(k_1(t),k_2(t),\dots,k_n(t)\bigr),\quad t\in(-\infty,\infty),$$

where $k_1, k_2, \dots, k_n$ *are certain continuous scalar functions on* $(-\infty,\infty)$. *Then the number* $1/R_{K,T}$ *is equal to the least of the absolute values of the numbers* μ *satisfying the relation*

$$\frac{T}{\mu}=\int_0^T s\exp\biggl[\mu\int_s^T\biggl(1-\frac{2\xi}{T}\biggr)k_\nu(\xi)\,\mathrm{d}\xi\biggr]\,\mathrm{d}s$$

for a certain $\nu\in\{1,2,\dots,n\}$.

PROOF. It remains to notice that, in the case considered, $\Omega_{\mu K,0}(t)$ is a diagonal matrix with the components

$$\exp\biggl[\mu\int_0^t\biggl(1-\frac{2\xi}{T}\biggr)k_\nu(\xi)\,\mathrm{d}\xi\biggr], \tag{5.49}$$

where $\nu\in\{1,2,\dots,n\}$. Applying Lemma 5.22 and substituting functions (5.49) into the expressions for the components of $\Omega_{\mu K,0}(t)$ in (5.32), we obtain the conclusion desired. □

REMARK 5.24. By using the well-known Krein–Rutman theorem on the maximal eigenvalue of a completely continuous positive operator (see Krein and Rutman [38], Theorem 6.2), one can show that, in formula (5.32), one may consider the positive real μ only.

COROLLARY 5.25. *For an arbitrary constant square matrix* K *of dimension* n, *the relation*

$$R_{K,T}\approx\frac{Tr(K)}{3.4161}$$

is true.

The assertion of Corollary 5.25 is known, e.g., from [64].

5.4. *Successive approximations and their convergence*

Having fixed some $\sigma\in(-\infty,\infty)$ and $z\in D_{\frac{T}{2}M}$, we introduce the sequence of functions $\{x_m(\cdot,z)\mid m\geqslant 0\}$ by the recurrence formula

$$x_{m+1}(t,z)=z+\int_\sigma^t f\bigl(s,x_m(s,z)\bigr)\,\mathrm{d}s-\frac{t-\sigma}{T}\int_\sigma^{T+\sigma} f\bigl(s,x_m(s,z)\bigr)\,\mathrm{d}s, \tag{5.50}$$

where $x_0(t,z) = z$ for all $t \in [\sigma, T+\sigma]$. Clearly, all the members of thus defined sequence belong to the space C_T^n.

LEMMA 5.26. *For all $x \in C_T^n$ and $t \in [\sigma, T+\sigma]$, the estimate*

$$\left|\int_\sigma^t \left[x(s) - \frac{1}{T}\int_\sigma^{T+\sigma} x(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s\right| \leqslant \frac{1}{2}\alpha_{1,T}(t-\sigma)\Big(\max_{s\in(-\infty,\infty)} x(s) - \min_{s\in(-\infty,\infty)} x(s)\Big) \tag{5.51}$$

is true, where $\alpha_{1,T}$ is the function given by formula (3.34).

PROOF. Consider the expression from the left-hand side of the desired inequality (5.51). Taking into account the fact that, for y from C_T^n, the relation

$$\int_\sigma^{T+\sigma} y(s)\,\mathrm{d}s = \int_0^T y(s)\,\mathrm{d}s$$

holds, and carrying out the change of variables $\eta = s - \sigma$ in each of the integrals, we obtain

$$\int_\sigma^t \left[x(s) - \frac{1}{T}\int_0^T x(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s = \int_0^{t-\sigma}\left[x(\xi+\sigma) - \frac{1}{T}\int_0^T x(\eta+\sigma)\,\mathrm{d}\eta\right]\mathrm{d}\xi.$$

In view of Lemma 2.3 from [70], the estimate

$$\left|\int_0^t \left[x(s) - \frac{1}{T}\int_0^T x(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s\right| \leqslant \frac{1}{2}\alpha(t)\Big[\max_{s\in(-\infty,\infty)} x(s) - \min_{s\in(-\infty,\infty)} x(s)\Big]$$

holds for all $x \in C_T^n$ and $t \in (-\infty, \infty)$, where the scalar function α is given by (3.34). This yields that, for $t \in [\sigma, T+\sigma]$,

$$\begin{aligned}
&\left|\int_\sigma^t \left[x(s) - \frac{1}{T}\int_0^T x(\sigma) d\xi\right]\mathrm{d}s\right| \\
&\quad\leqslant \frac{1}{2}\alpha(t-\sigma)\Big[\max_{\xi\in(-\infty,\infty)} x(\xi+\sigma) - \min_{\xi\in(-\infty,\infty)} x(\xi+\sigma)\Big] \\
&\quad= \frac{1}{2}\alpha(t-\sigma)\Big[\max_{\xi\in(-\infty,\infty)} x(\xi) - \min_{\xi\in(-\infty,\infty)} x(\xi)\Big],
\end{aligned}$$

which coincides with the required estimate (5.51). The lemma is proved. □

Recall that, throughout all the paper, the inequalities and the symbols max and min are understood in the sense of Remark 5.20.

LEMMA 5.27. *For an arbitrary z from $D_{\frac{T}{2}M}$, the values of all the functions of sequence* (5.50) *belong to D.*

PROOF. The assertion of Lemma 5.27 is established similarly to the argument in Section 2, Chapter 1 of [70]. For this purpose, one should use assumptions 1, 3, and 4. □

Having fixed a certain arbitrary $\sigma \in (-\infty, \infty)$, for all u from C_T^n and $t \in (-\infty, \infty)$, we put

$$(\Lambda_\sigma u)(t) := \int_\sigma^{t - Tk_\sigma(t)} u(s)\,\mathrm{d}s - \left(\frac{t-\sigma}{T} + k_\sigma(t)\right)\int_0^T u(s)\,\mathrm{d}s, \tag{5.52}$$

where $k_\sigma : (-\infty, \infty) \to \mathbb{Z}$ is the function defined by formula (5.29). By analogy with Lemma 5.21, it is not difficult to verify that $(\Lambda_\sigma u)(t + T) = (\Lambda_\sigma u)(t)$ for all $u \in C_T^n$ and $t \in (-\infty, \infty)$. Therefore, formula (5.52) defines a (bounded) linear operator $\Lambda_\sigma : C_T^n \to C_T^n$.

LEMMA 5.28. *For arbitrary non-negative u from C_T^n and all $t \in (-\infty, \infty)$, the componentwise inequality*

$$\left|(\Lambda_\sigma u)(t)\right| \leqslant (H_\sigma u)(t) \tag{5.53}$$

is true, where H_σ is the operator in C_T^n defined by formula (5.28).

PROOF. Let u belong to C_T^n, $u \geqslant 0$. Due to the T-periodicity of the functions Λ_σ and H_σ, it suffices to verify the validity of estimate (5.53) for $t \in [\sigma, T + \sigma]$ only.

According to (5.52) and (5.29), for $t \in [\sigma, T + \sigma]$, we have

$$(\Lambda_\sigma u)(t) = \left(1 - \frac{t-\sigma}{T}\right)\int_\sigma^t u(s)\,\mathrm{d}s + \frac{t-\sigma}{T}\int_t^{T+\sigma} u(s)\,\mathrm{d}s,$$

whence

$$\left|(\Lambda_\sigma u)(t)\right| \leqslant \left(1 - \frac{t-\sigma}{T}\right)\int_\sigma^t u(s)\,\mathrm{d}s + \frac{t-\sigma}{T}\int_t^{T+\sigma} u(s)\,\mathrm{d}s.$$

Recalling now the definition (5.28) of the operator H_σ, we arrive at (5.53). □

LEMMA 5.29. *For all $z \in D_{\frac{T}{2}M}$, $t \in (-\infty, \infty)$, and $m = 1, 2, \ldots$, the estimate*

$$\varrho_m(t, z) \leqslant \big(H_\sigma K \varrho_{m-1}(\cdot, z)\big)(t) \tag{5.54}$$

is true, where

$$\varrho_m(t, z) := \left| x_m(t, z) - x_{m-1}(t, z) \right| \tag{5.55}$$

and the functions $x_m(\cdot, z)$ $(m \geqslant 1)$ *are given by the recurrence formula* (5.50).

PROOF. According to Lemma 5.27, for all $z \in D_{\frac{T}{2}M}$, $t \in (-\infty, \infty)$, and $m = 1, 2, \ldots$, we have $x_m(t, z) \in D$. Therefore, in the estimate

$$\begin{aligned}
\varrho_{m+1}(t) \leqslant{} & \left(1 - \frac{t-\sigma}{T}\right) \int_\sigma^t \left[f\left(s, x_m(s, z)\right) - f\left(s, x_{m-1}(s, z)\right) \right] \mathrm{d}s \\
& + \frac{t-\sigma}{T} \int_t^{T+\sigma} \left[f\left(s, x_m(s, z)\right) - f\left(s, x_{m-1}(s, z)\right) \right] \mathrm{d}s, \\
& t \in [\sigma, T + \sigma],
\end{aligned}$$

which is an easy consequence of Corollary 5.50, we can use inequality (5.27) from assumption 2:

$$\begin{aligned}
\varrho_{m+1}(t) \leqslant{} & \left(1 - \frac{t-\sigma}{T}\right) \int_\sigma^t K(s) \left| x_m(s, z) - x_{m-1}(s, z) \right| \mathrm{d}s \\
& + \frac{t-\sigma}{T} \int_t^{T+\sigma} K(s) \left| x_m(s, z) - x_{m-1}(s, z) \right| \mathrm{d}s, \quad t \in [\sigma, T + \sigma].
\end{aligned}$$

Taking notation (5.55) and definition (5.28) of the operator H_σ into account, we arrive immediately at (5.54). Moreover, due to the T-periodicity of the functions considered, relation (5.54) holds for all $t \in (-\infty, \infty)$. The lemma is proved. □

LEMMA 5.30. *Under the assumption that*

$$R_{K,T} < 1, \tag{5.56}$$

for all $z \in D_{\frac{T}{2}M}$, *the sequence* $\{x_m(\cdot, z) \mid m \geqslant 0\}$ *given by formula* (5.50) *converges uniformly to a certain function* $x(\cdot, z) \in C_T^n$. *Furthermore, for all* $[\sigma, T + \sigma]$, *the pointwise and componentwise estimate*

$$\begin{aligned}
\left| x_m(\cdot, z) - x(\cdot, z) \right| \leqslant{} & \frac{1}{2} (H_\sigma K)^m (I - H_\sigma K)^{-1} \alpha_{1,T}(\cdot - \sigma) \\
& \times \left[\max_{s \in (-\infty, \infty)} f(s, z) - \min_{s \in (-\infty, \infty)} f(s, z) \right]
\end{aligned} \tag{5.57}$$

is true, in which the function $\alpha_{1,T} : \mathbb{R} \to \mathbb{R}$ *is given by formula* (3.34).

For all z indicated, the function $x(\cdot, z)$ is the unique solution of the non-linear integral equation

$$x(t) = z + \int_\sigma^t f\big(s, x(s)\big)\,\mathrm{d}s - \frac{t-\sigma}{T}\int_0^T f\big(s, x(s)\big)\,\mathrm{d}s,$$
$$t \in [\sigma, T+\sigma]. \tag{5.58}$$

This solution, moreover, satisfies the differential system (5.1) *if, and only if*

$$\int_0^T f\big(s, x(s, z)\big)\,\mathrm{d}s = 0. \tag{5.59}$$

The symbol I in inequality (5.57) stands for the identity operator in C_T^n.

PROOF. It is easy to see that the linear continuous operator $H_\sigma K : C_T^n \to C_T^n$ is positive in the sense that the (pointwise and componentwise) inequality $x \geqslant 0$ always yields $H_\sigma K x \geqslant 0$. Then, according to condition (5.56), the inverse operator $(I - H_\sigma K)^{-1}$ exists and, by virtue of the equality

$$(I - H_\sigma K)^{-1} = \sum_{m=0}^{+\infty} (H_\sigma K)^m, \tag{5.60}$$

is positive in the same sense. By virtue of the positivity of the operator $H_\sigma K$, Lemma 5.29 yields the pointwise inequality

$$\varrho_{m+1}(\cdot, z) \leqslant (H_\sigma K)^m \varrho_1(\cdot, z)$$

valid for all $m = 1, 2, 3, \ldots$. Therefore, according to (5.60), we have

$$\begin{aligned}
\big|x_{m+k}(\cdot, z) - x_m(\cdot, z)\big| &\leqslant (H_\sigma K)^m \sum_{\nu=1}^{k} (H_\sigma K)^\nu \varrho_1(\cdot, z) \\
&\leqslant (H_\sigma K)^m \sum_{\nu=1}^{+\infty} (H_\sigma K)^\nu \varrho_1(\cdot, z) \\
&= (H_\sigma K)^m (I - H_\sigma K)^{-1} \varrho_1(\cdot, z).
\end{aligned}$$

Thus, the sequence of continuous T-periodic vector-functions on $(-\infty, \infty)$ given by the recurrence formula (5.50) is fundamental in the uniform norm and, hence, is uniformly convergent.

On the other hand, it is clear from (5.55) that, for all (t, z) from $[\sigma, T+\sigma] \times D$,

$$\varrho_1(t,z) \leqslant \frac{1}{2}\alpha_{1,T}(t-\sigma)\Big[\max_{s\in(-\infty,\infty)} f(s,z) - \min_{s\in(-\infty,\infty)} f(s,z)\Big],$$

whence estimate (5.57) immediately follows.

Passing to the limit as $m \to +\infty$ in (5.50), we easily show that the relation

$$\lim_{m\to+\infty} \max_{t\in(-\infty,\infty)} \big|x_m(t,z) - x(t,z)\big| = 0$$

yields (5.58). The continuous differentiability of the function $x(\cdot,z)$ is obvious from the structure of equality (5.58). Differentiating both sides of the relation mentioned, we arrive at the system of non-linear integro-functional equations

$$x'(t,z) = f\big(t, x(t,z)\big) - \frac{1}{T}\int_0^T f\big(s, x(s,z)\big)\,\mathrm{d}s, \quad t \in (-\infty,\infty). \tag{5.61}$$

It is clear from the form of system (5.61) that every solution $x(\cdot,z)$, of system (5.61) satisfying equality (5.59) is also a solution of system (5.1), and *vice versa*. Thus, all the assertions of the lemma are proved. □

REMARK 5.31. It follows from assumption 5.19 that, under the conditions of Lemma 5.30, for all $m \geqslant 1$, the estimate

$$\big|x_m(\cdot,z) - x(\cdot,z)\big| \leqslant \frac{1}{2}(H_\sigma K)^m (I - H_\sigma K)^{-1}\alpha_{1,T}(\cdot-\sigma)M \tag{5.62}$$

holds on $[\sigma, T+\sigma]$. This estimate rougher but simpler than (5.57).

Note that, in the right-hand side of inequalities (5.57) and (5.62), the value of the operator $(H_\sigma K)^m (I - H_\sigma K)^{-1}$ is calculated on the function of the form

$$[\sigma, T+\sigma] \ni t \longmapsto \alpha_{1,T}(t-\sigma)c,$$

where $\alpha_{1,T}$ is the scalar function defined by formula (3.34) and $c \in \mathbb{R}^n$ is a constant vector independent of z.

REMARK 5.32. In the case where

$$K(t) \equiv K, \quad t \in (-\infty,\infty),$$

in assumption 2 from Section 5.2, condition (5.56), according to Corollary 5.25, takes the form

$$r(K) < \frac{3.4161\ldots}{T}, \tag{5.63}$$

where $r(K)$ is the greatest positive eigenvalue of the constant matrix K.

REMARK 5.33. The constant $r = 1/3.41613062\ldots$ has also appeared in [85] in the form $r = \inf_{t\in[0,1]} r(t)$, where $r(t)$, $t \in [0,1]$, is the convergence radius of the Poisson–Abel functional series

$$\sum_{k=0}^{\infty} \alpha_{k,1}(t)\lambda^k, \quad |\lambda| < r, \tag{5.64}$$

and $\alpha_{k,1}$, $k = 1, 2, \ldots$, are the functions given by formulae (3.35) for $T = 1$.

5.5. *Case of "large" Lipschitz constants*

Assuming the global Lipschitz condition 2 ((5.27) in Assumption 5.19), which, in other words, means the fulfilment of inequality (5.27) for arbitrary x and y from $\mathbb{R}^n$, one can drop condition (5.56). For this purpose, one can modify the recurrence formula (5.50) by using certain considerations from the paper of Ronto [61].

Let us consider problem (3.3), (3.4) under the following assumptions.

ASSUMPTION 5.34.

(1) $f : [0,T] \times D \to \mathbb{R}^n$ satisfies the Carathéodory conditions, i.e.,
   (a) The mapping $D \ni x \mapsto f(t,x)$ is continuous for almost every t from $[0,T]$;
   (b) The mapping $[0,T] \ni t \mapsto f(t,x)$ is Lebesgue measurable for all $x \in D$ and, moreover, the function

$$[0,T] \ni t \longmapsto M(t) := \sup_{x\in D} \bigl| f(t,x) \bigr| \tag{5.65}$$

   is Lebesgue integrable.

(2) There exists a matrix-valued function $K : [0,T] \to \mathrm{GL}_n(\mathbb{R})$ with Lebesgue integrable elements such that relation (5.27) holds for a.e. $t \in [0,T]$ and all $\{x_1, x_2\} \subset D$.

Let $\Phi : [0,T] \to \mathrm{GL}_n(\mathbb{R})$ be an arbitrary matrix-valued function with absolutely continuous elements possessing the property

$$\Phi(T) = \Phi(0) + \mathbb{1}_n. \tag{5.66}$$

It is obvious that such functions exist. One can always put, e.g.,

$$\Phi(t) := \operatorname{diag}\left(\frac{t^{\alpha_1} - \tau_1^{\alpha_1}}{T^{\alpha_1}}, \frac{t^{\alpha_2} - \tau_2^{\alpha_2}}{T^{\alpha_2}}, \ldots, \frac{t^{\alpha_n} - \tau_n^{\alpha_n}}{T^{\alpha_n}} \right), \quad t \in [0,T],$$

where $\{\tau_k \mid k = 1, 2, \ldots, n\} \subset \mathbb{R}$ and $\{\alpha_k \mid k = 1, 2, \ldots, n\} \subset (0, +\infty)$.

For any integrable function $x : [0,T] \to \mathbb{R}^n$, we put

$$(W_\Phi x)(t) := \bigl| \mathbb{1}_n - \Phi(t) \bigr| \int_0^t x(s)\,\mathrm{d}s + \bigl| \Phi(t) \bigr| \int_t^T x(s)\,\mathrm{d}s$$

at a.e. $t \in [0, T]$. It is clear that W_Φ can be regarded as a linear mapping from $L_1([0, T], \mathbb{R}^n)$ to $C([0, T], \mathbb{R}^n)$.

Let

$$\beta_\Phi := \max_{t \in [0,T]} (W_\Phi M)(t),$$

where $M : [0, T] \to \mathbb{R}^n$ is the integrable function (5.65) from Assumption 5.34.

ASSUMPTION 5.35. The set D_{β_Φ} is non-empty.

We recall that the set D_{β_Φ} is defined according to formula (1.3) in notation 1, Section 1. Assumption 5.35 means that the set D where f satisfies the Carathéodory and Lipschitz conditions is "large enough."

PROPOSITION 5.36. *Under Assumptions* 5.34 *and* 5.35:

(1) *For any solution x of the periodic problem* (3.3), (3.4), *there exists some $\xi \in \mathbb{R}^n$ such that*

$$x(t) = \xi + \int_0^t f\bigl(s, x(s)\bigr)\,\mathrm{d}s - \Phi(t) \int_0^T f\bigl(s, x(s)\bigr)\,\mathrm{d}s, \quad t \in [0, T]; \tag{5.67}$$

(2) *For any $\xi \in \mathbb{R}^n$, every solution x of Eq.* (5.67) *satisfies the T-periodic boundary condition* (3.4);

(3) *If $x(\cdot, \xi)$ is a solution of Eq.* (5.67) *and the value of ξ is such that*

$$\int_0^T f\bigl(s, x(s, \xi)\bigr)\,\mathrm{d}s = 0, \tag{5.68}$$

then $x(\cdot, \xi)$ also satisfies the differential system (3.3).

For the sake of simplicity, we now assume that

$$\Phi(0) = 0 \tag{5.69}$$

and, therefore, by (5.66), it follows that

$$\Phi(T) = \mathbb{1}_n. \tag{5.70}$$

THEOREM 5.37. (See [61, Theorem 2].) *Under Assumptions* 5.34 *and* 5.35, *there exists an absolutely continuous matrix-valued function $\Phi : [0, T] \to \mathrm{GL}_n(\mathbb{R})$ with properties* (5.69) *and* (5.70) *such that, for any $\xi \in D_{\beta_\Phi}$, Eq.* (5.67) *has a unique absolutely continuous solution $x(\cdot, \xi)$. For all such ξ, the function $x(\cdot, \xi)$ satisfies the relation*

$$\lim_{m \to \infty} \max_{t \in [0,T]} \bigl|x(t, \xi) - x_m(t, \xi)\bigr| = 0,$$

where, for any $t \in [0, T]$,

$$x_m(t,\xi) := \xi + \int_0^t f\big(s, x_{m-1}(s,\xi)\big)\,\mathrm{d}s - \Phi(t)\int_0^T f\big(s, x_{m-1}(s,\xi)\big)\,\mathrm{d}s, \tag{5.71}$$

and $x_0(\cdot,\xi) \in C([0,T], D)$ *is arbitrary. Furthermore, for all* $m = 1, 2, \ldots,$ *the error estimate*

$$|x - x_m| \leqslant \phi_m, \tag{5.72}$$

where

$$\phi_m := (I - W_\Phi K)^{-1}(W_\Phi K)^m |x_1 - x_0|, \tag{5.73}$$

holds coordinatewise and pointwise on $[0, T]$.

In (5.73), the symbol K stands for the operator of multiplication by the matrix-valued function $K : [0, T] \to \mathrm{GL}_n(\mathbb{R})$.

By analogy with formula (3.91), Section 3.2, the properties of sequence (5.71) described in Theorem 5.37 allow us to introduce the functions

$$D_{\beta_\Phi} \ni z \longmapsto \Delta_m(z) := \int_0^T f\big(t, x_m(t,z)\big)\,\mathrm{d}t \tag{5.74}$$

and consider the equation

$$\Delta_m(\xi) = 0$$

as an approximation of Eq. (5.68).

NOTATION 5.38. Let $G \subset \mathbb{R}^n$, $\Omega \subseteq G$, $\Omega \neq \varnothing$, and $X = (X_k)_{k=1}^n$ and $Y = (Y_k)_{k=1}^n$ be two functions from G to $\mathbb{R}^n$. We write $X \sqsupset_\Omega Y$ if and only if there exists a mapping $k : G \to \{1, 2, \ldots, n\}$ such that $X_{k(\xi)} > Y_{k(\xi)}$ for any $\xi \in \Omega$.

THEOREM 5.39. (See [61, Corollary 3].) *Let* $f : [0, T] \times D \to \mathbb{R}^n$ *satisfy Assumptions* 5.34 *and* 5.35. *Let* $\Phi : [0, T] \to \mathrm{GL}_n(\mathbb{R})$ *be an absolutely continuous matrix-valued function possessing properties* (5.69), (5.70) *and such that*

$$r(W_\Phi K) < 1. \tag{5.75}$$

Assume that the set D_{β_Φ} *contains a certain closed subdomain* Ω *such that, for a certain natural* m, *the relation*

$$|\Delta_m| \sqsupset_{\partial\Omega} \int_0^T K(t)\phi_m(t)\,\mathrm{d}t, \tag{5.76}$$

holds, where $\phi_m : [0, T] \to \mathbb{R}^n$ *is the function given by formula* (5.73)*. If moreover,* $\deg(\Delta_m, \Omega, 0) \neq 0$*, then the boundary value problem* (3.3), (3.4) *has a solution* x *with* $x(0) \in \Omega$*.*

Theorems 5.37 is established with the aid of Theorem 5.39. The proof of Theorem 5.39, in turn, is based upon the following statement [61, Lemma 1].

LEMMA 5.40. *For an arbitrary* $\varepsilon \in (0, \infty)$ *one can specify some* $\delta_\varepsilon \in (0, \infty)$ *such that* $r(W_\Phi K) < \varepsilon$*, provided that* $\max_{k,j=1,2,\ldots,n} \int_0^T |\Phi_{kj}(t)|\,\mathrm{d}t < \delta_\varepsilon$*.*

Here, Φ_{kj}, $k = 1, 2, \ldots, n$, stand for the corresponding elements of the matrix $\Phi = (\Phi_{kj})_{k,j=1}^n$. It follows from Lemma 5.40 that an absolutely continuous function $\Phi : [0, T] \to \mathrm{GL}_n(\mathbb{R})$ having properties (5.69), (5.70), and (5.75) exists even in the cases where the integrals $\int_0^T |K_{kj}(t)|\,\mathrm{d}t$, $k, j = 1, 2, \ldots, n$, of the elements of the variable matrix $K = (K_{kj})_{k,j=1}^n : [0, T] \to \mathrm{GL}_n(\mathbb{R})$ are arbitrarily large.

Thus, we can always suppose that condition (5.75) of Theorem 5.39 (unlike condition (5.56) of Lemma 5.30) is satisfied.

5.6. *Symmetries* (τ, E) *of solutions of the integral equation* (5.58)

Here, we describe certain conditions under which a certain (generally speaking, infinite) set of solutions of the n-parametric family of Eqs. (5.58) possess property (τ, E). For this purpose, we first state an important property of the linear operator $\Lambda_\sigma : C_T^n \to C_T^n$ given by formula (5.52).

LEMMA 5.41. *For an arbitrary point* $\sigma \in (-\infty, \infty)$ *and an arbitrary non-singular* n*-dimensional matrix* E*, the inclusion*

$$\Lambda_\sigma C_T^n(-2\sigma, -E^{-1}) \subset C_T^n(-2\sigma, E) \tag{5.77}$$

is true.

Inclusion (5.77) of Lemma 5.41 is proved by using [67, Lemma 5]. Recall that the linear set $C_T^n(-2\sigma, E)$ is defined by formula (5.3).

LEMMA 5.42. *Assume that the continuous function* $f : (-\infty, \infty) \times D \to \mathbb{R}^n$*, point* $\tau \in (-\infty, \infty)$*, and the non-singular* n*-dimensional matrix* E *are such that*

$$-f(-t - \tau, z) = Ef(t, Ez) \tag{5.78}$$

for all $z \in \mathbb{R}^n$ *and* $t \in (-\infty, \infty)$*. Then, for an arbitrary* (τ, E)*-proper function* $x \in C_T^n$*, the corresponding function*

$$(-\infty, \infty) \ni t \longmapsto f\big(t, x(t)\big)$$

possesses property $(\tau, -E^{-1})$.

Recall that the symmetry properties (τ, E) and $(\tau, -E^{-1})$ are described by Definition 5.1.

PROOF OF LEMMA 5.42. Let a function $x \in C_T^n$ have property (τ, E). According to Definition 5.1, this means that, for all t from $(-\infty, \infty)$, equality (5.2) is satisfied, whence

$$x(-t-\tau) = E^{-1}x(t), \quad t \in (-\infty, \infty). \tag{5.79}$$

Setting

$$u(t) := f\big(t, x(t)\big), \quad t \in (-\infty, \infty), \tag{5.80}$$

and taking relation (5.79) into account, we get

$$\begin{aligned} -E^{-1}u(-t-\tau) &= -E^{-1}f\big(-t-\tau, x(-t-\tau)\big) \\ &= -E^{-1}f\big(-t-\tau, E^{-1}x(t)\big). \end{aligned} \tag{5.81}$$

Using now condition (5.78) and definition (5.80), from equality (5.81) we find

$$-E^{-1}u(-t-\tau) = u(t), \quad t \in (-\infty, \infty).$$

The last equality means that u has property $(\tau, -E^{-1})$. Since the (τ, E)-proper function u is chosen arbitrarily, we have thus proved our lemma. □

REMARK 5.43. Condition (5.78) resembles condition (6.17) from Hale [28] that, in our notation, has form (5.78) with $\tau = 0$ and the matrix E possessing the properties

$$E^2 = \mathbb{1}_n, \qquad EP_0 = P_0E, \tag{5.82}$$

where P_0 is a certain projection operator defined on the set of continuous T-periodic functions. The former condition is, obviously, more general, because it involves the shift of the argument by an arbitrary τ. Moreover, we do not assume that the matrix E should satisfy (5.82) or any other related conditions used in the work cited.

Lemmata 5.30, 5.41, and 5.42 allow us to obtain the following statement.

THEOREM 5.44. *Assume that, for all real t, the function $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ satisfies condition* (5.78) *with certain number τ and non-singular n-dimensional matrix E. Let Assump-*

tion 5.19 *be true and, moreover, let inequality* (5.56) *hold. Then, for all* z *from* $D_{\frac{T}{2}M}$ *such that*[2]

$$Ez = z, \tag{5.83}$$

the integro-functional equation

$$\begin{aligned} x(t) = z &+ \int_{-\frac{\tau}{2}}^{t-T\lfloor T^{-1}(t+\frac{\tau}{2})\rfloor} f\big(s, x(s)\big)\,\mathrm{d}s \\ &- \frac{2(t - T\lfloor T^{-1}(t+\frac{\tau}{2})\rfloor) + \tau}{2T} \int_{-\frac{\tau}{2}}^{T-\frac{\tau}{2}} f\big(s, x(s)\big)\,\mathrm{d}s, \\ &t \in (-\infty, \infty), \end{aligned} \tag{5.84}$$

has a unique T*-periodic solution* $x(\cdot, z) : (-\infty, \infty) \to \mathbb{R}^n$*, and this solution possesses property* (τ, E).

Moreover, the fulfilment of equality (5.59) *is necessary and sufficient for the function* $x(\cdot, z)$ *to be a solution of the differential system* (5.1).

Note that the restriction of Eq. (5.84) to the interval $[-\tau/2, T - \tau/2]$ has the form

$$\begin{aligned} x(t) = z &+ \int_{-\frac{\tau}{2}}^{t} f\big(s, x(s)\big)\,\mathrm{d}s \\ &- \frac{2t+\tau}{2T} \int_{-\frac{\tau}{2}}^{T-\frac{\tau}{2}} f\big(s, x(s)\big)\,\mathrm{d}s, \quad t \in \left[-\frac{\tau}{2}, T - \frac{\tau}{2}\right], \end{aligned} \tag{5.85}$$

and that every solution of (5.85) satisfies the condition $x(-\tau/2) = x(T - \tau/2)$.

PROOF. It follows from (5.29) that

$$k_{-\frac{\tau}{2}}(t) = \left\lfloor \frac{1}{T}\left(t + \frac{\tau}{2}\right)\right\rfloor, \quad t \in (-\infty, \infty).$$

Therefore, as can be readily verified, Eq. (5.84) can be rewritten as

$$x = z + \Lambda_{-\frac{\tau}{2}} \mathfrak{f} x,$$

where $\mathfrak{f}$ is the Nemytsky operator generated by the function $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$,

$$u(\cdot) \longmapsto f\big(\cdot, u(\cdot)\big). \tag{5.86}$$

[2] A solution of Eq. (5.84) is unique for all $z \in D_{\frac{T}{2}M}$, not only for those satisfying condition (5.83). However, the fulfilment of the property (τ, E) is guaranteed only for those z indicated.

The T-periodicity of the function $f(\cdot, z)$ for z fixed ensures that the operator $\mathfrak{f}$ acts in the space C_T^n. We recall that the operator $\Lambda_{-\frac{\tau}{2}} : C_T^n \to C_T^n$ is defined by formula (5.52) for

$$\sigma = -\frac{\tau}{2}. \tag{5.87}$$

Let us set

$$x_{m+1} = z + \Lambda_{-\frac{\tau}{2}} \mathfrak{f} x_m, \quad m = 0, 1, 2, \ldots, \tag{5.88}$$

where $x_0 \equiv z$. It is obvious from (5.85) that the restriction of (5.88) to $[-\tau/2, T - \tau/2]$ coincides with (5.50) for σ given by (5.87).

Equality (5.83) that, by assumption, is satisfied by z, guarantees that the constant function x_0 belongs to $C_T^n(\tau, E)$ (see Definition 5.1). In view of Lemma 5.42, it follows from (5.78) and (5.87) that the function $\mathfrak{f} x_0$ possesses property $(\tau, -E^{-1})$, i.e., $u \in C_T^n(\tau, -E^{-1})$. By Lemma 5.41, this implies that, together with x_0, the function $\Lambda_{-\frac{\tau}{2}} \mathfrak{f} x_0$, also belongs to $C_T^n(\tau, E)$. Therefore, taking equality (5.88) and Lemma 5.2 into account, we conclude that $x_1 \in C_T^n(\tau, E)$. Arguing similarly, we show that all the members of the sequence of functions (5.88) possess property (τ, E).

Finally, applying Lemma 5.30 with σ given by (5.87) and taking condition (5.56) into account, we conclude that sequence (5.88) converges uniformly to a unique solution of the integro-functional Eq. (5.84). Since, by the consideration above, $x_m \in C_T^n(\tau, E)$ for all $m \geqslant 0$, we see that the corresponding limit function is also an element of the space $C_T^n(\tau, E)$, i.e., has property (τ, E).

It remains to note that the last assertion of Theorem 5.44 is also a consequence of Lemma 5.30. □

5.7. *Periodic solutions with property* (τ, E)

Now we are ready to establish the main theorem of the section. To prove it, besides the results of the preceding subsections, we need the following

LEMMA 5.45. *For an arbitrary function $x \in C_T^n$ having property (τ, G) with certain real τ and non-singular matrix G satisfying the condition*

$$\det(\mathbb{1}_n - G) \neq 0, \tag{5.89}$$

the equality

$$\int_0^T x(s)\,\mathrm{d}s = 0 \tag{5.90}$$

is true.

PROOF. Let us carry out the change of variables $s = -\xi - \tau$ in the integral from the left-hand side of (5.90). Taking into account the T-periodicity of the function $x \in C_T^n(\tau, G)$ and its property (τ, G), we get

$$\begin{aligned} G \int_0^T x(s)\,\mathrm{d}s &= -G \int_{-\tau}^{-T-\tau} x(-\xi - \tau)\,\mathrm{d}\xi \\ &= -G \int_0^{-T} x(-\xi - \tau)\,\mathrm{d}\xi \\ &= G \int_0^{T} x(-\xi - \tau)\,\mathrm{d}\xi \\ &= \int_0^T x(\xi)\,\mathrm{d}\xi, \end{aligned}$$

i.e.,

$$(\mathbb{1}_n - G) \int_0^T x(s)\,\mathrm{d}s = 0. \tag{5.91}$$

In view of (5.89), relation (5.91) yields the required equality (5.90). The lemma is proved. □

The statements obtained above lead us to the following theorem.

THEOREM 5.46. *Let us suppose that Assumption* 5.19[3] *hols for the functions* $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$, *inequality* (5.56) *is true, and, moreover,* f *satisfies condition* (5.78) *with certain real* τ *and non-singular* n*-dimensional matrix* E *such that*

$$\det(\mathbb{1}_n + E) \neq 0. \tag{5.92}$$

Then every solution x *of the system of differential Eqs.* (5.1) *for which the value* $x(-\frac{\tau}{2})$ *belongs to the set* $D_{\frac{\tau}{2}M}$ *and satisfies the equality*

$$x\left(-\frac{\tau}{2}\right) = Ex\left(-\frac{\tau}{2}\right), \tag{5.93}$$

is a T*-periodic solution possessing property* (τ, E).

PROOF. According to Theorem 5.44, under our conditions, the integro-differential equation (5.84) has a unique solution $x(\cdot, z)$ for an arbitrary $z \in \mathbb{R}^n$. For all vectors z satisfying equality (5.83), this solution, moreover, has property (τ, E).

[3] Namely, conditions (5.19)–(5.19) from Section 5.2, p. 517.

By Lemma 5.42, condition (5.78) imposed on the function f allows us to claim that the function

$$(-\infty, \infty) \ni t \longmapsto f\bigl(t, x(t, z)\bigr) \tag{5.94}$$

has property $(\tau, -E^{-1})$. Since, obviously,

$$\mathbb{1}_n + E^{-1} = E^{-1}(\mathbb{1}_n + E),$$

this, in view of (5.92), implies that the matrix $G := -E^{-1}$ and function (5.94) belonging to the subspace $C_T^n(\tau, G)$ satisfy the assumptions of Lemma 5.45. The lemma mentioned yields the relation

$$\int_{-\frac{\tau}{2}}^{T-\frac{\tau}{2}} f\bigl(s, x(s, z)\bigr)\,\mathrm{d}s = 0$$

which, by virtue of the T-periodicity of the functions considered, coincides with equality (5.59).

Thus, the solution $x(\cdot, z)$ of Eq. (5.84) for all real t satisfies the relation

$$x(t, z) = z + \int_{-\frac{\tau}{2}}^{t} f\bigl(s, x(s, z)\bigr)\,\mathrm{d}s,$$

whence it follows that this function is a solution of the initial-value problem

$$x'(t) = f\bigl(t, x(t)\bigr), \quad -\infty < t < \infty,$$
$$x\left(-\frac{\tau}{2}\right) = z.$$

On the other hand, we have proved above that $x(\cdot, z)$ lies in $C_T^n(\tau, E)$ and, in particular, is periodic with period T. It remains only to note that all these facts are true for an arbitrary z with property (5.83), which means that every solution of system (5.1) satisfying equality (5.93) belongs to the subspace $C_T^n(\tau, E)$. The theorem is proved. □

A review of the proof of Theorem 5.46 given above allows us to complement it by the following remark.

REMARK 5.47. The derivative of each of the (τ, E)-proper solutions of the differential system (5.1) that are dealt with in Theorem 5.46 has property $(\tau, -E^{-1})$.

In the case where the Lipschitz condition (5.27) for the function f in the right-hand side of system (5.1) is satisfied in the entire space and with a constant matrix, i.e., when (3.5) holds for all $t \in (-\infty, \infty)$ and $\{x_1, x_2\} \subset \mathbb{R}^n$, where $K \in \mathrm{GL}_n(\mathbb{R})$, the conditions of Theorem 5.46 are simplified. Namely, the following statement holds.

COROLLARY 5.48. *Let the continuous function* $f:\mathbb{R}^{n+1}\to\mathbb{R}^n$ *be* T*-periodic in the first variable and satisfy the Lipschitz condition* (3.5) *with a certain* n*-dimensional matrix* K, *whose spectral radius fulfils inequality* (5.63). *Assume also that, for certain real number* τ *and non-singular* n*-dimensional matrix* E *with property* (5.92), *the function* f *satisfies condition* (5.78).

Then every solution x *of the system of differential Eqs.* (5.1), *whose value at the point* $-\frac{\tau}{2}$ *satisfies relation* (5.93), *is a solution periodic with period* T *and possessing property* (τ, E).

A model example of a non-singular matrix E satisfying condition (5.92) of Theorem 5.46 is provided by the unit matrix $E=\mathbb{1}_n$, for which the validity of equality (5.2) means the τ-evenness of the function x. Recall that the notion of a τ-even function is described by Definition 5.4.

COROLLARY 5.49. *If the matrix* K *in the Lipschitz condition* (5.27) *is chosen to be constant, and the continuous function* $f:\mathbb{R}^{n+1}\to\mathbb{R}^n$ *is* T*-periodic with respect to the first variable and satisfies the condition*

$$-f(-t-\tau, z)=f(t,z),\quad -\infty<t<\infty, \tag{5.95}$$

then the fulfilment of inequality (5.63) *guarantees that all the solutions of the differential system* (5.1) *whose values at the point* $-\frac{\tau}{2}$ *lie in the set* $D_{\frac{T}{2}M}$, *are* τ*-even and periodic with period* T.

PROOF. According to Remark 5.32, inequality (5.63) guarantees the fulfilment of condition (5.56). It is also clear that (5.95) is a particular case of (5.78) with $E=\mathbb{1}_n$. Thus, the required assertion is a consequence of Theorem 5.46 for $E=\mathbb{1}_n$. □

The following statement is an immediate consequence of Corollary 5.49.

COROLLARY 5.50. *Let the continuous function* $f:\mathbb{R}^{n+1}\to\mathbb{R}^n$ *be* T*-periodic and odd in the first variable. If, moreover,* f *satisfies the Lipschitz condition* (3.5) *with a matrix* K *whose spectral radius* $r(K)$ *satisfies inequality* (5.63), *then all the solutions of the differential system* (5.1) *are even and periodic with period* T.

REMARK 5.51. Corollary 5.50 implies, in particular, Theorem 6 from [70, p. 326].

COROLLARY 5.52. *If the function* f *in Eq.* (5.1) *satisfies condition* (5.78) *with a non-singular matrix* E, *then the non-linear differential system* (5.1) *admits the trivial solution.*

PROOF. It is easy to see that, by virtue of (5.78), the function

$$(-\infty,\infty)\ni t\longmapsto f(t,0)$$

has property $(\tau, -E)$. Proposition 5.14 implies that

$$E^2 f(t, 0) = 0, \quad t \in (-\infty, \infty). \tag{5.96}$$

Since E^2 is non-singular, relation (5.96) yields $f(t, 0) = 0$ for all $t \in (-\infty, \infty)$. Therefore, system (5.1) admits the trivial solution $x \equiv 0$. □

6. Two-point boundary value problem with non-linear non-separated conditions

In Samoilenko and Ronto [91], Samoilenko and Taĭ [92], the methodology of the successive approximation method is extended in order to make possible studying the non-linear two-point boundary problem of the form

$$y'(t) = f\big(t, y(t)\big), \quad t \in [0, T], \tag{6.1}$$

$$g\big(y(0), y(T)\big) = 0, \tag{6.2}$$

for which purpose a general non-linear change of variable was applied to Eq. (6.1). Here, we use a simpler substitution, which, as is shown, essentially facilitates the subsequent application of the successive approximation method. In particular, all the assumptions are formulated in terms of the original problem, and not the transformed one.

6.1. *Reduction to a problem with linear conditions*

Consider the boundary value problem (6.1), (6.2), where the functions $f : [0, T] \times G \to \mathbb{R}^n$ and $g : G \times G \to \mathbb{R}^n$ are continuous, $G \subset \mathbb{R}^n$ being the closure of a bounded domain. Assume that, for $t \in [0, T]$ fixed, f satisfies the Lipschitz condition

$$\big|f(t, y_1) - f(t, y_2)\big| \leqslant K|y_1 - y_2|, \quad \{y_1, y_2\} \subset G, \tag{6.3}$$

where the square matrix K is supposed to have non-negative components. In (6.3), the absolute value sign and inequality are understood component-wise.

Let us introduce the traditional substitution

$$y(t) = x(t) + w, \tag{6.4}$$

where $w \in \Omega \subset \mathbb{R}^n$ is an unknown parameter. The domain, Ω, of w is chosen so that $D + \Omega \subset G$, whereas the new variable, x, is supposed to have range in D, the closure of a bounded subdomain of G. Substitution (6.4) allows one to rewrite (6.1), (6.2) as

$$x'(t) = f\big(t, x(t) + w\big), \quad t \in [0, T], \tag{6.5}$$

$$g\big(x(0) + w, x(T) + w\big) = 0. \tag{6.6}$$

Let us rewrite (6.6) in the form

$$Ax(0) + Bx(T) + g\big(x(0) + w, x(T) + w\big) = Ax(0) + Bx(T), \tag{6.7}$$

where A and B are some square matrices of dimension n such that $\det B \neq 0$.

The parameter w is natural to be determined from the additional equation

$$Ax(0) + Bx(T) + g\big(x(0) + w, x(T) + w\big) = 0. \tag{6.8}$$

Due to (6.7), condition (6.8) obviously implies that

$$Ax(0) + Bx(T) = 0. \tag{6.9}$$

Thus, the essentially non-linear problem (6.1), (6.2) is transformed into the equivalent problem (6.5), (6.6), and (6.8).

On the other hand, system (6.5), (6.6), (6.8) can be regarded as a collection of problems (6.9) for the differential system

$$x'(t) = f\big(t, x(t) + w\big), \tag{6.10}$$

parametrised by the vector parameter w belonging to a certain given set Ω. The essential advantage obtained thereby is that the boundary condition (6.9) is linear. Therefore, problems of the family (6.10), (6.9) can be studied by using the successive approximation method developed in [90,91].

Assume that

$$D_\beta := \big\{x \in \mathbb{R}^n \mid B\big(x, \beta(x)\big) \subset D\big\} \neq \varnothing, \tag{6.11}$$

where

$$\beta(x) := \frac{T}{4}\delta_G(f) + \big|(B^{-1}A + \mathbb{1}_n)x\big|. \tag{6.12}$$

Here, by definition,

$$\delta_G(f) := \max_{(t,y)\in[0,T]\times G} f(t, y) - \min_{(t,y)\in[0,T]\times G} f(t, y). \tag{6.13}$$

Moreover, we suppose that

$$r(K) < \frac{10}{3T}. \tag{6.14}$$

Introduce the sequence of functions

$$\begin{aligned} x_{m+1}(t, w, z) := z &+ \int_0^t f\big(s, x_m(s, w, z) + w\big)\,\mathrm{d}s \\ &- \frac{t}{T}\int_0^T f\big(s, x_m(s, w, z) + w\big)\,\mathrm{d}s - \frac{t}{T}(B^{-1}A + \mathbb{1}_n)z, \end{aligned} \tag{6.15}$$

where $m = 0, 1, \dots$ and $x_0(t, w, z) \equiv z$. It is easily seen that all the members of sequence (6.15) satisfy condition (6.9) for every $z \in D_\beta$ and $w \in \Omega$. Furthermore, $x_m(0, w, z) = z$ for all $m \in \mathbb{N}$.

By virtue of (6.9), a solution x of (6.10), (6.9) satisfies the condition

$$x(T) = -B^{-1}Ax(0)$$

and, therefore, Eq. (6.8) can be rewritten as

$$g\bigl(x(0) + w, -B^{-1}Ax(0) + w\bigr) = 0. \tag{6.16}$$

Summarising the above-said, we conclude that problem (6.10), (6.9), (6.8) is equivalent to system (6.10), (6.9), (6.16). We suggest to solve it sequentially: first solve (6.10), (6.9), and then try to determine whether (6.16) can be fulfilled.

THEOREM 6.1. *Assume conditions* (6.3), (6.11), *and* (6.14). *Then*:

(1) *Sequence* (6.15) *has a limit* $x^*(\cdot, w, z)$ *uniform in* $(t, w, z) \in [0, T] \times D \times D_\beta$;

(2) *The limit function* $x^*(\cdot, w, z)$ *is a solution of problem* (6.9) *for the "perturbed" differential system*

$$x'(t) = f\bigl(t, x(t) + w\bigr) + \Delta(w, z), \quad t \in [0, T],$$

where, by definition,

$$\Delta(w, z) := -\frac{1}{T}(B^{-1}A + \mathbb{1}_n) - \frac{1}{T}\int_0^T f\bigl(s, x^*(s, w, z)\bigr)\,\mathrm{d}s.$$

Furthermore, $x^*(0, w, z) = z$.

(3) *The error estimate*

$$\begin{aligned}\bigl|x^*(t, w, z) - x_m(t, w, z)\bigr| \leqslant \frac{20}{9}t\left(1 - \frac{t}{T}\right)Q^{m-1}(\mathbb{1}_n - Q)^{-1}\Biggl[\frac{1}{2}\delta_G(f) \\ + K\bigl|(B^{-1}A + \mathbb{1}_n)z\bigr|\Biggr]\end{aligned} \tag{6.17}$$

holds, where $Q := \frac{3T}{10}K$ *and* $\delta_G(f)$ *is as in* (6.13).

PROOF. It is carried out similarly to that of Theorem 1 from [68] and Theorem 2.1 from [91, p. 32]. □

The following statement shows the relation of the function $x^*(\cdot, w, z)$ to problem (6.10), (6.9).

THEOREM 6.2. *Under assumptions* (6.3), (6.11), *and* (6.14), *the function* $x^*(\cdot, w, z^*)$ *is a solution of the boundary value problem* (6.10), (6.9) *if, and only if* $z = z^*$ *satisfies the "determining equation,"*

$$\big(B^{-1}A + \mathbb{1}_n\big)z + \int_0^T f\big(s, x^*(s, w, z)\big)\,\mathrm{d}s = 0,$$

where w *is considered as a parameter.*

PROOF. Analogous to that of Theorem 2.4 from the book of Samoilenko and Ronto [91, p. 36]. □

THEOREM 6.3. *Assume conditions* (6.3), (6.11), *and* (6.14). *Then, for the function* $x^*(\cdot, w, z^*)$ *to be a solution of the boundary value problem* (6.1), (6.2), *it is sufficient that the parameters* $z = z^*$, $w = w^*$ *satisfy the system of determining equations*

$$(B^{-1}A + \mathbb{1}_n)z + \int_0^T f\big(s, x^*(s, w, z)\big)\,\mathrm{d}s = 0, \tag{6.18}$$

$$g(z + w, -B^{-1}Az + w) = 0. \tag{6.19}$$

In this case,

$$y^*(t) = x^*\big(t, w^*, z^*\big) + w^* \tag{6.20}$$

is a solution of the boundary value problem (6.1), (6.2).

PROOF. It is obvious from the form of substitution (6.4) that Eqs. (6.18) and (6.19) hold whenever the transformed boundary value problem (6.10), (6.9), (6.16) coincides with (6.1), (6.2). □

REMARK 6.4. In [92,91], a change of variable more general than (6.4) was used,

$$y(t) = x(t) + h(t, w). \tag{6.21}$$

However, when $\partial h(t, w)/\partial t \not\equiv 0$, substitution (6.21) leads one to a more complicated transformed boundary value problem, as well as more complicated recursion sequence and determining equations. Furthermore, the basic assumptions should then be made on the *transformed* differential equation.

REMARK 6.5. Considering (6.20), on can treat the function

$$y_m(t) = x_m(t, w_m, z_m) + w_m \tag{6.22}$$

as the mth approximation to the exact solution, $y^*(\cdot, w^*, z^*)$, of the boundary value problem (6.1), (6.2). In (6.22), w_m and z_m are solutions of the "approximate determining equations,"

$$(B^{-1}A + \mathbb{1}_n)z + \int_0^T f\bigl(s, x_m(s, w, z)\bigr)\,\mathrm{d}s = 0, \tag{6.23}$$

$$g(z + w, -B^{-1}Az + w) = 0, \tag{6.24}$$

and $x_m(\cdot, z, w)$ is given by (6.15). We do not consider the strict substantiation, referring the reader to [91] where similar techniques are described.

REMARK 6.6. It follows from the consideration in Section 7.2 that the convergence of the recursion sequence (6.15) can be proved under the condition $r(K) < 3.4161\ldots \cdot T^{-1}$, which is weaker than (6.14). However, estimate (6.17) does not hold in this case.

REMARK 6.7. By using an idea from [61], one can obtain similar statements based on replacing (6.15) by the sequence

$$\begin{aligned} x_{m+1}(t, w, z) := z - (A + B)z + \int_0^t f\bigl(s, x_m(s, w, z) + w\bigr)\,\mathrm{d}s \\ - \Phi(t)\int_0^T f\bigl(s, x_m(s, w, z) + w\bigr)\,\mathrm{d}s, \end{aligned}$$

where Φ is an arbitrary continuous matrix-valued function such that

$$A\Phi(0) + B\Phi(T) = \mathbb{1}_n.$$

In this case, the restrictive conditions of type (6.14) can be avoided. However, an analogue of condition (6.11) is more difficult to verify unless f in Eq. (6.1) is globally Lipschitzian.

EXAMPLE 6.8. Consider the differential equation

$$y''(t) + \frac{t}{16}y(t) + \frac{t}{4}\bigl(y'(t)\bigr)^2 = \frac{t^2}{128}, \quad t \in [0, 1], \tag{6.25}$$

under the non-linear boundary conditions

$$\begin{aligned} &y'(1)y'(0) + \frac{1}{4}y(0) = 0, \\ &y(1)y'(0) - \frac{1}{2}y'(1) = -\frac{7}{128}. \end{aligned} \tag{6.26}$$

We reduce Eq. (6.25) to the first order system

$$\begin{aligned} y_1'(t) &= y_2(t), \\ y_2'(t) &= \frac{t^2}{128} - \frac{t}{16} y_1(t) - \frac{t}{4} y_2^2(t); \end{aligned} \tag{6.27}$$

conditions (6.26) then rewrite as

$$\begin{aligned} y_2(1) y_2(0) + \frac{1}{4} y_1(0) &= 0, \\ y_1(1) y_2(0) - \frac{1}{2} y_2(1) &= -\frac{7}{128}. \end{aligned}$$

Put $A = B = I_2$ and $T = 1$. It is not difficult to verify that the assumptions of the theorems above hold in the domain $G := \{(y_1, y_2) \mid |y_1| \leqslant 1, |y_2| \leqslant \frac{1}{2}\}$.

Namely, relation (6.11) holds whenever β from (6.12) satisfies the component-wise inequality

$$\beta(x_1, x_2) \leqslant \begin{pmatrix} \frac{1}{4} \\ \frac{17}{256} \end{pmatrix} + 2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The vector function in the right-hand side of (6.27) satisfies (6.3) in G with $K = \begin{pmatrix} 0 & 1 \\ \frac{1}{16} & \frac{1}{4} \end{pmatrix}$. Since $r(K) = (1 + \sqrt{2})/4$, it is obvious that (6.14) holds.

Substitution (6.4) in (6.27) leads us to the system

$$\begin{aligned} x_1'(t) &= x_2(t) + w_2, \\ x_2'(t) &= \frac{t^2}{128} - \frac{t}{16}\big[x_1(t) + w_1\big] - \frac{t}{4}\big[x_2(t) + w_2\big]^2. \end{aligned}$$

In this case, the determining equation (6.24) has the form

$$\begin{aligned} w_2^2 - z_2^2 + \frac{1}{4}(z_1 + w_1) &= 0, \\ (w_1 - z_1)(w_2 + z_2) - \frac{1}{2}(w_2 - z_2) &= -\frac{7}{128}. \end{aligned} \tag{6.28}$$

Let us find the first approximation, $x_1 = \left(\begin{smallmatrix} x_{1,1} \\ x_{1,2} \end{smallmatrix}\right)$, in the sense of Remark 6.5. According to (6.15), we have

$$\begin{aligned} x_{1,1}(t, w_1, w_2, z_1, z_2) &= z_1 - 2t z_1, \\ x_{1,2}(t, w_1, w_2, z_1, z_2) &= z_2 + \frac{t^3}{384} - \frac{t^2}{32}(z_1 + w_1) - \frac{t^2}{8}(z_1 + w_2)^2 - \frac{t}{384} \\ &\quad + \frac{t}{32}(z_1 + w_1) + \frac{t}{8}(z_2 + w_2)^2 - 2t z_2. \end{aligned}$$

The first approximate determining equation (6.23) (i.e., that corresponding to $m = 1$), is equivalent to $\Delta_1(w, z) = 0$, where the function $\Delta_1 = \binom{\Delta_{1,1}}{\Delta_{1,2}}$ is given by the formula

$$\Delta_{1,1}(w_1, w_2, z_1, z_2) = \frac{385}{192} z_1 - \frac{1}{1536} + \frac{1}{192} w_1 + \frac{1}{48}(z_2 + w_2)^2 + w_2 \tag{6.29}$$

and

$$\begin{aligned}
\Delta_{1,2}(w_1, w_2, z_1, z_2) = &-\frac{1}{15360} z_1 z_2 w_2 + \frac{36863}{14155776} + \frac{107531}{10321920} z_1 \\
&+ \frac{46079}{23040} z_2 - \frac{322549}{10321920} w_1 + \frac{1}{5760} w_2 \\
&- \frac{1}{15360} w_1 z_2 w_2 - \frac{1}{15360} {z_2}^4 - \frac{1}{15360} {w_2}^4 \\
&- \frac{107509}{2580480} {z_2}^2 + \frac{107531}{1290240} z_2 w_2 - \frac{322549}{2580480} {w_2}^2 \\
&- \frac{1}{245760} {z_1}^2 - \frac{1}{122880} z_1 w_1 - \frac{1}{30720} z_1 {z_2}^2 \\
&- \frac{1}{30720} z_1 {w_2}^2 - \frac{1}{245760} {w_1}^2 - \frac{1}{30720} w_1 {z_2}^2 \\
&- \frac{1}{30720} w_1 {w_2}^2 - \frac{1}{3840} {z_2}^3 w_2 - \frac{1}{2560} {z_2}^2 {w_2}^2 \\
&- \frac{1}{3840} z_2 {w_2}^3 + \frac{1}{3840} z_2 z_1 + \frac{1}{3840} z_2 w_1 \\
&- \frac{1}{320} {z_2}^2 w_2 - \frac{3}{320} z_2 {w_2}^2 - \frac{1}{768} w_2 z_1 \\
&- \frac{1}{768} w_2 w_1 + \frac{1}{960} {z_2}^3 - \frac{1}{192} {w_2}^3.
\end{aligned} \tag{6.30}$$

Solving system (6.28) together with the approximate determining equations

$$\Delta_{1,1}(w_1, w_2, z_1, z_2) = 0, \qquad \Delta_{1,2}(w_1, w_2, z_1, z_2) = 0,$$

where $\Delta_{1,1}$ and $\Delta_{1,2}$ are given by (6.29) and (6.30), we obtain the following values for $z_1 = \binom{z_{1,1}}{z_{1,2}}$ and $w_1 = \binom{w_{1,1}}{w_{1,2}}$:

$$z_{1,1} \approx -0.06206890391, \qquad z_{1,2} \approx -0.00001967788609,$$
$$w_{1,1} \approx -0.0002202244703, \qquad w_{1,2} \approx 0.1247889518.$$

With these values of parameters, the first approximation, x_1, has the form

$$x_{1,1}(t) \approx -0.06228912838 + 0.1241378078t,$$

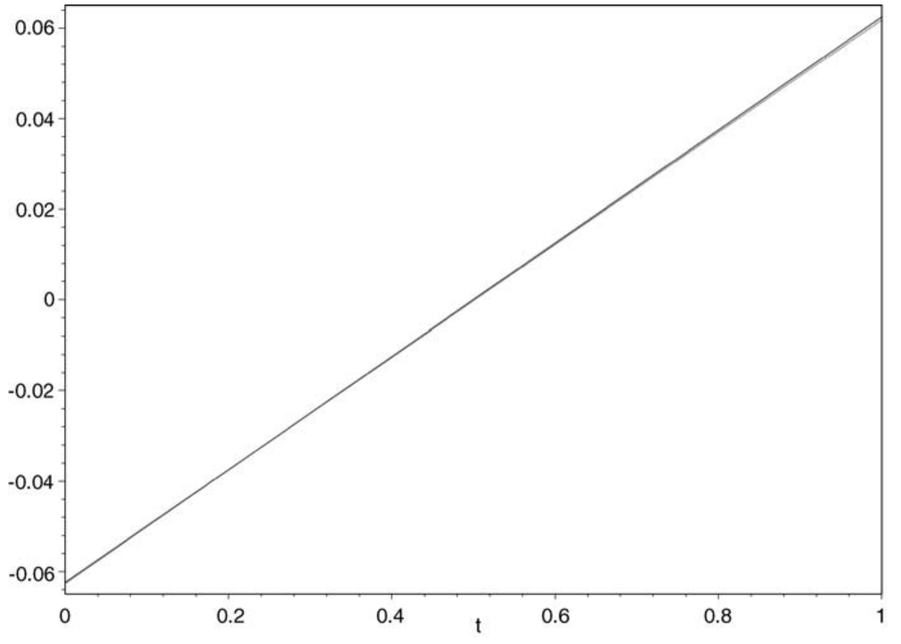

Fig. 4. The exact solution and the first approximation (Example 6.8, problem (6.25), (6.26)). The curves graphically almost coincide with one another.

$$x_{1,2}(t) \approx -0.196778861 \times 10^{-4} + \frac{1}{384}t^3 + 0.613798 \times 10^{-6}t^2 - 0.2565424693 \times 10^{-2}t.$$

Consequently, according to substitution (6.4), the solution, y, of problem (6.25), (6.26), up to the first approximation $y_1(t) = x_1(t) + w_1$, has the form

$$y_{1,1}(t) \approx -0.06250935285 + 0.1241378078t,$$

$$y_{1,2}(t) \approx 0.1247692739 + \frac{1}{384}t^3 + 0.613798 \times 10^{-6}t^2 - 0.2565424693 \times 10^{-2}t.$$

Note that the function $y^*(t) = \frac{t}{8} - \frac{1}{16}$ is an exact solution of problem (6.25), (6.26). As is seen from Figs. 4 and 5, the graphs of the exact solution and the first approximation almost coincide, whereas the deviation of their derivatives does not exceed 0.001. Thus, even the first iteration of the method gives a satisfactory approximation for the solution of the problem considered.

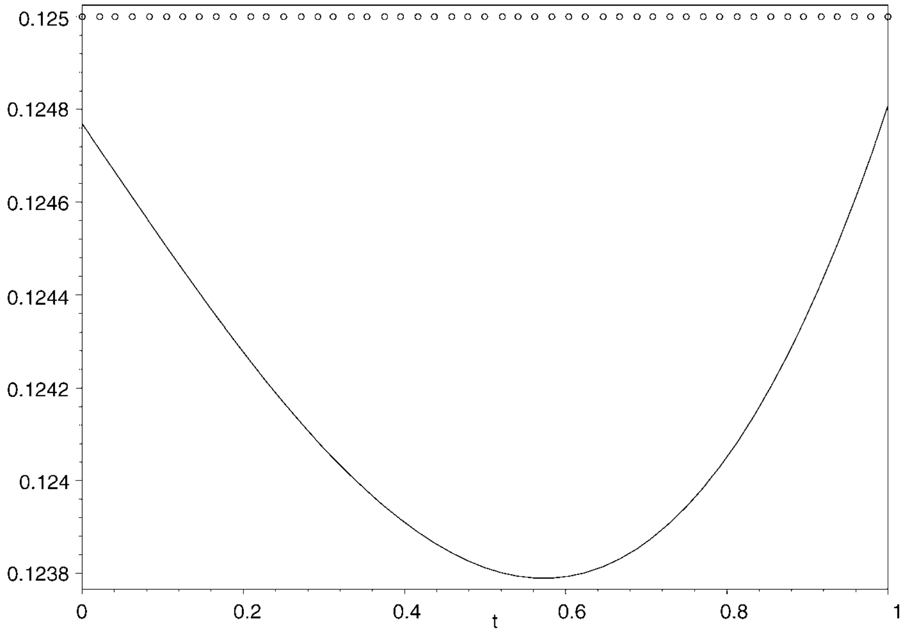

Fig. 5. The derivatives of the exact solution and the first approximation (Example 6.8, problem (6.25), (6.26)).

6.2. *A separated non-linear boundary condition*

It turns out that, in the case of separated, in a sense, boundary conditions, the successive approximation method can be applied without any change of variable.

Consider the two-point boundary value problem

$$x'(t) = f\big(t, x(t)\big), \quad t \in [0, T], \tag{6.31}$$

$$x(T) = a\big(x(0)\big), \tag{6.32}$$

where $a : D \to \mathbb{R}^n$ is continuous and f satisfies conditions (6.3) and (6.14) in the closure, D, of a bounded domain from $\mathbb{R}^n$.

Put $\gamma(x) := \frac{T}{4}\delta_D(f) + |a(x) - x|$, $x \in D$, and assume that

$$D_\gamma \neq \varnothing, \tag{6.33}$$

where the set D_γ is defined similarly to (6.11), whereas the number $\delta_D(f)$ is given by (6.13). Introduce the sequence of functions

$$\begin{aligned} x_{m+1}(t, z) := z &+ \int_0^t f\big(s, x_m(s, z)\big)\,\mathrm{d}s \\ &- \frac{t}{T}\int_0^T f\big(s, x_m(s, z)\big)\,\mathrm{d}s + \frac{t}{T}\big[a(z) - z\big], \end{aligned} \tag{6.34}$$

where $m = 0, 1, \ldots$ and $x_0(t, z) \equiv z \in D_\gamma$. Obviously, all the members of sequence (6.34) satisfy condition (6.32) for every $z \in D_\gamma$.

THEOREM 6.9. *Assume conditions* (6.3), (6.14), *and* (6.33). *Then*:

(1) *Sequence* (6.34) *converges uniformly in* $[0, T] \times D_\gamma$,

$$\lim_{n \to +\infty} \sup_{(t,z) \in [0,T] \times D_\gamma} \left\| x_n(t, z) - x^*(t, z) \right\| = 0.$$

1. *The function* $x^*(\cdot, z)$ *is a solution of problem* (6.32) *for the "perturbed" system*

$$x'(t) = f\bigl(t, x(t)\bigr) + \Delta(z), \quad t \in [0, T],$$

where

$$\Delta(z) := \frac{1}{T}\left(a(z) - z - \int_0^T f\bigl(s, x^*(s, z)\bigr)\,\mathrm{d}s\right).$$

Moreover, $x^*(\cdot, z)$ *satisfies* $x^*(0, z) = x_m(0, z) = z$ $(\forall m \in \mathbb{N})$.

(3) *The error estimate*

$$\left|x^*(t, w, z) - x_m(t, w, z)\right| \leqslant \frac{20}{9} t \left(1 - \frac{t}{T}\right) Q^{m-1} (\mathbb{1}_n - Q)^{-1} \times \left[\frac{1}{2}\delta_D(f) + K\left|a(z) - z\right|\right] \tag{6.35}$$

holds, where $Q := \frac{3T}{10} K$ *and* $\delta_D(f)$ *is defined by* (6.13).

PROOF. Analogous to that of Theorem 2.1 from [91, p. 32]. □

THEOREM 6.10. *Under conditions of Theorem* 6.9, *the limit function of sequence* (6.34) *is a solution of problem* (6.31), (6.32) *if, and only if the parameter* $z = z^*$ (*which stands for the initial value of the solution at* $t = 0$) *satisfies the determining equation*

$$a(z) - z = \int_0^T f\bigl(s, x^*(s, z)\bigr)\,\mathrm{d}s.$$

PROOF. Similar to that of Theorem 2.3 from [91, p. 36]. □

REMARK 6.11. If $a(z) \equiv z$ in (6.32), then (6.31), (6.32) is nothing but the T-periodic boundary value problem. In this case, sequence (6.34) is reduced to the well-known T-periodic successive approximations [89],

$$x_{m+1}(t, z) := z + \int_0^t f\bigl(s, x_m(s, z)\bigr)\,\mathrm{d}s - \frac{t}{T}\int_0^T f\bigl(s, x_m(s, z)\bigr)\,\mathrm{d}s, \quad t \in [0, T].$$

REMARK 6.12. Estimate (6.35) implies that the function $t \mapsto x_m(t, z_m)$ is natural to be taken as the mth approximation of the exact solution of problem (6.31), (6.32), when $z = z_m$ is a root of the mth approximate determining equation,

$$a(z) - z = \int_0^T f\big(s, x_m(s, z)\big)\,\mathrm{d}s. \tag{6.36}$$

EXAMPLE 6.13. Let us apply the technique described above to the problem

$$\begin{aligned} &x''(t) - \frac{t}{8}x'(t) + \frac{1}{2}\big(x'(t)\big)^2 = \frac{1}{4}, \quad t \in [0, 1],\\ &x(1) = \frac{1}{2}\big(x'(0)\big)^2 + \frac{1}{16},\\ &x'(1) = -x(0) + \frac{3}{16}. \end{aligned} \tag{6.37}$$

As usual, we rewrite the second order problem (6.37) in the form

$$\begin{aligned} &x_1'(t) = x_2,\\ &x_2'(t) = \frac{1}{4} + \frac{t}{8}x_2(t) - \frac{1}{2}x_2^2(t), \quad t \in [0, 1],\\ &x_1(1) = \frac{1}{2}x_2^2(0) + \frac{1}{16},\\ &x_2(1) = -x_1(0) + \frac{3}{16}. \end{aligned} \tag{6.38}$$

Consider (6.38) in the domain

$$(t, x_1, x_2) \in [0, 1] \times D, \quad D = \left\{ (x_1, x_2) \;\middle|\; |x_1| \leqslant 1, |x_2| \leqslant \frac{3}{4} \right\}. \tag{6.39}$$

It is not difficult to verify that the conditions of Theorem 6.9 are fulfilled with

$$K = \begin{pmatrix} 0 & 1 \\ 0 & \frac{7}{8} \end{pmatrix}, \qquad \gamma(x) \leqslant \begin{pmatrix} \frac{7}{16} + \frac{1}{2}x_2^2 - x_1 \\ \frac{1}{2} - x_1 - x_2 \end{pmatrix}.$$

The corresponding quantity (6.13) is estimated as

$$\delta_D(f) \leqslant \begin{pmatrix} \frac{3}{4} \\ \frac{5}{8} \end{pmatrix},$$

and K satisfies (6.14) because $r(K) = \frac{7}{8}$.

According to (6.34), we have the following formulae for $x_1 = \binom{x_{1,1}}{x_{1,2}}$:

$$x_{1,1}(t,\zeta_1,\zeta_2) = \zeta_1 + t\left(\frac{1}{2}\zeta_2^2 - \zeta_1 + \frac{1}{16}\right), \tag{6.40}$$

$$x_{1,2}(t,\zeta_1,\zeta_2) = \zeta_2 + \frac{t^2}{16}\zeta_2 - \frac{1}{16}t\zeta_2 - t\left(\zeta_1 + \zeta_2 - \frac{3}{16}\right). \tag{6.41}$$

The first approximate determining equations (6.36) have the form

$$\Delta_{1,1}(\zeta_1,\zeta_2) = 0, \qquad \Delta_{1,2}(\zeta_1,\zeta_2) = 0, \tag{6.42}$$

where

$$\Delta_{1,1}(\zeta_1,\zeta_2) = \frac{1}{2}\zeta_2^2 - \frac{47}{96}\zeta_2 - \frac{1}{2}\zeta_1 - \frac{1}{32},$$
$$\Delta_{1,2}(\zeta_1,\zeta_2) = -\frac{49}{48}\zeta_1 - \frac{33}{512} - \frac{88165}{89088}\zeta_2 + \frac{827}{5120}\zeta_2{}^2 - \frac{31}{192}\zeta_2\zeta_1 + \frac{1}{6}\zeta_1{}^2.$$

The solution, $\zeta = \binom{\zeta_1}{\zeta_2}$, of system (6.42) has the form

$$\zeta_1 = -\frac{1}{16}, \qquad \zeta_2 = 0. \tag{6.43}$$

Inserting (6.43) into (6.40) and (6.41), we obtain the first approximation

$$x_{1,1}(t) = \frac{t}{8} - \frac{1}{16}, \qquad x_{1,2}(t) = \frac{t}{4}. \tag{6.44}$$

Note that the vector function with components

$$x_1^*(t) = \frac{t^2}{8} - \frac{1}{16}, \qquad x_2^*(t) = \frac{t}{4} \tag{6.45}$$

is an exact solution of problem (6.38).

Let us construct the second approximation. By (6.34), we have

$$x_{2,1}(t,\zeta_1,\zeta_2) = \zeta_1 + \frac{1}{48}\zeta_2 t^3 - \frac{1}{2}t^2\zeta_1 + \frac{3}{32}t^2 - \frac{17}{32}t^2\zeta_2$$
$$+ \frac{49}{96}\zeta_2 t - \frac{1}{2}t\zeta_1 - \frac{1}{32}t + \frac{1}{2}t\zeta_2^2, \tag{6.46}$$

$$x_{2,2}(t,\zeta_1,\zeta_2) = \frac{95}{512}t - \frac{31}{192}t\zeta_2\zeta_1 + z_2 + \frac{17}{768}\zeta_2 t^3 - \frac{1}{32}t^2\zeta_2 - \frac{49}{48}t\zeta_1 - \frac{1733}{5120}t\zeta_2^2 + \frac{1}{512}t^3 - \frac{3041}{3072}\zeta_2 t - \frac{1}{2560}\zeta_2^2 t^5 + \frac{1}{64}t^4\zeta_2\zeta_1 - \frac{17}{48}t^3\zeta_2\zeta_1 + \frac{1}{2}t^2\zeta_2\zeta_1 - \frac{1}{1024}t^4\zeta_2 + \frac{17}{1024}t^4\zeta_2^2 + \frac{1}{48}t^3\zeta_1 - \frac{107}{512}t^3\zeta_2^2 - \frac{1}{6}t^3\zeta_1^2 + \frac{17}{32}t^2\zeta_2^2 + \frac{1}{6}t\zeta_1^2. \tag{6.47}$$

It is not difficult to verify that inserting into (6.46) and (6.47) even the roots (6.43) of the *first* determining system (6.42), we arrive at the exact solution (6.45) of problem (6.38).

7. A non-linear problem with another type of separated boundary conditions

Here, we suggest a constructive scheme for the investigation of non-linear differential systems (3.3) with non-linear two-point boundary conditions of the form (2.4).

The right-hand side of the equation is supposed to be Lipschitzian in a suitable bounded domain, Ω, and the spectrum of the corresponding Lipschitz matrix is assumed to be contained inside the circle of radius less than $r \approx 3.4161/T$. Under the latter assumption, the convergence of the method is proved by using some techniques developed in [70,64] and their appropriate modifications.

It is worth mentioning that we impose the only technical restriction (7.21) on the continuous mapping $b:\mathbb{R}^n \to \mathbb{R}^n$, and this condition is automatically satisfied whenever the domain Ω is large enough.

7.1. *Subsidiary lemmata*

The estimates related to the successive approximation method under consideration essentially use properties of the sequence of functions $\{\alpha_{k,T}:\ k \geqslant 0\} \subset C([0,T])$ given by the recurrence formula (3.35):

$$\alpha_{k+1,T}(t) = \left(1 - \frac{t}{T}\right)\int_0^t \alpha_{k,T}(s)\,\mathrm{d}s + \frac{t}{T}\int_t^T \alpha_{k,T}(s)\,\mathrm{d}s, \quad k = 0,1,\ldots, \tag{7.1}$$

and $\alpha_{0,T} \equiv 1$. Let us obtain some further properties of this sequence.

LEMMA 7.1.

(1) *The functions of sequence* (7.1) *are symmetric with respect to the point* $T/2$, *i.e., for all* $k \geqslant 0$,

$$\alpha_{k,T}(t) = \alpha_{k,T}(T-t), \quad t \in [0,T]; \tag{7.2}$$

$$\alpha_{k,T}(T/2 - t) = \alpha_{k,T}(T/2 + t), \quad t \in [0,T/2]; \tag{7.3}$$

(2) *Sequence* (7.1) *can be represented alternatively as*

$$\alpha_{k+1,T}(t) = \int_0^t \alpha_{k,T}(s)\,\mathrm{d}s + \frac{t}{T}\int_t^{T-t} \alpha_{k,T}(s)\,\mathrm{d}s \tag{7.4}$$

$$= \frac{t}{T}\int_0^{T-t} \alpha_{k,T}(s)\,\mathrm{d}s + \left(1-\frac{t}{T}\right)\int_{T-t}^{T} \alpha_{k,T}(s)\,\mathrm{d}s \tag{7.5}$$

$$= \int_0^{T-t} \alpha_{k,T}(s)\,\mathrm{d}s + \left(1-\frac{t}{T}\right)\int_{T-t}^{t} \alpha_{k,T}(s)\,\mathrm{d}s; \tag{7.6}$$

(3) *The relations*

$$\alpha_{k,T}(0) = \alpha_{k,T}(T) = 0; \qquad \alpha_{k,T}(t) > 0 \quad \textit{for all } t \in (0,T),$$

hold for every $k \geqslant 0$;

(4) *For every* $k \geqslant 1$, *the maximal value of* $\alpha_{k,T}(t)$ *is achieved at* $t = T/2$, *namely,*

$$\max_{t\in[0,T]} \alpha_{k,T}(t) = \alpha_{k,T}(T/2) = T^k \alpha_{k,1}(1/2). \tag{7.7}$$

PROOF. *Assertion* 1. Obviously, (7.2) holds for $k = 0$. Let us assume that (7.2) is true for some k and prove that this relation remains valid when k is replaced by $k+1$. Indeed, we have

$$\int_0^{T-t} \alpha_{k,T}(s)\,\mathrm{d}s = \int_0^{T-t} \alpha_{k,T}(T-s)\,\mathrm{d}s = \int_t^{T} \alpha_{k,T}(s)\,\mathrm{d}s \tag{7.8}$$

and, by (7.8),

$$\int_{T-t}^{T} \alpha_{k,T}(s)\,\mathrm{d}s = \int_0^{T} \alpha_{k,T}(s)\,\mathrm{d}s - \int_0^{T-t} \alpha_{k,T}(s)\,\mathrm{d}s = \int_0^{t} \alpha_{k,T}(s)\,\mathrm{d}s. \tag{7.9}$$

Considering (7.1), (7.8), and (7.9), we obtain

$$\begin{aligned}\alpha_{k,T}(T-t) &= \left(1-\frac{T-t}{T}\right)\int_0^{T-t} \alpha_{k,T}(s)\,\mathrm{d}s + \frac{T-t}{T}\int_{T-t}^{T} \alpha_{k,T}(s)\,\mathrm{d}s \\ &= \left(1-\frac{t}{T}\right)\int_0^{t} \alpha_{k,T}(s)\,\mathrm{d}s + \frac{t}{T}\int_t^{T} \alpha_{k,T}(s)\,\mathrm{d}s = \alpha_{k,T}(t),\end{aligned}$$

i.e., (7.2) holds. Equality (7.3) is an immediate consequence of (7.2) with t replaced by $t + T/2$.

Assertion 2. By virtue of (7.1) and (7.9), we have

$$\alpha_{k+1,T}(t) = \int_0^t \alpha_{k,T}(s)\,\mathrm{d}s - \frac{t}{T}\int_0^t \alpha_{k,T}(s)\,\mathrm{d}s$$

$$
\begin{aligned}
&+ \frac{t}{T}\int_t^{T-t} \alpha_{k,T}(s)\,\mathrm{d}s + \frac{t}{T}\int_{T-t}^{T} \alpha_{k,T}(s)\,\mathrm{d}s \\
&= \int_0^t \alpha_{k,T}(s)\,\mathrm{d}s + \frac{t}{T}\int_t^{T-t} \alpha_{k,T}(s)\,\mathrm{d}s,
\end{aligned}
$$

i.e. (7.4) holds. To establish relation (7.5), we note that (7.5) with t replaced by $T-t$ can be rewritten as

$$
\alpha_{k+1,T}(t) = \left(1 - \frac{t}{T}\right)\int_0^t \alpha_{k,T}(s)\,\mathrm{d}s + \frac{t}{T}\int_t^T \alpha_{k,T}(s)\,\mathrm{d}s,
$$

which coincides with the left-hand side of equality (7.1), i.e., with the $(k+1)$th member of the sequence defined by formula (7.1). Taking assertion 1 into account and using the fact that the change of variable $t \mapsto T-t$ is invertible, we show that representation (7.5) holds. Finally, (7.6) follows immediately when, in (7.4), we replace t by $T-t$ and take (7.2) into account.

Assertion 3. The equalities $\alpha_{k,T}(0) = \alpha_{k,T}(T) = 0$ $(k = 0, 1, \ldots)$ follow immediately from (7.1) and (7.2). Arguing by induction and using representation (7.4) of sequence (7.1), we show that all the functions $\alpha_0, \alpha_1, \ldots$ take positive values inside the interval $(0, T)$.

Assertion 4. Differentiating (7.4) with respect to t, we obtain

$$
\alpha'_{k,T}(t) = \left(1 - \frac{2}{T}t\right)\alpha_{k-1,T}(t) + \frac{1}{T}\int_t^{T-t} \alpha_{k-1,T}(s)\,\mathrm{d}s, \tag{7.10}
$$

whence it follows by induction that, for every k, the function $\alpha_{k,T}$ is increasing on $(0, T/2)$ and, by (7.2), is decreasing on $(T/2, T)$. Since $\alpha'_{k,T}(T/2) = 0$, we see that $T/2$ is the global maximum point of $\alpha_{k,T}$, and assertion 4 is thus proved.

It is obvious that, in order to prove the second part of equality (7.7), it will suffice to establish the following relation:

$$
\alpha_{m,T}(\sigma T) = T^m \alpha_{m,1}(\sigma) \quad \text{for all } \sigma \in [0,1]. \tag{7.11}
$$

By (7.1), we have

$$
\begin{aligned}
\alpha_{m,T}(\sigma T) &= (1-\sigma)\int_0^{\sigma T} \alpha_{m-1,T}(u)\,\mathrm{d}s + \sigma\int_{\sigma T}^{T} \alpha_{m-1,T}(u)\,\mathrm{d}u \\
&= T(1-\sigma)\int_0^{\sigma} \alpha_{m-1,T}(u)\,\mathrm{d}s + T\sigma\int_{\sigma}^{1} \alpha_{m-1,T}(u)\,\mathrm{d}u
\end{aligned}
$$

for all $\sigma \in [0,1]$. Introducing the mappings $A : C([0,1], \mathbb{R}^n) \to C([0,1], \mathbb{R}^n)$ and $\mathcal{S}_T : C([0,1], \mathbb{R}^n) \to C([0,T], \mathbb{R}^n)$ by the formulae

$$
(Ax)(\sigma) := (1-\sigma)\int_0^{\sigma} x(u)\,\mathrm{d}u + \sigma\int_{\sigma}^{1} x(u)\,\mathrm{d}u, \quad \sigma \in [0,1], \tag{7.12}
$$

and

$$(\mathcal{S}_T x)(\sigma) := x(\sigma T), \quad \sigma \in [0,1],$$

we arrive at the relations

$$\mathcal{S}_T \alpha_{m,T} = A \mathcal{S}_T \alpha_{m-1,T}, \quad m = 1, 2, \ldots,$$

whence it follows that

$$\mathcal{S}_T \alpha_{m,T} = T^m \cdot A^m \mathcal{S}_T \alpha_0 \equiv T^m \cdot A^m \mathcal{S}_T 1 \equiv T^m \cdot A^m 1. \tag{7.13}$$

On the other hand, equality (7.1) with $T = 1$ yields

$$\alpha_{m,1}(\sigma) = (1-\sigma) \int_0^\sigma \alpha_{m-1,1}(u)\,\mathrm{d}u + \sigma \int_\sigma^1 \alpha_{m-1,1}(u)\,\mathrm{d}u, \quad \sigma \in [0,1],$$

or, which is the same, $\alpha_{m,1} = A\alpha_{m-1,1}$, $m = 1, 2, \ldots$, whence

$$\alpha_{m,1} = A^m \alpha_{0,1} \equiv A^m 1, \quad m = 1, 2, \ldots. \tag{7.14}$$

Combining (7.13) and (7.14), we arrive at (7.11). The second part of (7.7) is an immediate consequence of relation (7.11). The lemma is proved. □

REMARK 7.2. Assertion 4 of Lemma 7.1 can also be derived from Corollary 10 of Rontó, Rontó, Samoilenko, and Trofimchuk [64]. Properties 1, 3, and 4 of sequence (7.1), established by Lemma 7.1, are also given by Samoilenko [85] for $T = 1$.

LEMMA 7.3. *The relations*

$$\begin{aligned}\lim_{m\to+\infty} \max_{t\in[0,T]} \frac{\alpha_{m,T}(t)}{\alpha_{m+1,T}(t)} &= \lim_{m\to+\infty} \frac{\max_{t\in[0,T]} \alpha_{m,T}(t)}{\max_{t\in[0,T]} \alpha_{m+1,T}(t)} \\ &= \lim_{m\to+\infty} \frac{\alpha_{m,T}(T/2)}{\alpha_{m+1,T}(T/2)} = \frac{1}{Tr(A)}\end{aligned}$$

hold, where $r(A)$ is the spectral radius of operator (7.12).

PROOF. In view of (7.11), for all $m = 0, 1, 2, \ldots$,

$$\frac{\alpha_{m,T}(T/2)}{\alpha_{m+1,T}(T/2)} = \frac{\alpha_{m,1}(1/2)}{T\alpha_{m+1,1}(1/2)}.$$

By [64, Lemma 10], one has $\lim_{m\to+\infty} \sqrt[m]{\alpha_{m,1}(1/2)} = r(A)$, whereas [64, Corollary 12] implies that $\lim_{m\to+\infty} \sqrt[m]{\alpha_{m,1}(\sigma)/\alpha_{1,1}(\sigma)} = r(A)$ uniformly in $\sigma \in [0,1]$. Since

$\lim_{m\to\infty} c_m = c$ implies $\lim_{m\to\infty} \sqrt[m]{c_m} = \lim_{m\to\infty} c_{m+1}c_m^{-1} = c$, we arrive at the desired equality. □

REMARK 7.4. The value of the spectral radius $r(A)$ can be found numerically (see, e.g., [64,85]). Note that $r(A) \approx 0.292728853011 \approx 1/3.416130626393$.

Lemma 7.5 below extends Lemma 3.16.

LEMMA 7.5. *For the sequence of functions defined by relation* (7.1),[4] *given an arbitrarily small* $\varepsilon \in (0, +\infty)$, *one can specify an* $m_\varepsilon \in \mathbb{N}$ *such that*

$$\alpha_{m+1,T}(t) \leqslant T\big(r(A) + \varepsilon\big)\alpha_{m,T}(t), \quad m \geqslant m_\varepsilon, \tag{7.15}$$

and, consequently,

$$\alpha_{m+1,T}(t) \leqslant T^{m-m_\varepsilon+1}\big(r(A) + \varepsilon\big)^{m-m_\varepsilon+1}\alpha_{m_\varepsilon,T}(t), \quad m \geqslant m_\varepsilon. \tag{7.16}$$

PROOF. Estimate (7.15) is easily proved with the help of Lemma 7.3, whereas (7.16) is obtained immediately from (7.15) by the induction argument. □

The latter lemma will be used to establish error estimates for the successive approximations to be constructed below. However, when obtaining the error estimates on practice, one may want the above logic somewhat reversed. Namely, having the mth iteration (whose error, as will be shown below, is estimated with the help of the function $\alpha_{m-1,T}$), specify a (possibly negative) number ε_m such that the estimate $\alpha_{m+1,T}(t) \leqslant T(r(A) + \varepsilon_m)\alpha_{m,T}(t)$ holds for all $t \in [0, T]$. This aim can obviously be achieved by putting $\varepsilon_m := \alpha_{m+1,1}(1/2)/\alpha_{m,1}(1/2) - r(A)$, a few values of ε_m being shown in Table 1.

REMARK 7.6. Lemma 7.5 improves the result of Lemma 4 from Rontó and Mészáros [69] because the former, together with the table below, shows, in particular, that

$$\alpha_{m+1}(t) \leqslant T\big(r(A) + \frac{1}{1000}\big)\alpha_{m,T}(t)$$

for all $t \in [0, T]$ and $m \geqslant 3$, whereas $r(A) + \frac{1}{1000} \approx 0.29372885 < \frac{3}{10}$.

We also need a generalisation of Lemma 3.13.

LEMMA 7.7. *For every continuous* $x : [0, T] \to \mathbb{R}$, *the inequality*

$$\begin{aligned}&\left|\int_0^{\theta(t)} \left(x(\tau) - \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s\right)\mathrm{d}\tau\right| \\ &\quad\leqslant \frac{1}{2}\alpha_{1,T}(t)\Big[\max_{\tau\in[0,T]} x(\tau) - \min_{\tau\in[0,T]} x(\tau)\Big]\end{aligned} \tag{7.17}$$

[4] Or, which is the same, by (7.4), (7.5), or (7.6)

Table 1

m	approximate ε_m	$r(A)+\varepsilon_m$
1	$0.406044803222 \times 10^{-1}$	$\frac{1}{3}$
2	$0.727114698891 \times 10^{-2}$	$\frac{3}{10}$
3	$0.921940639708 \times 10^{-3}$	$\frac{37}{126} \approx 0.293650793650$
4	$0.639397817078 \times 10^{-4}$	$\frac{65}{222} \approx 0.292792792792$
5	$-0.158028381223 \times 10^{-5}$	$\frac{161}{550} \approx 0.292727272727$
6	$-0.956604258782 \times 10^{-6}$	$\frac{6827}{23322} \approx 0.292727896407$
7	$-0.101923715775 \times 10^{-6}$	$\frac{195849}{669046} \approx 0.292728751087$
8	$-0.981324932500 \times 10^{-9}$	$\frac{3248737}{11098110} \approx 0.292728852030$
9	$0.114183386672 \times 10^{-8}$	$\frac{975725035}{3333204162} \approx 0.292728854153$
10	$0.153987313839 \times 10^{-9}$	$\frac{571245741}{1951450070} \approx 0.292728853165$
11	$0.488928250983 \times 10^{-11}$	$\frac{2961468157897}{10116762073110} \approx .292728853016$
12	$-0.130772159250 \times 10^{-11}$	$\frac{5201443062517}{17768808947382} \approx 0.292728853010$
...	...	...
$+\infty$	0	$r(A) \approx 0.292728853011$

holds for all $t \in [0, T]$, *where* $\theta : [0, T] \to [0, T]$ *is defined either with* $\theta(t) := t$ *or* $\theta(t) := T - t$.

PROOF. When $\theta(t) \equiv t$, the statement constitutes inequality (4) of Lemma 3 from [69]. When $\theta(t) \equiv T - t$, by the above-said, we have

$$\left| \int_0^{T-t} \left(x(\tau) - \frac{1}{T} \int_0^T x(s)\,\mathrm{d}s \right) \mathrm{d}\tau \right| \\ \leqslant \frac{1}{2}\alpha_{1,T}(T-t)\Big[\max_{\tau\in[0,T]} x(\tau) - \min_{\tau\in[0,T]} x(\tau) \Big].$$

Since, in view of Lemma 7.1 (assertion 1), we have $\alpha_{m,T}(t) = \alpha_{m,T}(T-t)$ for all $m \geqslant 0$ and $t \in [0, T]$, the latter inequality yields (7.17). $\square$

7.2. *Successive approximations for a problem with separated boundary conditions*

We consider the two-point boundary value problem

$$x'(t) = f\big(t, x(t)\big), \quad t \in [0, T], \tag{7.18}$$

$$x(0) = b\big(x(T)\big), \tag{7.19}$$

where $f:[0,T]\times\Omega\to\mathbb{R}^n$ and $b:\Omega\to\mathbb{R}^n$ are continuous and Ω is the closure of a bounded domain. We suppose that, for every $t\in[0,T]$ fixed, f satisfies the componentwise Lipschitz condition

$$\bigl|f(t,y_1)-f(t,y_2)\bigr|\leqslant K|y_1-y_2| \text{ for all } \{y_1,y_2\}\subset\Omega, \tag{7.20}$$

where K is a square matrix of dimension n having non-negative components. We recall that, in (7.20) and similar relations below, the signs $|\cdot|$ and $\leqslant$ are understood componentwise.

Assume that

$$\Omega(\beta):=\bigl\{x\in\mathbb{R}^n \mid H_b(x,\beta)\subset\Omega\bigr\}\neq\varnothing, \tag{7.21}$$

where

$$H_b(x,\beta):=\bigl\{u\in\mathbb{R}^n \mid \max\bigl(|u-x|,|u-b(x)|\bigr)\leqslant\beta\bigr\} \tag{7.22}$$

with $\beta:=\frac{T}{4}\delta_\Omega(f)$,

$$\delta_\Omega(f):=\max_{(t,x)\in[0,T]\times\Omega} f(t,x)-\min_{(t,x)\in[0,T]\times\Omega} f(t,x), \tag{7.23}$$

and

$$r(K)<\frac{1}{Tr(A)}. \tag{7.24}$$

Recall that A is the linear operator on the space $C([0,T],\mathbb{R}^n)$ defined by formula (7.12), and $1/r(A)\approx 3.4161306$.

In connection with the study of problem (7.18), (7.19), we introduce the sequence of functions

$$\begin{aligned} x_{m+1}(t,z) :={}& \frac{t}{T}z+\left(1-\frac{t}{T}\right)b(z)+\int_0^t f\bigl(s,x_m(s,z)\bigr)\,\mathrm{d}s \\ &-\frac{t}{T}\int_0^T f\bigl(s,x_m(s,z)\bigr)\,\mathrm{d}s, \quad m=0,1,2,\ldots, \end{aligned} \tag{7.25}$$

or, which is the same,

$$\begin{aligned} x_{m+1}(t,z) :={}& \frac{t}{T}z+\left(1-\frac{t}{T}\right)b(z)-\int_t^T f\bigl(s,x_m(s,z)\bigr)\,\mathrm{d}s \\ &+\left(1-\frac{t}{T}\right)\int_0^T f\bigl(s,x_m(s,z)\bigr)\,\mathrm{d}s, \quad m=0,1,2,\ldots, \end{aligned}$$

where

$$x_0(t, z) := \frac{t}{T} z + \left(1 - \frac{t}{T}\right) b(z) \quad \text{for all } t \in [0, T], \tag{7.26}$$

and $z = \operatorname{col}(z_1, z_2, \ldots, z_n) \in \Omega(\beta)$ is a parameter.

It can be easily verified that, for every $m \in \mathbb{N} \cup \{0\}$,

$$x_m(T, z) = z \quad \text{and} \quad x_m(0, z) = b(z), \tag{7.27}$$

i.e., functions (7.25) satisfy the boundary condition (7.19) for all $z \in \Omega(\beta)$.

THEOREM 7.8. *Assume that conditions* (7.20), (7.21), *and* (7.24) *hold for the boundary value problem* (7.18), (7.19). *Then*:

(1) *Sequence* (7.25) *has a limit $x^*(\cdot, z)$ uniform in $(t, z) \in [0, T] \times \Omega(\beta)$.*

(2) *The limit function $x^*(\cdot, z)$ is a solution of the "perturbed" problem*

$$\begin{aligned} &x'(t) = f\big(t, x, (t)\big) + \frac{1}{T} \Delta(z), \quad t \in [0, T], \\ &x(0) = b\big(x(T)\big) \end{aligned} \tag{7.28}$$

such that $x^*(T, z) = z$, where

$$\Delta(z) := z - b(z) - \int_0^T f\big(s, x^*(s, z)\big)\, \mathrm{d}s. \tag{7.29}$$

(3) *Given an arbitrarily small positive ε, one can choose a number $m_\varepsilon \in \mathbb{N}$ such that, for all $m \geqslant m_\varepsilon$, the following error estimate holds*:

$$\big|x_m(t, z) - x^*(t, z)\big| \leqslant \frac{1}{2} \alpha_{m_\varepsilon, T}(t) Q_\varepsilon^{m - m_\varepsilon} (\mathbb{1}_n - Q_\varepsilon)^{-1} Q_\varepsilon \delta_\Omega(f), \tag{7.30}$$

where $\delta_\Omega(f)$ is given by the component-wise relation (7.23) *and*

$$Q_\varepsilon := T\big(r(A) + \varepsilon\big) \cdot K. \tag{7.31}$$

REMARK 7.9. The value of $\alpha_{m_\varepsilon, T}(t)$ in (7.30) can be estimated from above by using formula (7.7).

PROOF OF THEOREM 7.8. We shall prove that, under the conditions assumed, (7.25) is a Cauchy sequence in the Banach space $C([0, T], \mathbb{R}^n)$ equipped with the usual uniform norm. First, we show that $x_m(t, z) \in \Omega$ for all $(t, z) \in [0, T] \times \Omega(\beta)$ and $m \in \mathbb{N}$.

Indeed, in view of Lemma 7.7, we have

$$x_1(t, z) - \frac{t}{T} z - \left(1 - \frac{t}{T}\right) b(z) = \int_0^t \left[f(s, z) - \int_0^T f(\tau, z)\, \mathrm{d}\tau \right] \mathrm{d}s$$

and, hence,

$$\left|x_1(t,z) - tT^{-1}z - \left(1 - tT^{-1}\right)b(z)\right| \leqslant \frac{1}{2}\alpha_{1,T}(t)\delta_\Omega(f)$$
$$\leqslant \frac{T}{4}\delta_\Omega(f) \equiv \beta. \tag{7.32}$$

It is easy to see from (7.21) and (7.22) that $z \in \Omega(\beta)$ if, and only if, for every $u \in \mathbb{R}^n$, the relation $\max_{t\in[0,T]} |tT^{-1}z + (1 - tT^{-1})b(z) - u| \leqslant \beta$ implies $u \in \Omega$. Therefore, by virtue of (7.32), we conclude that $x_1(t,z) \in \Omega$ whenever $(t,z) \in [0,T] \in \Omega(\beta)$. The fact that $x_m(t,z) \in \Omega$ for all $(t,z) \in [0,T] \in \Omega(\beta)$ and $m \in \mathbb{N}$ is proved quite similarly.

Now, consider the difference

$$x_{m+1}(t,z) - x_m(t,z) = \int_0^t \left[f\left(s, x_m(s,z)\right) - f\left(s, x_{m-1}(s,z)\right)\right] \mathrm{d}s$$
$$- \frac{t}{T}\int_0^T \left[f\left(s, x_m(s,z)\right) - f\left(s, x_{m-1}(s,z)\right)\right] \mathrm{d}s \tag{7.33}$$

and introduce the notation

$$d_m(t,z) := \left|x_m(t,z) - x_{m-1}(t,z)\right|, \quad m = 1, 2, \ldots. \tag{7.34}$$

By virtue of identity (7.33) and the Lipschitz condition (7.20), we have

$$d_{m+1}(t,z) \leqslant K\left[\left(1 - \frac{t}{T}\right)\int_0^t d_m(s,z)\,\mathrm{d}s + \frac{t}{T}\int_t^T d_m(s,z)\,\mathrm{d}s\right] \tag{7.35}$$

for every $m \geqslant 0$. According to (7.32),

$$d_1(t,z) \equiv \left|x_1(t,z) - tT^{-1}z - \left(1 - tT^{-1}\right)b(z)\right| \leqslant \frac{1}{2}\alpha_{1,T}(t)\delta_\Omega(f). \tag{7.36}$$

Iterating (7.35) and taking into account (7.1) and (7.36), we obtain that, for all $m \geqslant 0$, $t \in [0,T]$, and $z \in \Omega_\beta$,

$$d_{m+1}(t,z) \leqslant \frac{1}{2}K^m \alpha_{m+1,T}(t)\delta_\Omega(f). \tag{7.37}$$

By Lemma 7.5, for an arbitrary $\varepsilon \in (0, +\infty)$, estimate (7.16) holds for the function α_{m+1} [see (7.1)] whenever m is not less than some m_ε. Fixing the number ε and combining (7.37) and (7.16), we obtain

$$d_{m+1}(t,z) \leqslant \frac{1}{2}\alpha_{m_\varepsilon,T}(t)\, Q_\varepsilon^{m-m_\varepsilon+1}\delta_\Omega(f) \tag{7.38}$$

for all $t \in [0, T]$, $z \in \Omega(\beta)$, and $m \geqslant m_\varepsilon$. Therefore,

$$
\begin{aligned}
\left|x_{m+j}(t,z) - x_m(t,z)\right| &\leqslant \sum_{i=0}^{j-1} \left|x_{m+j-i}(t,z) - x_{m+j-i-1}(t,z)\right| \\
&= \sum_{i=1}^{j} d_{m+i}(t,z) \quad \text{by (7.38)} \\
&\leqslant \frac{1}{2}\alpha_{m_\varepsilon,T}(t) \sum_{i=0}^{j-1} Q_\varepsilon^{m-m_\varepsilon+i+1}\delta_\Omega(f) \\
&= \frac{1}{2}\alpha_{m_\varepsilon,T}(t) Q_\varepsilon^{m-m_\varepsilon} Q_\varepsilon \sum_{i=0}^{j-1} Q_\varepsilon^i \delta_\Omega(f).
\end{aligned}
\tag{7.39}
$$

Since ε in (7.31) can be chosen arbitrarily small, relation (7.39) together with assumption (7.24) and definition (7.31) imply $\sum_{k=0}^{+\infty} Q_\varepsilon^k = (\mathbb{1}_n - Q_\varepsilon)^{-1}$ and $\lim_{m\to\infty} Q_\varepsilon^m = 0$. This means that (7.25) is a Cauchy sequence and, hence, assertion 7.8 holds.

Passing to the limit as $m \to +\infty$ in equality (7.25), we show that the function $x^*(\cdot, z) := \lim_{m\to+\infty} x_m(\cdot, z)$ satisfies the following integral equation:

$$
\begin{aligned}
x(t) = \frac{t}{T}z + \left(1 - \frac{t}{T}\right)b(z) + \int_0^t f\big(\tau, x(\tau)\big)\,\mathrm{d}\tau \\
- \frac{t}{T}\int_0^T f\big(\tau, x(\tau)\big)\,\mathrm{d}\tau.
\end{aligned}
\tag{7.40}
$$

It follows from (7.40) and (7.27) that $x^*(\cdot, z)$ satisfies the two-point boundary condition (7.19). It is also obvious from (7.40) that $x^*(T, z) = z$, which means that $x^*(\cdot, z)$ is a solution of problem (7.28), i.e., assertion 7.8 holds. Estimate (7.30) of assertion 7.8 is an immediate consequence of (7.39). □

REMARK 7.10. In a number of earlier works in this area (see, e.g., [70] for a survey), conditions of the form $r(K) < cT^{-1}$ with $2 < c < 1/r(A)$ were assumed instead of (7.24).

REMARK 7.11. Depending on the form of a particular boundary value problem, one may want to choose a different starting approximation, x_0, for the iteration process (7.25). For instance, if a "slowly varying" solution is expected, one may put $x_0(t, z) := z$, which was often done in the earlier works on the numerical-analytic method. In the latter case, on the assumption that

$$
z \in \left\{u \in \mathbb{R}^n \;\middle|\; B\left(u, \frac{T}{4}\delta_\Omega(f)\right) \subset \Omega\right\} \neq \varnothing,
$$

the convergence of sequence (7.25) as well as estimate (7.30) of Theorem 7.8 can easily be proved, e.g., by using Lemma 4 from [70, p. 384]. For this purpose, it suffices to replace (7.38) by the inequality

$$d_{m+1}(t,z) \leqslant \alpha_{m_\varepsilon,T}(t)\left[\frac{1}{2} Q_\varepsilon^{m-m_\varepsilon+1}\delta_\Omega(f) + K\, Q_\varepsilon^{m-m_\varepsilon}\bigl|b(z)+z\bigr|\right],$$

put $a_m := T\alpha_{m,1}(\frac{1}{2})K^{m-1}\delta_\Omega(f)$, $b_m := T\alpha_{m-1,1}(\frac{1}{2})K^{m-1}|b(z)+z|$, $u_m := \mathrm{col}(a_m, b_m)$ and $\phi_1(t) := \phi_2(t) := 1$, and notice that, for an arbitrary fixed positive ε and all m greater than some $m_\varepsilon \in \mathbb{N}$, estimate (7.15) of Lemma 7.5 implies the component-wise relation $u_{m+1} \leqslant T[r(A)+\varepsilon]\,\mathrm{diag}\,(K,K)u_m$. Then, similarly to [70, p. 385], assumption (7.24) implies that $\lim_{m\to+\infty} d_m(t,z) = 0$ uniformly in $(t,z) \in [0,T] \times \Omega(\beta)$, which, by (7.34), means that (7.25) is a Cauchy sequence.

Now we show that, in view of Theorem 7.8, the boundary value problem (7.18), (7.19) can be formally interpreted as a family of initial value problems for differential equations with “additively forced” right-hand side members. Namely, consider the Cauchy problem

$$x(T) = z \tag{7.41}$$

for the system

$$x'(t) = f\bigl(t, x(t)\bigr) + \mu, \quad t \in [0,T], \tag{7.42}$$

where $\mu \in \mathbb{R}^n$ and $z \in \Omega(\beta)$ are parameters.

THEOREM 7.12. *Under the conditions of Theorem* 7.8, *the solution of the initial value problem* (7.42) *satisfies the two-point condition* (7.19) *if, and only if* $\mu = \frac{1}{T}\Delta(z)$, *where* $\Delta : \Omega(\beta) \to \mathbb{R}^n$ *is the mapping defined by* (7.29).

REMARK 7.13. It is easy to show that the Lipschitz condition (7.20) implies that (7.42) has a unique solution for all $(\mu, z) \in \mathbb{R}^n \times \Omega(\beta)$.

PROOF OF THEOREM 7.12. It follows from the proof of Theorem 7.8 that, for every fixed $z \in \Omega(\beta)$, the function

$$x = x^*(\cdot, z) = \lim_{m\to+\infty} x_m(\cdot, z) \tag{7.43}$$

[see (7.25)] satisfies the integral equation (7.40). This implies immediately that (7.43) is the unique (see Remark 7.13) solution of the initial value problem (7.41) for the system

$$x'(t) = f\bigl(t, x(t)\bigr) + \frac{1}{T}\Delta(z), \quad t \in [0,T], \tag{7.44}$$

and, in addition, (7.43) satisfies the two-point condition (7.19). Since (7.44) coincides with problem (7.42) corresponding to $\mu = \frac{1}{T}\Delta(z)$. The fact that function (7.43) is not solution of (7.42) for any other value of μ, is obvious, e.g., from Eq. (7.40). □

COROLLARY 7.14. *Under the conditions of Theorem* 7.8, *the limit function* (7.43) *of sequence* (7.25) *is a solution of the two-point boundary value problem* (7.18), (7.19) *if, and only if the parameter* $z \in \Omega(\beta)$ *involved in* (7.25) *and* (7.43) *satisfies the following "determining equation"*:

$$z - b(z) = \int_0^T f\big(s, x^*(s, z)\big)\, \mathrm{d}s. \tag{7.45}$$

In Corollary 7.14, the parameter z can be interpreted as the value at $t = T$ of a possible solution of problem (7.18), (7.19).

PROOF. It suffices to apply Theorem 7.12 and notice that the differential equation in (7.44) coincides with (7.18) if, and only if z satisfies $\Delta(z) = 0$, i.e., when relation (7.45) holds. □

REMARK 7.15. On practise, it is natural to fix some natural m and consider the "approximate determining equation"

$$z - b(z) = \int_0^T f\big(s, x_m(s, z)\big)\, \mathrm{d}s. \tag{7.46}$$

In case Eq. (7.46) has an isolated root, say $z = z_m$, in some open subdomain of $\Omega(\beta)$, one can prove that, under certain additional conditions, the exact determining equation (7.45) is also solvable and, hence, the two-point problem (7.18), (7.19) has a solution x satisfying $x(T) \in \Omega(\beta)$; in addition, the function

$$\xi_m(t) := x_m(t, z_m), \quad t \in [0, T], \tag{7.47}$$

can be regarded as the "mth approximation" to the solution mentioned [see estimate (7.30)]. For that purpose, one can use some topological degree techniques (cf. Theorem 3.1 in [70, p. 43]) or the methods for the approximate solution of non-linear equations in Banach spaces extensively developed in the book of Krasnoselskii et al. [37] (see, e.g., Theorem 19.2 in [37, p. 281]). Here, we do not consider this problem in more detail.

We apply the above techniques to the following

EXAMPLE 7.16. Consider the two-point problem

$$\begin{aligned} &x''(t) - \frac{t}{8}x'(t) + \frac{1}{2}\big(x'(t)\big)^2 + \frac{1}{2}x(t) = \frac{9}{32} + \frac{t^2}{16}, \quad t \in [0, 1], \\ &x(0) = \big(x'(1)\big)^2, \\ &x'(0) = x'(1) - x(1) - \frac{1}{16}. \end{aligned} \tag{7.48}$$

By setting $x_1 := x$ and $x_2 := x'$, system (7.48) is rewritten as

$$
\begin{aligned}
&x_1'(t) = x_2(t),\\
&x_2'(t) = \frac{9}{32} + \frac{t^2}{16} + \frac{t}{8}x_2(t) - \frac{1}{2}\bigl(x_2(t)\bigr)^2 - \frac{1}{2}x_1(t),\\
&x_1(0) = \bigl(x_2(1)\bigr)^2,\\
&x_2(0) = x_2(1) - x_1(1) - \frac{1}{16}.
\end{aligned}
\tag{7.49}
$$

Let us consider the first order problem (7.49) in the domain $(t, x_1, x_2) \in [0,1] \times \Omega$ with $\Omega := \{(x_1, x_2) \mid |x_1| \leqslant 1, |x_2| \leqslant \frac{3}{4}\}$. It is not difficult to verify that the corresponding quantity (7.23) is estimated as $\delta_\Omega(f) \leqslant \operatorname{col}(\frac{3}{2}, \frac{5}{4})$, and (7.20) holds with $\beta = \frac{1}{4}\delta_\Omega(f) \leqslant \operatorname{col}(\frac{3}{8}, \frac{5}{16})$, because $x \in \Omega(\beta)$ whenever $|x|$ is small enough. Condition (7.21) is satisfied with $K := \left(\begin{smallmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{8} \end{smallmatrix}\right)$ and, by Remark 7.4, the spectral radius of the matrix K satisfies inequality (7.24): $r(K) = \frac{7+\sqrt{177}}{16} \approx 1.269 < 1/r(A)$.

Let us see how the "zero approximation" (in the sense of Remark 7.15), behaves. According to (7.26), we have

$$
\begin{aligned}
&x_{0,1}(t,z) = \bigl(z_1 - {z_2}^2\bigr)t + {z_2}^2,\\
&x_{0,2}(t,z) = \left(z_1 + \frac{1}{16}\right)t + z_2 - z_1 - \frac{1}{16},
\end{aligned}
\tag{7.50}
$$

where $x_k := \operatorname{col}(x_{k,1},\ x_{k,2})$ for $k = 0, 1, \ldots$. The corresponding system of approximate determining equations is written as

$$
\begin{aligned}
&\frac{3}{2}z_1 - {z_2}^2 + \frac{1}{32} - z_2 = 0,\\
&-\frac{3}{32}z_2 + \frac{31}{24}z_1 - \frac{365}{1536} + \frac{1}{6}{z_1}^2 - \frac{1}{2}z_2 z_1 + \frac{3}{4}{z_2}^2 = 0.
\end{aligned}
$$

It has a positive solution

$$
z_1 \approx 0.1801357504, \qquad z_2 \approx 0.2425992362, \tag{7.51}
$$

which, substituted into (7.50) according to (7.47), yields the zeroth approximation to the solution of problem (7.48):

$$
\xi_0(t) \approx 0.1212813610 \cdot t + 0.05885438940, \quad t \in [0,1]. \tag{7.52}
$$

Let us construct the first and the second approximations. According to formula (7.25), we have

$$
x_{1,1}(t,z) = \frac{1}{2}\left(z_1 + \frac{1}{16}\right)t^2 + \left(\frac{1}{2}z_1 - {z_2}^2 - \frac{1}{32}\right)t + {z_2}^2, \tag{7.53}
$$

and

$$
\begin{aligned}
x_{1,2}(t,z) = {}& \left(\frac{3}{128} - \frac{1}{6}\left(z_1 + \frac{1}{16}\right)^2 + \frac{1}{24}z_1\right)t^3 \\
&+ \left(\frac{1}{16}z_2 - \frac{5}{16}z_1 - \frac{1}{256} - \frac{1}{2}\left(z_2 - z_1 - \frac{1}{16}\right)\left(z_1 + \frac{1}{16}\right)\right. \\
&\left. + \frac{1}{4}{z_2}^2\right)t^2 + \left[-\frac{1}{16}z_2 + \frac{61}{48}z_1 + \frac{11}{256} - \frac{1}{4}{z_2}^2 + \frac{1}{6}\left(z_1 + \frac{1}{16}\right)^2\right. \\
&\left. + \frac{1}{2}\left(z_2 - z_1 - \frac{1}{16}\right)\left(z_1 + \frac{1}{16}\right)\right]t + z_2 - z_1 - \frac{1}{16}.
\end{aligned}
$$

The corresponding two approximate determining equations (7.46) are

$$
\frac{281}{192}z_1 - \frac{23}{24}{z_2}^2 + \frac{75}{2048} + \frac{1}{24}{z_1}^2 - \frac{191}{192}z_2 - \frac{1}{12}z_2z_1 = 0 \tag{7.54}
$$

and

$$
\begin{aligned}
&\frac{5524531}{4423680}z_1 - \frac{32309}{327680}z_2 + \frac{18407}{24576}{z_2}^2 + \frac{98797}{645120}{z_1}^2 - \frac{28847}{61440}z_2z_1 \\
&\quad + \frac{1}{945}{z_1}^4 - \frac{1}{240}{z_2}^3z_1 + \frac{1}{960}{z_2}^4 - \frac{53}{1280}{z_2}^3 + \frac{2467}{120960}{z_1}^3 + \frac{1}{160}{z_2}^2{z_1}^2 \\
&\quad - \frac{1}{240}{z_1}^3z_2 - \frac{61}{768}z_2{z_1}^2 + \frac{391}{3840}{z_2}^2z_1 = \frac{16965793}{70778880},
\end{aligned} \tag{7.55}
$$

respectively. Computing a non-negative solution of (7.54), (7.55), we obtain

$$
z_1 \approx 0.1875156187, \qquad z_2 \approx 0.2500152909, \tag{7.56}
$$

and, inserting (7.56) into (7.53), we arrive at the first approximation:

$$
\xi_1(t) \approx 0.06250764568 + 0.163710^{-6}t + 0.1250078094t^2, \quad t \in [0,1]. \tag{7.57}
$$

In a similar manner, one can establish that the second approximate system of determining equations has a positive solution

$$
z_1 \approx 0.1874363008, \qquad z_2 \approx 0.2499449264 \tag{7.58}
$$

and, by (7.47) and (7.58), the corresponding function of the second approximation has the form

$$
\begin{aligned}
\xi_2(t) \approx {}& 0.06247246623 - 0.2235410^{-5}t + 0.005208996698t^4 \\
&- 0.01041383249t^3 + 0.1301709057t^2, \quad t \in [0,1].
\end{aligned} \tag{7.59}
$$

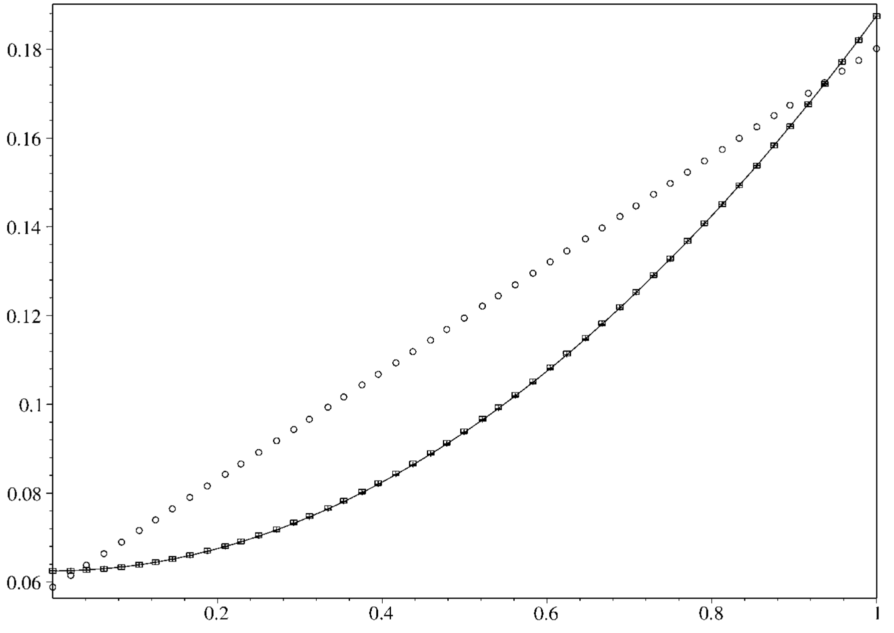

Fig. 6. The exact solution of problem (7.48) and functions (7.52), (7.57), and (7.59).

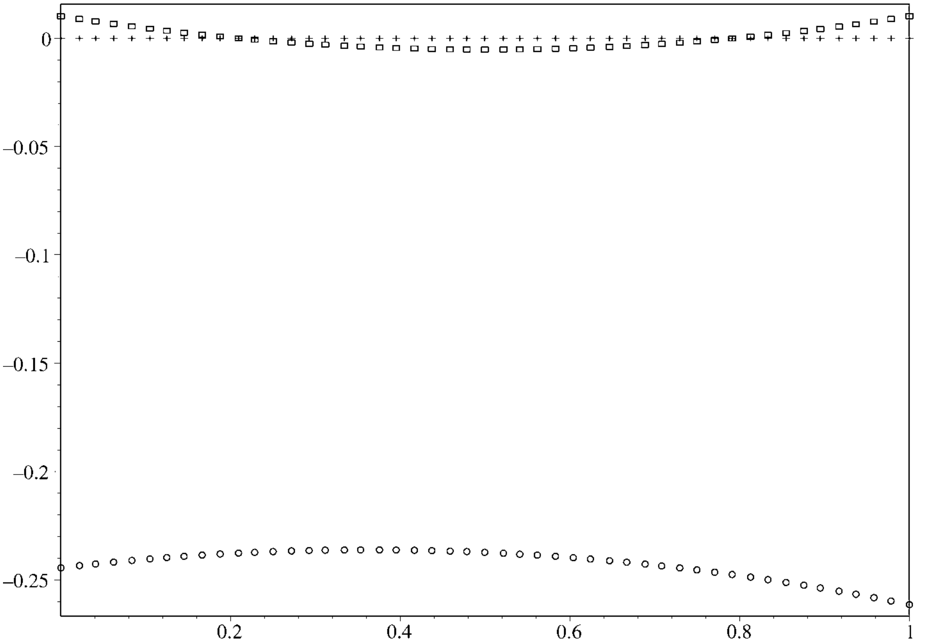

Fig. 7. Functions (7.61) for $k = 0, 1, 2$.

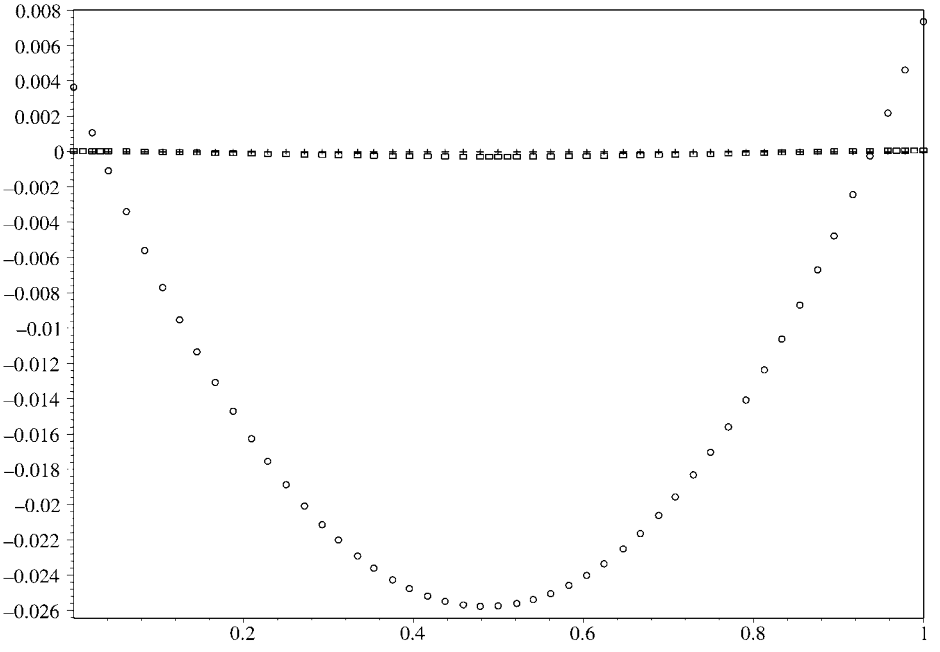

Fig. 8. The difference between the exact solution (7.60) of problem (7.48) and functions (7.52), (7.57), and (7.59).

Note that the two-point problem (7.48) has a solution

$$x(t) = \frac{t^2}{8} + \frac{1}{16}, \quad t \in [0, 1], \tag{7.60}$$

which, as is seen from Fig. 6, is well enough approximated by functions (7.52), (7.57), and (7.59) constructed above according to formula (7.47).

The functions

$$t \mapsto \xi_k''(t) - \frac{t}{8}\xi_k'(t) + \frac{1}{2}\left(\xi_k'(t)\right)^2 + \frac{1}{2}x\xi_k(t) - \frac{t^2}{16} - \frac{9}{32}, \quad k = 0, 1, 2, \tag{7.61}$$

obtained as a result of substitution of functions (7.52), (7.57), and (7.59) into the given differential equation, are plotted on Fig. 7, and the error of the approximate solutions with respect to function (7.60) is illustrated by Fig. 8. On all the figures, the symbols “circle”, “cross”, and “box” indicate the curves related to the zero, first, and second approximations, respectively. The graph of the exact solution (7.60) of problem (7.48) is drawn with the solid line [which, in fact, almost coincides with the graphs of functions (7.57) and (7.59)].

The computation shows that, e.g., function (7.59) satisfies the differential equation in (7.48) “up to 0.01” and differs from the exact solution (7.60) at most by 0.001.

8. Parametrisation method for three-point boundary value problems

Here, a three-point boundary-value problem for a system of non-linear differential equations is transformed to certain family of two-point problems the solutions of which are investigated by using successive approximation techniques.

Numerous approaches to investigation of boundary value problems with unknown parameters are available. Various iteration methods are used for such problems in the papers of Goma [26,25,27], Hosabekov [30], Kurpel′ and Marusjak [41]. In the works of Akhmedov, Svarichevskaya, and Yagubov [3], Luchka [45], Luchka and Zakharĭĭchenko [46], Samoilenko, Luchka, and Listopadova [93,94], the method of averaging of functional corrections and other projection-iterative methods are developed. Estimates of the number of solutions of certain singular two-point problems with parameters were obtained by Fečkan [19,20] by using the Nielsen fixed point theory. Various versions of the continuation method are studied, e.g., in the book of Gaines and Mawhin [24]. The are many works where different kinds of two-point and multipoint boundary value problems are studied by using numerical methods (see, e.g., the books of Ascher, Mattheij, and Russell [6], Keller [35] and the bibliography given therein for a survey).

It is important to note that the majority of works dealing with the numerical finding of solutions of boundary value problems are based upon the shooting method (see, e.g., [35, 6,10,1]). In particular, in the book of Keller [35], the two-point problem

$$y'(x) = f(x, y(x), \lambda_1, \lambda_2, \ldots, \lambda_m), \quad x \in [a, b], \tag{8.1}$$

$$Ay(a) + By(b) = \gamma, \tag{8.2}$$

is considered, where $y : [a, b] \to \mathbb{R}^n$, A and B are given matrices of dimension $(m+n) \times n$, and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are unknown parameters. Recall that the idea of the shooting method for the two-point problem (8.1), (8.2) is to replace it by the Cauchy problem for the same equation subjected to the initial conditions

$$y(a) = s. \tag{8.3}$$

In this case, the solution $y(\cdot, s, \lambda)$ of problem (8.1), (8.3) contains the unknown parameters $\lambda = \mathrm{col}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $s = \mathrm{col}(s_1, s_2, \ldots, s_n)$ whose values are to be determined from the system of $n + m$ equations

$$Ay(a, s, \lambda) + By(b, s, \lambda) = \gamma. \tag{8.4}$$

It is then stated that the two-point problem (8.1), (8.2) has the same number of solutions as the system of numerical equations (8.4).

The main complication in this approach obviously lies in the numerical construction and investigation of system (8.4), which is possible only if the initial value problems (8.1), (8.3) for arbitrary values of λ and s have a unique solution *well-defined on the entire interval* $[a, b]$. In this relation, it is usually assumed that the function f involved in the right-hand side of (8.1) is continuous on the entire space and satisfies the Lipschitz condition with respect to the second argument on the entire $\mathbb{R}^n$.

The rigorous investigation of boundary value problems by using numerical methods tends to avoid the case where the range of solutions in question is restricted to a certain given (possibly, bounded) closed $D \subset \mathbb{R}^n$ and, correspondingly, one or another regularity assumption is imposed on the function f on the set $[a, b] \times D \times \mathbb{R}^m$ only. For example, in the case of the scalar two-point boundary value problem (8.2) for the equation

$$x'(t) = \bigl(x(t)\bigr)^2, \quad t \in [a, b], \tag{8.5}$$

the global Lipschitz condition for the right-hand side of (8.5) is obviously not satisfied. The local solution of the corresponding Cauchy problem (8.5), (8.3) has the form

$$x(t, s) = -\frac{1}{t - s^{-1} - a}, \quad t \in [a, b], \tag{8.6}$$

where s is a real parameter. It is clear from (8.6) that if $0 \leqslant s^{-1} \leqslant b - a$ then the solution of the initial value problem (8.5), (8.3) is undefined on the entire interval $[a, b]$ and, therefore, it does not make sense even to construct Eq. (8.4), not talking of its solution.

In many cases, the complications indicated can be overcome by using the approach of [89,70,60,66].

8.1. *Problem setting*

Let us consider the problem on finding a solution of the differential system (2.1) satisfying the three-point non-separated linear boundary conditions

$$Ax(0) + Bx(\xi) + x(T) = d, \tag{8.7}$$

where $f : [0, T] \times D \to \mathbb{R}^n$, D is the closure of a bounded domain in $\mathbb{R}^n$, $\{A, B\} \subset \mathrm{GL}_n(\mathbb{R})$, and $\xi \in (0, T)$ is given. It is easy to see that the tree-point boundary condition (2.5) with a non-singular matrix C can be brought to form (8.7) (it suffices to replace A, B, and d in (8.7) by $C^{-1}A$, $C^{-1}B$, and $C^{-1}d$, respectively). Conditions (8.7) are non-separated unless the matrices A and B are both equal to zero.

Here we show an approach to the investigation of problem (2.1), (8.7), which is based upon artificial introduction of a suitable number of parameters into the boundary conditions and allows one to construct approximate solutions in an analytic form.

We assume that the function $f : [0, T] \times D \to \mathbb{R}^n$ is continuous and, in the second variable, satisfies the Lipschitz condition (3.5) with a certain n-dimensional square matrix K with non-negative elements.

We restrict our consideration by the class of differential systems (2.1) for which the spectral radius $r(K)$ of the non-negative matrix K appearing in the Lipschitz condition (3.5) satisfies inequality (6.14).

8.2. *Transformation to a two-point problem with a parameter in boundary conditions*

Let us replace formally the value of the unknown solution of problem (2.1), (8.7) at the point ξ by the n-dimensional vector of parameters $(\lambda_k)_{k=1}^n$ that varies in a certain set $\Lambda \subset D$:

$$x(\xi) = \lambda. \tag{8.8}$$

Every function x satisfying relations (8.8) and (8.7) obviously has the property

$$Ax(0) + x(T) = d - B\lambda. \tag{8.9}$$

Let us consider problem (2.1), (8.9), in which the two-point boundary condition (8.9) contains an unknown parameter $\lambda \in \Lambda$.

REMARK 8.1. The set of solutions of the original three-point problem (2.1), (8.7), clearly, coincides with the set of those solutions of the two-point problem (2.1), (8.9) for which the parameter λ satisfies the additional condition (8.8).

For any $\{z, \lambda\} \subset \mathbb{R}^n$, we put

$$\beta(z, \lambda) := \frac{T}{4}\delta_D(f) + \left|d - B\lambda - (A + \mathbb{1}_n)z\right|, \tag{8.10}$$

where $\mathbb{1}_n$ is the unit matrix of dimension n and $\delta_D(f)$ is defined by equality (6.13). We shall assume in the sequel that the set $\Gamma \subseteq D$ given by the formula

$$\Gamma := \left\{z \in D\colon\ B\big(z, \beta(z, \lambda)\big) \subset D \text{ for all } \lambda \in \Lambda\right\} \tag{8.11}$$

is non-empty: $\Gamma \neq \varnothing$. Recall that, here and above, notation (12) of Section 1 is used.

With the parametrised problem (2.1), (8.9), we associate the sequence of functions

$$\begin{aligned} x_{m+1}(t, z, \lambda) := z + \int_0^t f\big(s, x_m(s, z, \lambda)\big)\,\mathrm{d}s - tT^{-1}\int_0^T f\big(s, x_m(s, z, \lambda)\big)\,\mathrm{d}s \\ + tT^{-1}[d - B\lambda - Az - z], \end{aligned} \tag{8.12}$$

where $\lambda \in \Lambda$, $z \in \Gamma$, $m = 0, 1, 2, \ldots$, $x_0(t, z, \lambda) \equiv z$. It is easy to see that $x_m(0, z, \lambda) = z$ for any $m \geqslant 0$, $\lambda \in \Lambda$, and $z \in \Gamma$ and, moreover,

$$x_m(T, z, \lambda) = d - B\lambda - Az, \tag{8.13}$$

i.e., function (8.12) satisfies the two-point condition (8.9).

REMARK 8.2. The two-point condition (8.9) obviously explicitly only p parameters among $\lambda_1, \lambda_2, \ldots, \lambda_n$, where $p = \operatorname{rank} B$. Therefore, relation (8.8), according to which

the three-point condition (8.7) is replaced by the two-point boundary condition (8.9), can be replaced by the relation

$$Bx(\xi) = B\lambda. \tag{8.14}$$

In other words, it is sufficient to consider only those solutions of the two-point problem (2.1), (8.9), for which the additional condition (8.14) is satisfied. In this case, one completely excludes from consideration the $n - p$ parameters the values of which one need not to determine. If $p = n$ (i.e., $\det B \neq 0$), then conditions (8.14) and (8.8) have the same meaning.

One could also construct a similar scheme by replacing the three-point condition (8.7) by the condition

$$Ax(0) + x(T) = d - \mu \tag{8.15}$$

and seeking for those solutions x of the two-point problem (2.1), (8.15), for which $Bx(\xi) = \mu$.

8.3. *Convergence of successive approximations*

Let us present conditions sufficient for the uniform convergence of the recurrence sequence (8.12) and establish the relation between its limit and the sets of solutions of problems (2.1), (8.7) and (2.1), (8.9).

THEOREM 8.3. *Assume that the continuous function $f : [0, T] \times D \to \mathbb{R}^n$ satisfies condition* (3.5) *with the matrix K possessing property* (6.14). *Moreover, let the matrices A, B and the vector d be such that $\Gamma \neq \varnothing$.*

Then:

(1) *Sequence* (8.12) *converges uniformly in $t \in [0, T]$ for all $(z, \lambda) \in \Gamma \times \Lambda$;*

(2) *For any $(z, \lambda) \in \Gamma \times \Lambda$, the limit function*

$$x^*(t, z, \lambda) := \lim_{m \to +\infty} x_m(t, z, \lambda) \tag{8.16}$$

of sequence (8.12) *is the unique solution of the integral equation*

$$\begin{aligned} x(t) := z + \int_0^t f\big(s, x(s)\big)\, \mathrm{d}s - tT^{-1} \int_0^T f\big(s, x(s)\big)\, \mathrm{d}s \\ + tT^{-1}[d - B\lambda - Az - z], \quad t \in [0, T], \end{aligned} \tag{8.17}$$

or, which is the same, of the integro-differential equation

$$x'(t) = f\big(t, x(t)\big) + T^{-1}\Delta(z, \lambda), \quad t \in [0, T], \tag{8.18}$$

where

$$\Delta(z,\lambda) := d - B\lambda - Az - z - \int_0^T f\big(s, x^*(s,z,\lambda)\big)\,\mathrm{d}s. \tag{8.19}$$

For $(z,\lambda) \in \Gamma \times \Lambda$, *this solution satisfies the boundary conditions* (8.9);

(3) *For all* $(t,z,\lambda) \in [0,T] \times \Gamma \times \Lambda$, *the estimate*

$$\big|x^*(t,z,\lambda) - x_m(t,z,\lambda)\big| \leqslant h(t,z,\lambda), \tag{8.20}$$

is true, where

$$h(t,z,\lambda) := \frac{20t}{9}\left(1 - \frac{t}{T}\right) Q^m (\mathbb{1}_n - Q)^{-1}\big[\delta_D(f) + K|d - B\lambda - Az - z|\big]$$

and $Q := \frac{3T}{10}K$.

Proof. Let us show that sequence (8.12) is a Cauchy sequence in the Banach space $C([0,T],\mathbb{R}^n)$. Let us first prove that the values of each of the functions (8.12) for all $(z,\lambda) \in \Gamma \times \Lambda$ are contained in D. By Lemma 7.7, for any continuous $x : [0,T] \to \mathbb{R}^n$, the estimate

$$\left|\int_0^t \left(x(\tau) - \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s\right)\mathrm{d}\tau\right| \leqslant \frac{1}{2}\alpha_{1,T}(t)\Big[\max_{\tau\in[0,T]} x(\tau) - \min_{\tau\in[0,T]} x(\tau)\Big]$$

is true, where the function $\alpha_{1,T}$ has form (3.34). It follows from Lemma 7.1 that $\max_{t\in[0,T]} \alpha_{1,T}(t) = T/2$. Therefore, for $m = 0$, relation (8.12), in view of (8.10), implies that

$$\big|x_1(t,z,\lambda) - z\big| \leqslant \frac{1}{2}\alpha_{1,T}(t)\delta_D(f) + |d - B\lambda - z - Az| \leqslant \beta(z,\lambda),$$

for any $(t,z,\lambda) \in [0,T] \times \Gamma \times \Lambda$. Here the value $\delta_D(f)$ is given by formula (6.13). Therefore, in view of (8.11), it follows that $x_m(t,z,\lambda) \in D$ for all such (t,z,λ) and any $m = 0,1,2,\ldots$. Arguing similarly to the proof of [71, Theorem 1], one can show that, for all $m \geqslant 0$, $j \geqslant 0$ and $(t,z,\lambda) \in [0,T] \times \Gamma \times \Lambda$

$$\begin{aligned}\big|x_{m+j}(t,z,\lambda) - x_m(t,z,\lambda)\big| \leqslant \frac{10t}{9}\alpha_{1,T}(t) Q^m \Bigg[&\frac{1}{2}\sum_{i=0}^{j-1} Q^i \delta_D(f) \\ &+ K\sum_{i=0}^{j-1} Q^{i-1}|d - B\lambda - z - Az|\Bigg].\end{aligned} \tag{8.21}$$

Since, by (6.14), $\lim_{m\to+\infty} Q^m = 0$ and $\sum_{i=0}^{+\infty} Q^i = (\mathbb{1}_n - Q)^{-1}$, it is obvious from (8.21) that (8.12) is a Cauchy sequence in the uniform norm. Passing to the limit when j

tends to $+\infty$ in (8.21), we obtain that the uniform limit (8.16) of sequence (8.12) satisfies inequality (8.20). Passing to the limit as $m \to +\infty$ in (8.12) and (8.13), we conclude that function (8.16) satisfies Eqs. (8.17), (8.18) and condition (8.9). □

REMARK 8.4. It is clear from the form of Eq. (8.17) that, under assumptions of Theorem 8.3, its unique solution (8.16) for all $(z, \lambda) \in \Gamma \times \Lambda$ satisfies condition

$$x^*(0, z, \lambda) = z. \tag{8.22}$$

REMARK 8.5. In some papers (e.g., in [41,100,101]), parametrised problems of type (2.1), (8.9) are studied by means of construction of two separate iteration processes for the unknown function and the vector of parameters.

The form of the "perturbed" equation (8.18) suggests the idea that the parametrised boundary value problem (2.1), (8.9) might be interpreted as the family of the Cauchy problems

$$x(0) = z \tag{8.23}$$

for the differential equations of the form

$$x'(t) = f\bigl(t, x(t)\bigr) + \mu, \quad t \in [0, T], \tag{8.24}$$

where the vector of parameters μ takes values in a suitable set. More precisely, the following statement holds.

THEOREM 8.6. *Let conditions of Theorem* 8.3 *be satisfied and let* $(z, \lambda) \in \Gamma \times \Lambda$ *and* $\mu \in \mathbb{R}^n$. *Then a solution of the Cauchy problem* (8.24), (8.23) *satisfies the two-point boundary conditions* (8.9) *if and only if the parameter* μ *in* (8.24) *is given by the equation*

$$\mu = \frac{1}{T} \Delta(z, \lambda), \tag{8.25}$$

where $\Delta : \Gamma \times \Lambda \to \mathbb{R}^n$ *is the function defined by formula* (8.19).

REMARK 8.7. It follows from Theorem 8.3 that, under the conditions indicated above, for any $(z, \lambda) \in \Gamma \times \Lambda$, there exists a limit function (8.16) of the recurrence sequence (8.12) and, therefore, mapping (8.19) is well-defined.

PROOF OF THEOREM 8.6. *Sufficiency.* Let us put $\mu = T^{-1}\Delta(z, \lambda)$, in (8.24), where $z \in \Gamma$ and $\lambda \in \Lambda$ are some fixed vectors, and Δ is given b formula (8.19). According to Remark 8.7, the expression $\Delta(z, \lambda)$ makes sense for all $(z, \lambda) \in \Gamma \times \Lambda$. Theorem 8.3 implies that, for the given z and λ, the uniform limit (8.16) of the corresponding sequence (8.12) coincides with the unique solution of the two-point problem (8.18), (8.9). As indicated in Remark 8.4, this function also satisfies the initial condition (8.22), i.e., is a solution of the Cauchy problem (8.24), (8.23) corresponding to the value (8.25) of the parameter μ.

Necessity. Let $z \in \Gamma$, $\lambda \in \Lambda$, and $\bar{\mu} \in \mathbb{R}^n$ be arbitrary vectors, and $\bar{x} : [0, T] \to \mathbb{R}^n$ be a solution of the Cauchy problem (8.23) for the equation

$$x'(t) = f\big(t, x(t)\big) + \bar{\mu}, \quad t \in [0, T], \tag{8.26}$$

satisfying, for the given value of λ, the two-point condition (8.9). Assume that there exists another value $\bar{\bar{\mu}}$, of the parameter μ, for which a certain solution $\bar{\bar{x}}$ of problem (8.23) for the equation

$$x'(t) = f\big(t, x(t)\big) + \bar{\bar{\mu}}, \quad t \in [0, T], \tag{8.27}$$

satisfies the two-point condition (8.9). It is obvious from (8.23), (8.26), and (8.27) that the functions $\bar{x}$ and $\bar{\bar{x}}$ satisfy the integral equations

$$\bar{x}(t) = z + \int_0^t f\big(s, \bar{x}(s)\big)\, \mathrm{d}s + \bar{\mu}t, \quad t \in [0, T], \tag{8.28}$$

and

$$\bar{\bar{x}}(t) = z + \int_0^t f\big(s, \bar{\bar{x}}(s)\big)\, \mathrm{d}s + \bar{\bar{\mu}}t, \quad t \in [0, T], \tag{8.29}$$

respectively. For $t = T$, relations (8.28), (8.29) yield

$$T\bar{\mu} = \bar{x}(T) - z - \int_0^T f\big(s, \bar{x}(s)\big)\, \mathrm{d}s \tag{8.30}$$

and

$$T\bar{\bar{\mu}} = \bar{\bar{x}}(T) - z - \int_0^T f\big(s, \bar{\bar{x}}(s)\big)\, \mathrm{d}s. \tag{8.31}$$

The function $\bar{x}$, by assumption, satisfies the two-point condition (8.9),

$$A\bar{x}(0) + \bar{x}(T) = d - B\lambda,$$

and the initial condition $\bar{x}(0) = z$, whence

$$\bar{x}(T) = d - B\lambda - Az.$$

Similarly, on can verify that

$$\bar{\bar{x}}(T) = d - B\lambda - Az.$$

Therefore, (8.30) and (8.31) yield

$$\bar{\mu} = \frac{1}{T}\left(d - B\lambda - Az - z - \int_0^T f\big(s, \bar{x}(s)\big)\, \mathrm{d}s\right) \tag{8.32}$$

and

$$\bar{\bar{\mu}} = \frac{1}{T}\left(d - B\lambda - Az - z - \int_0^T f\big(s, \bar{\bar{x}}(s)\big)\,\mathrm{d}s\right). \tag{8.33}$$

Substituting (8.32) and (8.33) into (8.28) and (8.29), respectively, we obtain that

$$\begin{aligned}\bar{x}(t) = z &+ \int_0^t f\big(s, \bar{x}(s)\big)\,\mathrm{d}s \\ &+ \frac{t}{T}\left[d - B\lambda - Az - z - \int_0^T f\big(s, \bar{x}(s)\big)\,\mathrm{d}s\right]\end{aligned} \tag{8.34}$$

and

$$\begin{aligned}\bar{\bar{x}}(t) = z &+ \int_0^t f\big(s, \bar{x}(s)\big)\,\mathrm{d}s \\ &+ \frac{t}{T}\left[d - B\lambda - Az - z - \int_0^T f(s, \bar{\bar{x}}(s))\,\mathrm{d}s\right]\end{aligned} \tag{8.35}$$

for any $t \in [0, T]$. We recall that here $z \in \Gamma$ and $\lambda \in \Lambda$. Therefore, similarly to the proof of Theorem 8.3, by using the form of Eqs. (8.34), (8.35) and definition (8.11) of the set Γ, we can show that all the values of the functions $\bar{\bar{x}}$ and $\bar{x}$ are contained in D:

$$\bar{x}\big([0, T]\big) \cup \bar{\bar{x}}\big([0, T]\big) \subset D. \tag{8.36}$$

It is obvious from (8.34) and (8.35) that

$$\begin{aligned}\bar{\bar{x}}(t) - \bar{x}(t) = &\int_0^t \big[f\big(s, \bar{\bar{x}}(s)\big) - f\big(s, \bar{x}(s)\big)\big]\,\mathrm{d}s \\ &- \frac{t}{T}\int_0^T \big[f\big(s, \bar{\bar{x}}(s)\big) - f\big(s, \bar{x}(s)\big)\big]\,\mathrm{d}s, \quad t \in [0, T],\end{aligned}$$

and, hence, by virtue of relation (8.36) and the Lipschitz condition (3.5), the function

$$r(t) := \big|\bar{\bar{x}}(t) - \bar{x}(t)\big|, \quad t \in [0, T],$$

satisfies the integral inequality

$$\begin{aligned}r(t) &\leqslant K\left[\int_0^t r(s)\,\mathrm{d}s + \frac{t}{T}\int_0^T r(s)\,\mathrm{d}s\right] \\ &\leqslant K\alpha_{1,T}(t)\max_{s\in[a,b]} r(s), \quad t \in [0, T],\end{aligned} \tag{8.37}$$

where the function $\alpha_{1,T}$ is given by formula (3.34). Using (8.37) sequentially, we arrive at the inequality

$$r(t) \leqslant K^m \alpha_{m,T}(t) \max_{s\in[a,b]} r(s), \quad t \in [0, T], \tag{8.38}$$

where the natural m is arbitrary and $\alpha_{m,T}$ is defined by relation (3.35). According to Lemma 2.4 from [70], functions (3.35) for all $t \in [0, T]$ and $k = 1, 2, \ldots$ admit estimate (3.36). Consequently, it follows from (8.38) that, for any natural m,

$$r(t) \leqslant \frac{10}{9} K\alpha_{1,T}(t)\left(\frac{3T}{10}K\right)^{m-1} \max_{s\in[a,b]} r(s), \quad t \in [0, T]. \tag{8.39}$$

Letting in (8.39) m tend to $+\infty$ and taking condition (6.14) into account, we conclude that $r \equiv 0$ on $[0, T]$, i.e., the functions $\bar{x}$ and $\bar{\bar{x}}$ coincide with one another, and hence $\bar{\mu} = \bar{\bar{\mu}}$. The contradiction obtained proves that $\mu = T^{-1}\Delta(z, \lambda)$ is the unique value of the parameter μ in Eq. (8.24), for which the solution of the initial value problem (8.24), (8.23) satisfies the two-point condition (8.9). □

REMARK 8.8. It follows from Lemma 7.5 that for an arbitrarily small positive ε there always exists a number $k_\varepsilon \in \mathbb{N}$ such that functions (3.35), starting with the k_εth one, admit the estimate

$$\alpha_{k,T}(t) \leqslant \left(\frac{T}{q} + \varepsilon\right)^{k-k_\varepsilon} \alpha_{k_\varepsilon,T}(t), \quad t \in [0, T],\ k \geqslant k_\varepsilon, \tag{8.40}$$

where $q \approx 3.416131$. Using (8.40) instead of estimate (3.36) and modifying appropriately the proofs of Theorems 8.3 and 8.6, one ca show that inequality (6.14) can be replaced by the weaker condition $r(K) < qT^{-1}$. All the related statements are preserved in this case at the expense of certain technical modifications of the argument (e.g., the form of the function h in estimate (8.20)).

Let us make clear how the function (8.16) defined in Theorems 8.3 and 8.6 is related to the set of solutions of the two-point problem (2.1), (8.9) containing a free parameter $\lambda \in \Lambda$.

PROPOSITION 8.9. *Under conditions of Theorem* 8.3, *the function* $x^*(\cdot, z, \lambda)$ *given by formula* (8.16) *for* $(z, \lambda) \in \Gamma \times \Lambda$, *is a solution of the two-point problem* (2.1), (8.9) *with the parameter* λ *if and only if* z *and* λ *satisfy the relation*

$$\int_0^T f\big(s, x^*(s, z, \lambda)\big)\,\mathrm{d}s = d - B\lambda - Az - z. \tag{8.41}$$

Function (8.16) *is a solution of the original three-point problem* (2.1), (8.7) *if and only if the pair* (z, λ) *satisfies condition* (8.41), *and, moreover,*

$$x^*(\xi, z, \lambda) = \lambda. \tag{8.42}$$

PROOF. It follows from Theorem 8.3 that for all $(z, \lambda) \in \Gamma \times \Lambda$ the function $x^*(\cdot, z, \lambda)$ is a solution of the two-point problem (8.18), (8.9). Equation (8.18) coincides with Eq. (2.1) if and only if z and λ satisfy condition $\Delta(z, \lambda) = 0$, i.e., equality (8.41) holds. The second assertion follows from Remark 8.1. □

8.4. *Practical realisation*

When putting the suggested scheme to practise, it is natural to fix a certain iteration number, m, in (8.12) and regard $x_m(\cdot, z, \lambda)$ as an approximation to the unknown function $x^*(\cdot, z, \lambda)$, the existence of which is stated in Theorem 8.3. In this case, instead of Eqs. (8.41), (8.42), the "approximate determining equations"

$$\int_0^T f\bigl(s, x_m(s, z, \lambda)\bigr)\,\mathrm{d}s = d - B\lambda - Az - z \tag{8.43}$$

and

$$\begin{aligned} z + \int_0^t f\bigl(s, x_m(s, z, \lambda)\bigr)\,\mathrm{d}s - tT^{-1}\int_0^T f\bigl(s, x_m(s, z, \lambda)\bigr)\,\mathrm{d}s \\ + tT^{-1}[d - B\lambda - Az - z] = \lambda. \end{aligned} \tag{8.44}$$

arise, from which we can find suitable values of the pair of parameters $(z, \lambda) \in \Gamma \times \Lambda$. If the system of $2n$ Eqs. (8.43), (8.44) has an isolated solution $(\bar{z}, \bar{\lambda}) \in \Gamma \times \Lambda$, then, by using topological and functional-analytic methods (see, e.g., Theorem 3.1 in [70] and Theorem 19.2 in [37]), under the appropriate additional assumptions, one can prove that system (8.41), (8.42) is also solvable and, thus establishing the existence of a solution of the original three-point problem. The function

$$\bar{x}_m(t) := x_m(\cdot, \bar{z}, \bar{\lambda}), \quad t \in [0, T], \tag{8.45}$$

can then be regarded as an approximation to a solution of problem (2.1), (8.7).

8.5. *Example of a two-dimensional three-point problem*

Let us consider the three-point boundary value problem

$$x_1'(t) = x_2(t), \tag{8.46a}$$

$$x_2'(t) = \frac{1}{8}tx_2 - \frac{1}{2}{x_2}^2 - \frac{1}{2}x_1 + \frac{9}{32} + \frac{1}{16}t^2, \quad t \in [0, 1], \tag{8.46b}$$

$$x_2(0) + x_1\left(\frac{1}{2}\right) + x_1(1) = \frac{9}{32}, \tag{8.46c}$$

$$x_1(0) + x_2(1) = \frac{5}{16}. \tag{8.46d}$$

Conditions (8.46c), (8.46d), clearly, can be written in form (8.7) with $T = 1$ and

$$A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad d := \begin{bmatrix} \frac{9}{32} \\ \frac{5}{16} \end{bmatrix}, \quad \xi := \frac{1}{2}. \tag{8.47}$$

Let us restrict our consideration by those solutions of system (8.46a), (8.46b), all the values of which are contained in the set

$$D := \left\{ (x_1, x_2) \colon \, |x_1| \leqslant 1, |x_2| \leqslant \frac{3}{4} \right\}. \tag{8.48}$$

It is not difficult to verify that, for all $t \in [0, 1]$ and $\{u, v\} \subset D$, the function $f : [0, 1] \times D \to \mathbb{R}^2$ given by the formula

$$f(t, x_1, x_2) := \begin{bmatrix} x_2 \frac{1}{8} t x_2 - \frac{1}{2} {x_2}^2 - \frac{1}{2} x_1 + \frac{9}{32} + \frac{t^2}{16} \end{bmatrix}, \quad (t, x_1, x_2) \in [0, 1] \times D,$$

satisfies the Lipschitz condition (3.5) with the matrix

$$K := \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{7}{8} \end{bmatrix}. \tag{8.49}$$

The greatest eigenvalue of matrix (8.49) is equal to $(7 + \sqrt{177})/16 \approx 1.27,$ which is less than $\frac{10}{3}$, and, hence, inequality (6.14) holds.

Let us seek for the solutions of the three-point problem (8.46) among the solutions of system (8.46a), (8.46b) satisfying the parametrised two-point conditions

$$x_2(0) + x_1(1) = \frac{9}{32} - \lambda_1, \tag{8.50}$$

$$x_1(0) + x_2(1) = \frac{5}{16}. \tag{8.51}$$

Conditions (8.50), (8.51), obviously, coincide with (8.9) for A, B, and d given by (8.47). For the domain where the two-dimensional vector of parameters $\lambda = (\lambda_1, \lambda_2)$ varies, we take, e.g., the rectangle

$$\Lambda := \left\{ (\lambda_1, \lambda_2) \colon \, |\lambda_1| \leqslant \frac{1}{4}, |\lambda_2| \leqslant \frac{1}{2} \right\} \tag{8.52}$$

contained in set (8.48). A solution (x_1, x_2) of problem (8.46a), (8.46b), (8.50), (8.51) is a solution of the original three-point problem (8.46) if and only if the additional conditions $x_1(\frac{1}{2}) = \lambda_1$ and $x_2(\frac{1}{2}) = \lambda_2$ are satisfied.

Let us use the approach based on Theorem 8.3, for which purpose verify the fulfilment of its assumptions. It is easy to see that the value $\delta_D(f)$ in the case considered is given by the equality

$$\delta_D(f) = \frac{1}{2}\begin{bmatrix} 3 \\ \frac{355}{128} \end{bmatrix}. \tag{8.53}$$

Furthermore, it is clear from (8.48), (8.52), (8.47), and (8.53) that, for our problem, the components of the function $\binom{\beta_1}{\beta_2} : D \times \Lambda \to \mathbb{R}^2$ defined by equality (8.10) have the form

$$\beta_1(z_1, z_2, \lambda_1, \lambda_2) = \frac{3}{8} + \left|\frac{9}{32} - \lambda_1 - z_1 - z_2\right|,$$
$$\beta_2(z_1, z_2, \lambda_1, \lambda_2) = \frac{355}{1024} + \left|\frac{5}{15} - z_1 - z_2\right|.$$

Therefore, according to the definition (8.11) of the set Γ, a point (z_1, z_2) from D belongs to Γ if and only if

$$\left|\frac{9}{32} - \lambda_1 - z_1 - z_2\right| \leqslant \frac{5}{8}, \tag{8.54}$$

$$\left|\frac{5}{15} - z_1 - z_2\right| \leqslant \frac{413}{1024} \tag{8.55}$$

for all $\lambda_1 \in [-\frac{1}{4}, \frac{1}{4}]$. The set Γ of such pairs $(z_1, z_2) \in D$ is obviously non-empty. Thus, all the conditions of Theorem 8.3 are satisfied in this case and, therefore, for any $(z, \lambda) \in \Gamma \times \Lambda$, on $[0, 1]$, the function (8.16) discussed in Theorem 8.6 and Proposition 8.9 is well-defined.

Let us construct approximate solutions of problem (8.46) by using the approach indicated in Section 8.4. For this purpose, we compute analytically several members of the recurrence function sequence (8.12) with the help of Maple [29]. Let us set the zeroth iteration to be equal identically to z, $z \in \Gamma$. Then for $m = 1$ relation (8.12) yields

$$x_{11}(t, z_1, z_2, \lambda_1, \lambda_2) = z_1 + \frac{9}{32}t - t\lambda_1 - z_1 t - z_2 t, \tag{8.56}$$

$$x_{12}(t, z_1, z_2, \lambda_1, \lambda_2) = z_2 + \frac{7}{24}t + \frac{1}{48}t^3 + \frac{1}{16}z_2 t^2 - \frac{17}{16}z_2 t - z_1 t \tag{8.57}$$

for any $t \in [0, 1]$, $(z_1, z_2) \in \Gamma$, and λ_1, λ_2 satisfying the inequalities

$$|\lambda_1| \leqslant \frac{1}{4}, \qquad |\lambda_2| \leqslant \frac{1}{2}.$$

Here and below, the symbols x_{k1} and x_{k2} designate, respectively, the first and the second components of the vector x_k.

For $m = 1$, the system approximate determining equations (8.43), (8.44) corresponding to problem (8.46) consists of the equations

$$\frac{25}{192} - \frac{143}{96}z_2 - \frac{1}{2}z_1 - \lambda_1 = 0, \tag{8.58}$$

$$\frac{40391}{483840} - \frac{28157}{23040}z_2 - \frac{583}{720}z_1 + \frac{827}{5120}z_2^2 - \frac{31}{192}z_2 z_1 + \frac{1}{6}z_1^2 - \frac{1}{4}\lambda_1 = 0, \tag{8.59}$$

$$\frac{5}{8}z_1 + \frac{313}{3072} - \frac{3}{8}z_2 - \frac{3}{2}\lambda_1 = 0, \tag{8.60}$$

and

$$\begin{aligned}&\frac{129067}{294912}z_2 + \frac{98569}{589824} - \frac{899}{1536}z_1 - \frac{63}{1024}z_2^2 \\ &\quad + \frac{1}{1024}z_2 z_1 + \frac{1}{16}z_1^2 - \frac{1}{16}\lambda_1 - \lambda_2 = 0.\end{aligned} \tag{8.61}$$

REMARK 8.10. The rank of the matrix B corresponding to the two-point conditions (8.46c), (8.46d) rewritten in form (8.7) is equal to 1 (the conditions do not contain explicitly $x_2(\frac{1}{2})$). Therefore, in view of Remark 8.2, the determining equations — both those given above and the other ones — need not be solved with respect to the parameter λ_2 because its value does not play any role in for problem considered. In this example we, however, write down all the equations completely in order to illustrate the approach.

Solving the system of Eqs. (8.58)–(8.60), (8.61) numerically, we obtain

$$z_1 \approx 0.06775109879, \qquad z_2 \approx 0.006168107627, \tag{8.62}$$

$$\lambda_1 \approx 0.08714487359, \qquad \lambda_2 \approx 0.125. \tag{8.63}$$

Substituting (8.62), (8.63) into (8.56) and (8.57), we find a function of form (8.45) – the "first approximation" to a solution of the three-point problem (8.46) corresponding to the values (8.62), (8.63) of the parameters z_1, z_2 and λ_1, λ_2 computed above:

$$\begin{aligned}\bar{x}_{11}(t) &= 0.06775109879 + 0.1201859200t, \\ \bar{x}_{12}(t) &= 0.006168107627 + 0.2173619535t \\ &\qquad + \frac{t^3}{48} + 0.0003855067267t^2.\end{aligned} \tag{8.64}$$

Note that the given three-point problem (8.46) has the solution

$$x_1(t) = \frac{t^2}{8} + \frac{1}{16}, \quad x_2(t) = \frac{t}{4}, \quad t \in [0, 1], \tag{8.65}$$

passing through the point $(\frac{1}{16}, 0)$ of the set Γ at the moment of time 0.

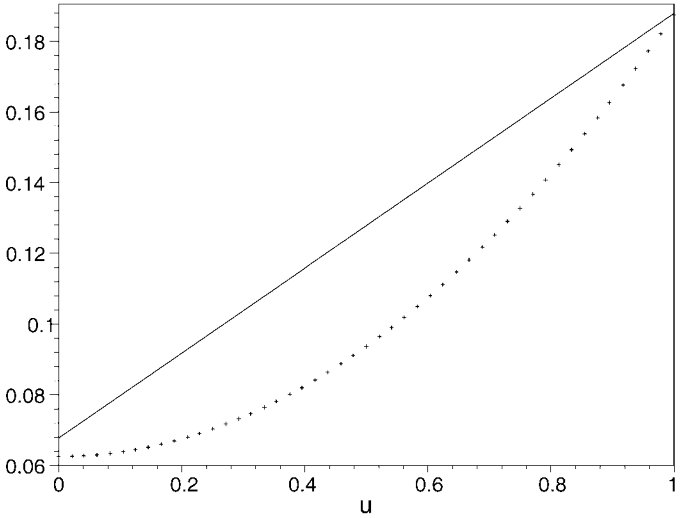

Fig. 9. The first component of the exact solution (8.65) and its “first approximation” (8.64).

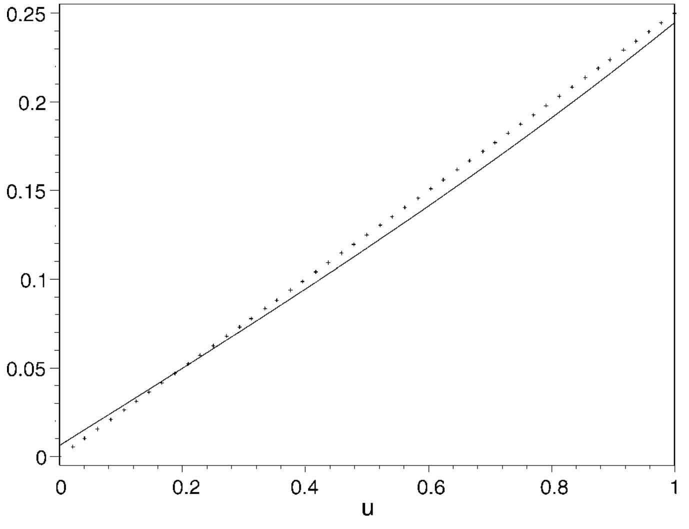

Fig. 10. The second component of the exact solution (8.65) and its “first approximation” (8.64).

Figs. 9 and 10 show the graphs of the first and the second components of the exact solution (8.65) of problem (8.46) (dotted line) and its “first approximation” (8.64) (solid line). The components of the deviation of the “first approximation” (8.64) from solution (8.65), i.e., the functions $x_1 - \bar{x}_{11}$ and $x_2 - \bar{x}_{12}$, are shown, respectively, on Figs. 13 and 14.

In a similar way, by using formulae (8.56) and (8.57) that have been derived above for the first iteration, one can find the second iteration $x_2(\cdot, z_1, z_2, \lambda_1, \lambda_2)$ (i.e., $m = 2$ in formula

(8.12)), the components of which have the form

$$x_{21}(t,z_1,z_2,\lambda_1,\lambda_2)=z_1+\frac{t^4}{192}+\frac{t^3}{48}z_2+\frac{7t^2}{48}-\frac{17t^2}{32}z_2-\frac{t^2}{2}z_1 -\frac{47t}{96}z_2+\frac{25t}{192}-\frac{t}{2}z_1-t\lambda_1,\quad t\in[0,1], \tag{8.66}$$

and

$$\begin{aligned} x_{22}(t,z_1,z_2,\lambda_1,\lambda_2)=z_2&+\frac{176471}{483840}t-\frac{28157}{23040}z_2t+\frac{17t^3}{288}z_2+\frac{t^2}{6}z_2+\frac{t^2}{4}z_1\\ &-\frac{t^6z_2}{4608}-\frac{943t}{720}z_1-\frac{1733}{5120}z_2^2t+\frac{17t^5z_2}{3840}+\frac{t^5z_1}{240}-\frac{t^5z_2^2}{2560}\\ &-\frac{t^4z_2}{128}+\frac{17t^4z_2^2}{1024}-\frac{107t^3z_2^2}{512}+\frac{t^3z_1}{18}-\frac{t^3z_1^2}{6}+\frac{65t^3}{3456}\\ &-\frac{9t^2}{128}-\frac{t^5}{1440}-\frac{t^7}{32256}+\frac{t^2}{2}z_2z_1+\frac{17t^2}{32}z_2^2+\frac{t^2}{4}\lambda_1\\ &+\frac{t}{6}z_1^2-\frac{t}{4}\lambda_1+\frac{t^4}{64}z_2z_1-\frac{31}{192}tz_2z_1\\ &-\frac{17}{48}t^3z_2z_1,\quad t\in[0,1], \end{aligned}\tag{8.67}$$

and construct the corresponding system of Eqs. (8.43), (8.44) for determining the values of z_1, z_2, λ_1, and λ_2. The system mentioned, as the computation shows, has the approximate solution

$$z_1\approx 0.06242777432,\qquad z_2\approx 0.0001215436768, \tag{8.68}$$

$$\lambda_2\approx 0.125,\qquad \lambda_1\approx 0.09364329365. \tag{8.69}$$

Substituting (8.68), (8.69) in (8.66) and (8.67), we obtain the “second approximation,” the components of which are given by the formulae

$$\begin{aligned}\bar{x}_{21}(t)=0.06242777432&+\frac{t^4}{192}+0.2532159933\times 10^{-5}t^3\\ &+0.1145548760t^2+0.0052916467t,\quad t\in[0,1],\end{aligned}\tag{8.70}$$

and

$$\begin{aligned}\bar{x}_{22}(t)=&-0.2637666597\times 10^{-7}t^6+0.2600559795t+0.0001215436768\\ &-0.8307568908\times 10^{-6}t^4+0.02163102627t^3-0.03127067403t^2\\ &-0.0004337906399t^5-\frac{t^7}{32256},\quad t\in[0,1],\end{aligned}\tag{8.71}$$

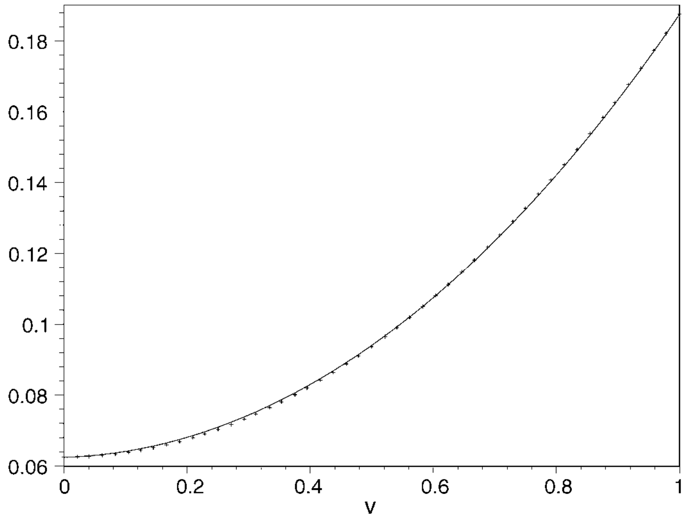

Fig. 11. The first component of solution (8.65) and its “second approximation” (8.70).

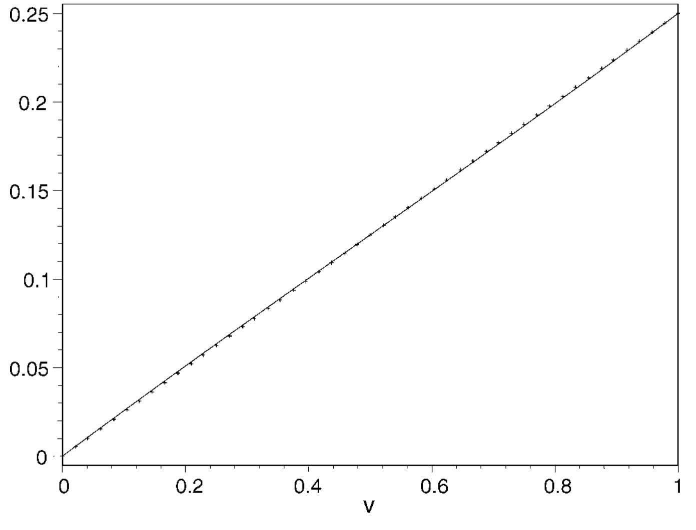

Fig. 12. The second component of solution (8.65) and its “second approximation” (8.71).

respectively.

Figure 11 shows the graph of the first component of the solution (8.65) of problem (8.46) (dotted line) and its “second approximation” (8.70) (solid line). On Fig. 12, the graphs of the second component of solution (8.65) (dotted line) and its “second approximation” (8.71) (solid line) are presented. The curves almost coincide with one another. The error of the second approximation is drawn on Figs. 15 and 16.

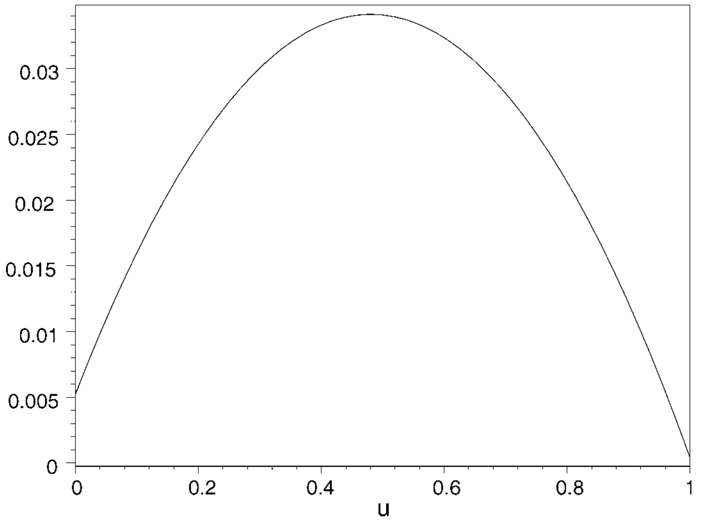

Fig. 13. Error of the first component of the “first approximation” (8.64) with respect to solution (8.65).

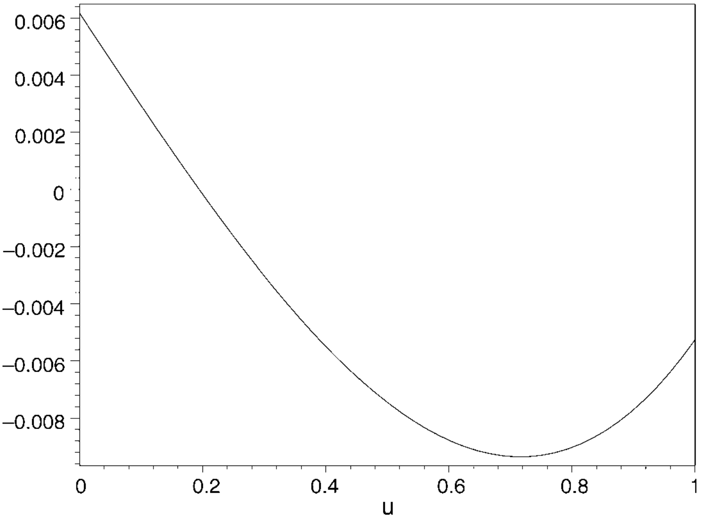

Fig. 14. Error of the second component of the “first approximation” (8.64) with respect to solution (8.65).

The computation shows that the third approximation constructed according to the scheme indicated above gives the absolute error which does not exceed 0.00035 for the first component and 0.0001 for the second one.

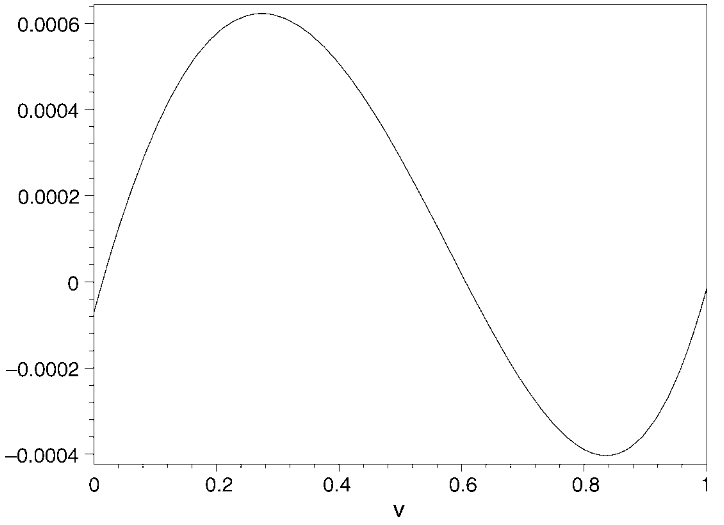

Fig. 15. Error of the “second approximation” (8.70) to the first component of solution (8.65).

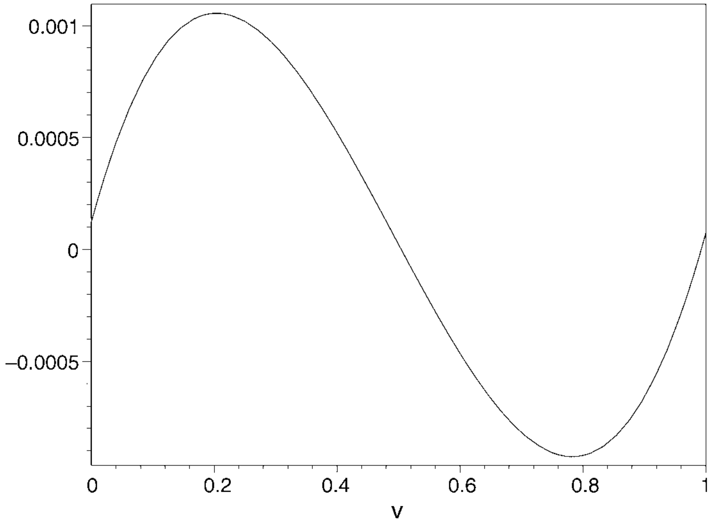

Fig. 16. Error of the “second approximation” (8.71) to the second component of solution (8.65).

9. Historical remarks

The basic concepts of the successive approximation method considered here had been first introduced by Samoilenko [82,83] in connection with the study of the periodic boundary value problem (2.1), (2.2) and was referred to as the “numerical-analytic method” and “periodic successive approximations method”. Some its aspects, as was noticed afterwards,

have common points with other methods oriented to the investigation of periodic solutions of non-autonomous systems of differential equations.

The survey papers [74,76,77,75,78–80] analyse the history of the method and many interesting works inspired by it. Among the other works concerning the successive approximation method for boundary value problems, we mention the paper [74] and the Appendix of the book [70] which analyse the relation of the method to the Lyapunov–Schmidt equation [48], with the Cesari–Hale method for weakly non-linear differential equations [12], with the Cesari scheme for essentially non-linear systems of ordinary differential equations [11], and with the integral operator M_J introduced by Mawhin [81, pp. 98–107] as a mapping in C_T whose fixed points coincide with the solutions of a T-periodic boundary value problem for system (2.1). For example, the result on the existence of T-periodic solutions established in [81] (see Theorem 2.3, Corollary 4.4, and Remark 4.5 in [81]) can also be obtained with the help of the successive approximations method discussed here. In [70, Appendix] by Rontó, Samoilenko, and Trofimchuk, such a result is presented as Theorem 10, in the formulation borrowed from [72].

Note that a result close to the above-mentioned Theorem 10 from [70] had been first proved by Mawhin in 1969 for a locally Lipschitzian equation by using the Cesari method [81]. The statement indicated can be regarded as a consequence of the coincidence degree theory (see [81, p. 225] and [24]).

The improvement of the applicability conditions of the successive approximations method under consideration and estimates appearing in Theorems 3.17, 3.22, 3.23, 3.28, 3.36, 4.2, 6.1, and 8.3, Lemma 3.26, and inequality (4.25), involving the matrix Q of form (3.9), is closely related to the refinement of the constant q. Clearly, the more q is, the wider the domain of application of the iteration scheme will be. The following list describes, in the chronological order, the constants used in various estimates related to the successive approximation method:

1965 Samoilenko [82,83]; $q = 3.1$.

1976 Samoilenko and Ronto [88]; $q = \pi$.

1982 Samoilenko and Laptinskii [86]; $q = 3.416\ldots$. The proof has a gap which can be eliminated by using the Krein–Rutman theorem [38].

1985 Evhuta and Zabreiko [15,16]; $q = 3.416\ldots$.

1990 Trofimchuk [104]; $q = 3.416\ldots$. It is also proved that $q \leqslant 2\pi$. The estimates obtained are rather complicated and difficult to apply due to the need to compute the values of some integral operators.

1992 Kwapisz [42]; $q = \sqrt{10} = 3.1622\ldots$.

1996 Rontó and Mészáros [69]; $q = \frac{10}{3}$.

1996 Rontó, Rontó, Samoilenko, and Trofimchuk [64]; $q = 3.416\ldots$. Convenient techniques for the computation of constants in estimates are suggested.

2001 Perov, Dikareva, Oleinikova, and Portnov [57]; $q = \pi$. Three methods of proof are presented. In the first one, certain inaccuracies in some auxiliary statements used in [82,83,88] are corrected, whereas the second one uses the Steklov inequality [55, 56,14]. The third approach suggested allowed the authors to improve the a priori estimates by determining the maximal eigenvalues of certain integral operators.

2003 Samoilenko [85]; $q = 3.416\ldots$. The value of the constant was determined as the convergence radius of the Poisson–Abel series (5.64).

Using the idea of the successive approximations method first introduced for the study of the periodic solutions, many authors developed its versions adopted to the investigation of other kinds of boundary value problems for various types of ordinary differential, integro-differential, and more general functional differential equations and their systems. Many such papers are discussed in [74,76,77,75,78–80] and in the Appendix of [70].

Section A3.1 in [70, Appendix] analyses, in particular, the analogues of the successive approximations (3.29) constructed directly for the scalar second-order non-linear Lipschitzian differential equation

$$x''(t) + \omega^2 x(t) = f\bigl(t, x(t), x'(t)\bigr), \quad t \in [0, T], \tag{9.1}$$

without reduction of (9.1) to an equivalent system of first order equations. The convergence corresponding conditions turned out to be significantly worse compared to their counterparts for first order systems [70, p. 339]. The scheme of the successive approximations was adjusted for finding the periodic solutions of second order equations by Strizhak [102] and Samoilenko and Ronto [89]. Some modifications of the method were applied by Mitropolskii, Khoma, and Gromyak [50] in order to prove existence theorems to the T-periodic problem

$$x(0) = x(T), \qquad x'(0) = x'(T), \tag{9.2}$$

for Eq. (9.1) with $\omega \neq 0$, both in the non-resonance and resonance cases. The conditions of convergence of the scheme from [50] in the non-resonance case are rather restrictive.

Existence and uniqueness results for the non-resonance T-periodic boundary value problem (9.1), (9.2) were established by Chornyĭ [13]. Another scheme, which is also applicable in the cases where the Lipschitz constants for f in (9.1) are large, was suggested by Perestyuk and Ronto [53]. Problem (9.2) for some types of second order differential equations was studied by Shlapak [97,98], Sobkovich [99], Ronto and Martynyuk [73], and many others (see, e.g., [70, Appendix] for references).

Modified versions of the successive approximation scheme of form (3.29) for the study of T-periodic solutions of the delay differential system

$$x'(t) = f\bigl(t, x(t), x(t-\delta)\bigr), \quad t \in [0, T], \tag{9.3}$$

was first introduced by Martynyuk and Samoilenko [47], Mitropol′skiĭ, Samoilenko, and Martynyuk [51] and the role of the delay, δ, pointed out.

Versions of the successive approximations obtained from (3.29) by a certain change of variables were used by Augustynowicz and Kwapisz [7], Kwapisz [42,44,43] for the investigation of solutions of the neutral type functional differential equation

$$x'(t) = f\bigl(t, x\bigl(\alpha(t)\bigr), x'\bigl(\beta(t)\bigr)\bigr), \quad t \in [0, T], \tag{9.4}$$

subjected to the two-point boundary conditions of the form

$$Ax(0) + Bx(T) = d,$$

where $\{A, B\} \subset \mathrm{GL}_n(\mathbb{R})$, $d \in \mathbb{R}^n$, and α, β are functions mapping the interval $[0, T]$ into itself (see Section A3.2.2 in [70, p. 369]).

Techniques based upon successive approximations of the kind indicated were used by Baĭnov and Sarafova [8] for integro-differential equations and Sarafova [95,96,9] Ronto [62] for differential equations with "maxima." Zavalykut [107], Zavalykut and Nurzhanov [106] extended the techniques discussed to the more general differential operator equations of the form

$$x'(t) = (Ax)(t), \quad t \in (-\infty, \infty),$$

where A is a continuous mapping defined on the space C_T (see also [70, Section A3.3]).

Successive approximation techniques were applied for impulsive systems by Samoilenko [84], Samoilenko and Perestyuk [87], Kurbanbaev [40], Perestyuk and Shovkoplyas [54].

In [78], the application of the successive approximations method to difference equations is analysed. The paper [79] deals with its application to various non-linear boundary value problems (see also [60,39]). The paper [80] gives a survey of works devoted to successive approximation techniques for differential equations in Banach spaces, certain types of implicit differential equations, and to boundary value problems containing a control parameter (see also Akhmet and Zafer [4], Jankowski [34,32,33,31], Ronto, Ronto, and Shchobak [65]).

10. Exercises

Here, we suggest several exercises related to the techniques discussed.

EXERCISE 10.1. Study in which sense the family of sets $\{\Omega_{m,N}\}$ tends to the set Ω^* in Remark 3.37.

EXERCISE 10.2. For Eq. (4.96) in Example 4.9, by using the scheme (4.96), construct the functions $y_1(\theta, c, \mu)$ and $y_2(\theta, c, \mu)$. Write down the corresponding determining equations (4.45) and construct the 1st and the 2nd approximate periodic solutions (4.56) of Eq. (4.96). Find the corresponding values of c and T, compare them to ε and 2π (cf. (4.111)).

EXERCISE 10.3. Investigate the properties of the sequence $\alpha_{m,T}$, $m \geqslant 0$, given by formula (3.35). Prove the equality

$$\begin{aligned}\lim_{m\to\infty} \max_{t\in[0,T]} \frac{\alpha_{m,T}(t)}{\alpha_{m+1,T}(t)} &= \lim_{m\to\infty} \frac{\max_{t\in[0,T]} \alpha_{m,T}(t)}{\max_{t\in[0,T]} \alpha_{m+1,T}(t)} \\ &= \lim_{m\to\infty} \frac{\alpha_{m,T}(T/2)}{\alpha_{m+1,T}(T/2)} = \frac{1}{Tr(\mathcal{A})},\end{aligned}$$

where $\mathcal{A}$ is the linear operator in the space $C([0,1])$ given by the formula

$$(\mathcal{A}x)(s) := (1-s)\int_0^s x(\xi)\,\mathrm{d}\xi + s\int_s^1 x(\xi)\,\mathrm{d}\xi, \quad s \in [0,1]. \tag{10.1}$$

EXERCISE 10.4. Compute approximately the spectral radius $r(\mathcal{A})$ of operator (10.1). Hint: $r(\mathcal{A}) \approx 0.29$, see [60].

EXERCISE 10.5. Show how is it possible to relax condition (3.7). Hint: use Problems 10.3 and 10.4 and see [60] to show that the constant $3/10$ in (3.7) can be replaced by $r(\mathcal{A})$.

EXERCISE 10.6. For the periodic boundary value problem for the differential equation of the form

$$x'(t) = f\big(t, x, x'(t)\big), \quad t \in [0, T],$$

construct a suitable successive approximation method instead of that based upon the iteration scheme (3.29). Hint: see Section 3.10 in [80].

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CHAPTER 6

Analytic Ordinary Differential Equations and Their Local Classification

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Abstract

The article is devoted to the local theory of analytic differential equations. We describe classification of meromorphic systems $\dot{x} = A(t)x$ near regular and irregular singular point. We present a local theory of non-linear holomorphic equations $\dot{x} = V(x)$, with the proof of

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the resolution theorem and its application to the center–focus problem, with Ecalle–Voronin–Martinet–Ramis moduli and with Briuno–Yoccoz classification of non-resonant saddles. Finally, we present formal classification of nilpotent singularities and prove analyticity of the Takens prenormal form.

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1. Introduction

The origins of the theory of Ordinary Differential Equations lie in the classical mechanics. The second Newton's principle of dynamics states that the further configurations of a mechanical system are determined by the initial positions and velocities via a second order differential equation. Therefore the theory is essentially real, because the time is real. But in the conservative mechanical systems the direction of time is not that important, the corresponding Newton's equations are time–reversible. Even in the non-conservative case the knowledge of the initial positions and velocities at $t = 0$ allows to reconstruct the history of the system.

Therefore, in the philosophical sense, the theory of Ordinary Differential Equations resembles other mathematical theories: the Riemannian geometry, the real algebraic geometry, the Morse theory of functions, etc. In all these theories the extension to the complex domain turned out very effective. For example, the Morse theory describes the change of topology of level surfaces of a function during passage through a critical value; but in the case of holomorphic function one can pass along a loop around the critical value and the resulting change in the topology of level surface is described by the Picard–Lefschetz formula.

It is not surprising that also the theory of Ordinary Differential Equations has its holomorphic analogue. It is the analytic theory of Ordinary Differential Equations, which does not concentrate on solutions in the form of power series, but rather treats the time as complex and represents solutions as embeddings of Riemann surfaces into a complex phase space. This approach was used already in the works of K. Gauss (on the hypergeometric equations) and of B. Riemann (in the so-called Riemann–Hilbert problem).

The aim of this article is to present the local theory of analytic Ordinary Differential Equations. It is an analogue of the singularity theory of holomorphic functions (generalization of the Morse Lemma). It turns out that the classification of singularities of differential equations is more complicated than in the geometrical case. It is related with the Stokes phenomenon for linear differential systems, when some asymptotic expansions of solutions are valid only in some sectors (in the complex time plane) and the 'differences' between the expansions in adjacent sectors constitute functional moduli of the singularity. There exists also a non-linear version of the Stokes phenomenon in the orbital classification of holomorphic planar vector fields near elementary singular point. In the case of non-elementary singularity the situation is even more complicated, e.g. in the nilpotent case only the formal orbital classification is known.

2. Linear differential equations

The subject in this chapter are the non-autonomous linear differential systems

$$\dot{z} = A(t)z, \quad z \in \mathbb{C}^n, \tag{2.1}$$

and the linear higher order differential equations

$$x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = 0, \quad x \in \mathbb{C}. \tag{2.2}$$

In the above two equations the 'time' t usually takes value in the complex plane $\mathbb{C}$. Often we will consider Eqs. (2.1) and (2.2) locally, then $t \in (\mathbb{C}, 0)$.

The entries of the matrix $A(t)$ and the coefficients $a_i(t)$ are meromorphic functions.

The equations of this type are met very often. For example, the *Gauss hypergeometric equation*

$$t(t-1)\ddot{x} + \big[(\alpha+\beta+1)t - \gamma\big]\dot{x} + \alpha\beta x = 0 \tag{2.3}$$

belongs to this class. Probably it is the most important linear differential equation. Most of the differential equations appearing in the theory of special functions, e.g. the Bessel equation, the Weber equation, the Legendre equation, the Hermite equation, the Jacobi equation, are obtained from the hypergeometric equation by application of some limit process.

Algebraic functions are also solutions of such differential equations. It follows from the Riemann theorem about multivalued holomorphic functions with regular singularities (see [89]). For example, the function $x(t) = \sqrt{t} + \sqrt[3]{t-1}$ satisfies the equation

$$(t-3)\frac{\mathrm{d}}{\mathrm{d}t}\big(6t(t-1)\dot{x}\big) = 3(3t-1)\big[3(t-1)\dot{x} - x\big] + 2(4t-3)[2t\dot{x} - x].$$

DEFINITION 2.1. A point t_* at which the matrix $A(t)$ in (2.1) or one of the coefficients $a_i(t)$ in (2.2) has a pole is the *singular* point for the equation. In the case of local situation we shall assume that $t_* = 0$.

The singular point $t = 0$ of Eq. (2.1) or (2.2) is called *regular* if any its solution $\phi(t)$ has at most polynomial growth in any sector with vertex at $t = 0$. It means that

$$|\phi(t)| < C_N/|t|^N, \quad |t| \to 0,\ \alpha < \arg t < \beta.$$

Otherwise the singular point is called *irregular*.

The division of the singular points of systems (2.1) into regular and irregular is very rough. Our aim is to classify such systems with respect to equivalence relations defined below.

DEFINITION 2.2. Two local differential systems $\dot{z}_1 = A_1(t)z_1$ and $\dot{z}_2 = A_2(t)z_2$ are called *holomorphically equivalent* near the singular point $t = 0$ if there exists a holomorphic matrix $H(t)$ such that, after the change $z_2 = H(t)z_1$, the first system is transformed to the other system. In other words,

$$A_2 = \dot{H}H^{-1} + HA_1H^{-1}.$$

These systems are called *meromorphically equivalent* (respectively, *formally equivalent* or *formally meromorphically equivalent)* if there exists a meromorphic matrix $H(t)$ (respectively a formal series $H(t) \sim \sum_{j=0}^{\infty} H_j t^j$ or $\sum_{j=-m}^{\infty} H_j t^j$) which transforms one system to another system. In the formal cases this holds at the level of formal expansions of these systems in powers of t.

2.1. *Regular singularities*

We begin the study of singularities of the meromorphic differential equations with the one-dimensional phase space, $n = 1$. It means that we have the equation

$$t^r \dot{z} = a(t)z$$

where $a(t)$ is an analytic function near 0.

If $r = 0$ then the solutions are analytic; in fact, the point $t = 0$ is not singular for this equation.

If $r > 0$ then we assume that the coefficient $a_0 \neq 0$ in the expansion $a(t) = a_0 + a_1 t + \cdots$. The general solution is written in the form

$$z(t) = C \cdot \exp\left[\int^t a(s)s^{-r}\, \mathrm{d}s\right].$$

If $r = 1$ then we get $z(t) = Ct^{a_0} \times \Psi(t)$, where Ψ is an analytic function. This means the regularity of the singular point.

If $r > 1$ then $z(t) = C\mathrm{e}^{R(1/t)} t^b \times \Psi(t)$, Ψ analytic, where

$$R(x) = \frac{a_0}{1-r} x^{r-1} + \frac{a_1}{2-r} x^{r-2} + \cdots - a_{r-2}x,$$

$b = a_{r-1}$. Here the singularity is irregular, because for $\mathrm{Re}(a_0 t^{1-r}) < 0$ the solution diverge faster than any power of t. We proved the following

PROPOSITION 2.3. *If $n = 1$ then the singular point of Eq.* (2.1) *is regular if and only if the function $A(t)$ has pole of order exactly* 1.

In general, Eq. (2.1) *is analytically equivalent to the equation $t^r \dot{z} = (\sum_0^{r-1} a_j t^j)z$.*

Consider now the case of n-dimensional system with the following singularity

$$t\dot{z} = Az,$$

where A is a constant matrix. The fundamental matrix of solutions of this system is

$$t^A = \mathrm{e}^{A \ln t},$$

which grows at most polynomially (regularity). This fact is generalized in the following way.

PROPOSITION 2.4. *If $A(t) = A_0/t + A_1(t)$, where $A_0 = \mathrm{const} \neq 0$ and $A_1(t)$ is a holomorphic matrix-valued function, then the point $t = 0$ is regular.*

PROOF. We put $t = \rho \mathrm{e}^{\mathrm{i}\theta}$, where $\theta = \mathrm{const}$ and ρ tends to 0. Let $\phi(t) \not\equiv 0$ be a solution, and $r(\rho) = |\phi(t)|_{\theta=\mathrm{const}}|$.

From the identity $\frac{\partial\phi}{\partial\rho} = e^{i\theta}\dot{\phi} = (\tilde{A}(t)/\rho)\phi$ (where $\tilde{A}$ is holomorphic) we get the inequality $\frac{dr}{d\rho} > -\frac{Kr}{\rho}$ for $0 < \rho < \rho_0$, with some constant K and ρ_0.

From this we see that the graph of the function $r(\rho)$ must lie in the domain $\{r < \rho^{-K}\}$ for ρ near 0. □

DEFINITION 2.5. System (2.1), i.e. $\dot{z} = A(t)z$, has singularity of the *Fuchs type* at $t = 0$ if $A(t)$ has simple pole at 0. Equation (2.2), i.e. $x^{(n)} + \cdots + a_n(t)x = 0$, has singularity of the *Fuchs type* at $t = 0$ if all the functions $t^j a_j(t)$ are holomorphic near 0.

Proposition 2.4 says that, if the system (2.1) has singularity of the Fuchs type, then this singularity is regular. The following example shows that the converse is not true.

EXAMPLE 2.6. The *Euler equation*

$$t^n x^{(n)} + c_1 t^{n-1} x^{(n-1)} + \cdots + c_n x = 0,$$

where c_i are constants, by means of the change $t = e^u$, $\frac{d}{dt} = e^{-u}\frac{d}{du}$, $\frac{d^2}{dt^2} = e^{-2u}(\frac{d^2}{du^2} - \frac{d}{du})$, … is reduced to the equation

$$\frac{d^n x}{du^n} + d_1 \frac{d^{n-1} x}{du^{n-1}} + \cdots + d_0 x = 0$$

with constant coefficients. Its general solution has the form $x = \sum a_{\alpha,k} e^{\alpha u} u^k = \sum a_{\alpha,k} t^\alpha (\ln t)^k$. Thus the point $t = 0$ is regular. On the other hand, if we rewrite the Euler equation in form of the linear system

$$\dot{z} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -c_n/t^n & -c_{n-1}/t^{n-1} & \dots & \dots & -c_1/t \end{pmatrix} z$$

(where $z_1 = x, z_2 = \dot{x}, \ldots$) then the corresponding matrix $A(t)$ has non-simple pole.

In the case of linear equation of n-th order the situation is much more clear.

THEOREM 2.7. *The singular point* $t = 0$ *for Eq.* (1.2) *is regular if and only if it is of Fuchs type*.

Before proof of this theorem we need one important definition.

DEFINITION 2.8. Let t_0 be a non-singular point of Eq. (2.1) or (2.2), defined near the singularity $t = 0$. Let $\mathcal{V}$ be the space of its solutions defined near t_0. As t turns around the point 0, with the beginning and end at t_0, the solutions are prolonged analytically and define the *monodromy operator* $\mathcal{M} : \mathcal{V} \to \mathcal{V}$: if $\phi(t)$ is a (local) solution then $\mathcal{M}\phi(t) = \phi(e^{2\pi i} t)$.

Let $\phi_1(t), \dots, \phi_n(t)$ be a basis of $\mathcal{V}$. This basis defines the *fundamental matrix* $\mathcal{F} = (\phi_1, \dots, \phi_n)$ in the case (2.1) and

$$\mathcal{F} = \begin{pmatrix} \phi_1 & \dots & \phi_n \\ \vdots & \ddots & \vdots \\ \phi_1^{(n-1)} & \dots & \phi_n^{(n-1)} \end{pmatrix}$$

in the case (2.2). Then the monodromy operator, written in this basis, is the *monodromy matrix* M satisfying $\mathcal{F}(t)M = \mathcal{F}(\mathrm{e}^{2\pi\mathrm{i}}t)$. Here the matrix M depends on the choice of $\mathcal{F}$ but locally does not depend on the point t_0.

PROOF OF THEOREM 2.7. (We follow the proof of Theorem 5.2 in the book of E. Coddington and N. Levinson [24].)

Assume that the Fuchs conditions, i.e. all $t^j a_j(t)$ are holomorphic, hold. We are going to show that the singularity $t = 0$ is regular.

Following the latter example we perform a transformation reducing equation (2.2) to a system (1.1) with the Fuchs type singularity. Then Proposition 2.4 will give us the regularity property.

The change is following

$$z_1 = x, \quad z_2 = (t\,\mathrm{d}/\mathrm{d}t)x, \quad z_3 = (t\,\mathrm{d}/\mathrm{d}t)^2 x, \dots, z_n = (t\,\mathrm{d}/\mathrm{d}t)^{n-1} x.$$

It gives $\dot z_1 = (1/t)z_2$, $\dot z_2 = (1/t)z_3, \dots, \dot z_{n-1} = (1/t)z_n$ and $z_j = t^j x^{(j)}$ plus a combination of $t^k x^{(k)}$, $k < j$. Thus $t^j x^{(j)}$ can be expressed as linear combination of z_k, $k \leqslant j$.

The last equation from the promised system is $\dot z_n = \frac{\mathrm{d}}{\mathrm{d}t}(t^{n-1}x^{(n-1)}) +$ plus combination of $\dot z_i$, $i < n$. The term $\frac{\mathrm{d}}{\mathrm{d}t}(t^{n-1}x^{(n-1)})$ contains $(n-1)z_n/t$ and $t^{n-1}x^{(n)} = -t^{n-1}\sum a_j x^{(n-j)} = -t^{-1}\sum(a_j t^j)(x^{(n-j)}t^{n-j})$. The functions $a_j t^j$ are holomorphic by the assumption and $x^{(n-j)}t^{n-j}$ are expressed by means of z_i's. All this shows that $t\dot z_n$ is analytic in t and z.

Assume that $t = 0$ is a regular point. We want to show that the functions $b_j(t) = a_j(t)t^j$ are analytic.

We use the monodromy operator. If $\mathcal{F}$ is the fundamental matrix and $\mathcal{F} \to \mathcal{F}M$ is the monodromy operation and $G(t) = t^{-\ln M/2\pi\mathrm{i}}$, then the matrix-function $\mathcal{F}G$ is univalent. By the regularity of $\mathcal{F}$ it is a meromorphic function. Thus $\mathcal{F} =$ (meromorphic matrix) $\times\, t^{\ln M/2\pi\mathrm{i}}$.

The monodromy matrix M has always some (left) eigenvector $v = (v_1, \dots, v_n)$, $vM = \lambda v$. Using it we can find a solution $\phi(t) = \sum v_i\phi_i$ of the form $t^\mu \times$ (analytic function), where the exponent μ satisfies $\mathrm{e}^{2\pi\mathrm{i}\mu} = \lambda$. Note that the function $\phi(t)$ does not contain logarithms in its expansion.

Other solutions are searched for in the form $x = \phi \cdot y$. For $u = \dot y$ we obtain an equation of order $n - 1$.

Indeed, if we define the differential operator $P = (\mathrm{d}/\mathrm{d}t)^n + \sum a_j (\mathrm{d}/\mathrm{d}t)^{n-j}$ then $P(\phi y) = P(\phi) \cdot y$ plus expressions depending on $\dot{y}$. The original equation for x gives an equation is of the form

$$\sum c_j(t) u^{(j)} = 0,$$

where $c_0 = a_0 = 1$ and

$$c_j = a_j + \mathrm{const} \cdot a_{j-1} \cdot \dot{\phi}/\phi + \mathrm{const} \cdot a_{j-2} \cdot \ddot{\phi}/\phi + \cdots, \quad \phi^{(l)}/\phi \sim \mathrm{const} \cdot t^{-l}.$$

Next we use induction with respect to the order n. In the case $n = 1$ the statement was proved in Proposition 2.3; the regularity implies Fuchs property.

Assume that we know that any regular equation of order $n - 1$ satisfies the Fuchs conditions and consider our system.

Let ψ_i be the system of independent solutions of the equation for u. Then $\psi_i = \frac{\mathrm{d}}{\mathrm{d}t}(\phi_i/\phi)$, where ϕ_i are independent solutions of the equation for x. Because all ϕ_i are regular (by assumption) and $\phi \sim t^\mu$ then also $\psi_i(t)$ behave regularly. Therefore the equation for u has regular singularity at $t = 0$. By the induction assumption the coefficients $c_j(t)$ have good behavior, $c_j t^j$ are analytic. Because of the above relation between a_j's and c_k's we find that also the functions $a_j t^j$ are holomorphic. □

PROPOSITION 2.9. *The point $t = 0$ is a regular singular point for the system $\dot{z} = A(t)z$ if and only if this system is meromorphically equivalent to the system $\dot{z} = (C/t)z$ where the matrix C is the logarithm of the monodromy matrix M, $M = \mathrm{e}^{2\pi \mathrm{i} C}$.*

PROOF. Let $\mathcal{F}(t)$ be the fundamental matrix of the system with the matrix A. Define $H(t) = \mathcal{F}(t) t^{-C}$. It is univalent and meromorphic matrix-valued function (by the regularity). It is easy to check that $H(t)$ is the conjugating transformation.

The reverse implication is obvious. □

Our next aim is to improve the latter result. Let us rewrite system (2.1) in the form

$$t^r \dot{z} = B(t) z, \quad B(0) \neq 0, \tag{2.4}$$

with an analytic matrix $B(t)$. This system is associated with the following autonomous system in the extended phase space

$$z' = B(t) z, \quad t' = t^r, \tag{2.5}$$

where $' = \mathrm{d}/\mathrm{d}\tau$. Note that the graphs of solutions to Eq. (2.4) form the phase curves of Eq. (2.5). Assume also that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the matrix $B(0)$.

The point $z = 0$, $t = 0$ is an equilibrium point of system (2.5). If $r = 1$, i.e. in the Fuchs case, then the eigenvalues of the linear part at $(0, 0)$ of system (2.5) are $\lambda_1, \ldots, \lambda_n, 1$. If $r > 1$, then the eigenvalues of system (2.5) are $\lambda_1, \ldots, \lambda_n, 0$.

DEFINITION 2.10. The system $\lambda_1, \dots, \lambda_n$ of eigenvalues of $B(0)$ is *resonant* if:
(a) $\lambda_i - \lambda_j \in \mathbb{Z}$ for some $i \neq j$ in the case $r = 1$;
(b) $\lambda_i - \lambda_j = 0$ for some $i \neq j$ in the case $r > 1$.

Now we use some results from the theory of normal forms for germs of analytic vector fields $(\mathbb{C}^n, 0)$

$$\dot{x} = V(x) = Ax + \cdots,$$

where A is a constant matrix of the linearization of the vector field at 0 (with the eigenvalues $\mu_1, \dots \mu_n$). The *Poincaré–Dulac theorem* (Theorem 3.17 from the next section) says that:

there exists a change $x \to y = G(x)$, where $G(x)$ is given by formal power series, which transforms the above vector field to a formal vector field $\dot{y} = W(y)$, such that the expansion of each component $W_j(y)$ of W is of the form

$$W_i(y) = \mu_i y_i + \sum a_{i,k} y^k$$

and the summation runs over multiindices $k = (k_1, \dots, k_n)$ which satisfy the resonance relations

$$\mu_i = \mu_1 k_1 + \cdots + \mu_n k_n.$$

Moreover, if the components $V_1(x), \dots, V_k(x)$ are linear with respect to the variables $x_1, \dots, x_k$ then also the components $G_1(x), \dots, G_k(x)$ and $W_1(y), \dots, W_k(y)$ are linear with respect to suitable variables.

We apply this result to vector field (2.5), where we use changes of the form $(z, t) \to (H(t)z, t)$. (We note that the idea to apply the Poincaré–Dulac theorem to linear systems was used firstly in the review article by Arnold and Il'yashenko [5].) The reader can notice that the resonant relations from the Poincaré–Dulac theorem correspond to the relations between the eigenvalues of the matrix $B(0)$ defined in Definition 2.10. The corresponding resonant terms in the normal form are

$$t^l z_j e_i,$$

i.e. if $\lambda_i = \lambda_j + l$, where $(e_1, \dots, e_n)$ is the eigenvector basis for the matrix $B(0)$.

COROLLARY 2.11. *If the singular point $t = 0$, with $r = 1$ or $r > 1$, is non-resonant then the Poincaré–Dulac theorem says that system* (2.4) *is formally equivalent to the system $t^r \dot{z} = B(0)z$.*

On the other hand, Proposition 2.9 says about the meromorphic equivalence. But the formal equivalence means that, in the meromorphic matrix $H(t) = H_{-d} t^{-d} + \cdots$, all the terms with negative power of t vanish. This gives the following result.

THEOREM 2.12. *The system $t\dot{z} = B(t)z$ with non-resonant Fuchsian singular point is analytically equivalent to $t\dot{z} = B(0)z$.*

Of course, in the above theorem we can assume that the matrix $B(0)$ is diagonal. Hence the normal form variables are completely separated. In the resonant case we can divide the eigenvalues into groups such that:

- the differences $\lambda_i - \lambda_j \notin \mathbb{Z}$ for eigenvalues from different groups,
- the differences $\lambda_i - \lambda_j \in \mathbb{Z}$ for eigenvalues from the same group.

We order the eigenvalues in each group such that the differences $\lambda_i - \lambda_j$ are non-negative integers for $j < i$, $\lambda_i - \lambda_j \in \mathbb{Z}_+$. We have the following result whose proof is postponed to Subsection 2.3 below.

THEOREM 2.13. *If a Fuchsian singular point is resonant then it is analytically equivalent to a system of the form*

$$t\dot{z}_i = \lambda_i z_i + \sum_{\lambda_i - \lambda_j \in \mathbb{Z}_+} a_{ij} t^{\lambda_i - \lambda_j} z_j, \quad i = 1, \dots, n, \ a_{ij} = \text{const.}$$

REMARK 2.14. The system from the thesis of Theorem 2.13 can be easily integrated. Its general solution has the form $z_i(t) = t^{\lambda_i} \times$ (polynomial of $\log t$). For more details we refer the reader to [24].

REMARK 2.15. If a singular point $t = 0$ is regular for Eq. (2.2) and $a_i(t) = \nu_i t^{-i} + \cdots$ then we can seek solutions ϕ such that $\phi(t) \sim Ct^{\lambda}$. We arrive to the equation

$$P(\lambda) = \lambda(\lambda - 1) \cdots (\lambda - n + 1) + \lambda \cdots (\lambda - n + 2) \cdot \nu_1 + \cdots + \nu_n = 0$$

which is called the *defining equation* for (2.2).

The defining equation corresponds to the characteristic equation for the matrix $B(0)$ from the system (2.4). Namely, we apply the transformation of Eq. (2.2) to a Fuchsian system from the proof of Theorem 2.7. This Fuchsian system has the form (2.4) with $r = 1$.

The roots of the defining equation allow to determine the first terms of the asymptotic of solutions of the differential equation.

If the system $(\lambda_i, \dots, \lambda_n)$ of roots is not resonant in the sense of Definition 2.10 (for $r = 1$) then the general solution is a linear combination of functions of the form $\phi_i(t) = t^{\lambda_i} \times$ (analytic function). Otherwise we have also functions of the form $t^{\lambda_i} \times (\log t)^j \times$ (analytic function).

EXAMPLE 2.16. Consider the Gauss hypergeometric equation (2.3), i.e. $t(t-1)\ddot{x} + [(\alpha + \beta + 1)t - \gamma]\dot{x} + \alpha\beta x = 0$. It has singular points at $t = 0$, $t = 1$ and also at $t = \infty$, all Fuchsian.

The defining equation at $t = 0$ is $\lambda(\lambda - 1 + \gamma) = 0$ and the two linearly independent solutions are $\phi_1 = F(\alpha, \beta; \gamma; t)$ and $\phi_2 = t^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; t)$, where $F(\alpha, \beta; \gamma; t) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} t^n$, $(a)_n = a(a+1) \cdots (a + n - 1)$, is the so-called the *hypergeometric series*. This holds when γ is not integer and for integer γ one of

these functions is not defined. For example, when $\gamma = 0$ the solution ϕ_2 is the same but $\phi_1 = tF(\alpha, \beta; 1; t) \cdot \log t + G(t)$, where $G(t)$ is some uniquely defined series.

For $t = 1$ the defining equation is $\lambda(\lambda - \gamma + \alpha + \beta) = 0$ and the (generically) independent solutions are $F(\alpha, \beta; \alpha + \beta + 1 - \gamma; 1 - t)$ and $t^{\gamma-\alpha-\beta} F(\gamma - \beta, \gamma - \alpha; 1 - \alpha - \beta + \gamma; 1 - t)$.

Near $t = \infty$ one rewrites the Gauss equation in the local variable $\tau = 1/t$. The defining equation is $(\lambda - \alpha)(\lambda - \beta) = 0$ and the functions $\tau^\alpha F(\alpha, 1 - \gamma + \alpha; 1 + \alpha - \beta; \tau)$, $\tau^{-\beta} F(\beta, 1 - \gamma + \beta; 1 + \beta - \alpha; \tau)$ constitute the basis of the space of solutions.

2.2. *Irregular singularities*

Firstly we find a formal normal form for linear systems (2.4) with an irregular singularity.

THEOREM 2.17. *If $r > 1$ and $B(0)$ is non-resonant then the system $t^r \dot{z} = B(t)z$ is formally equivalent to the system*

$$t^r \dot{w} = D(t)w, \tag{2.6}$$

where the matrix $D(t)$ is diagonal, $D = \operatorname{diag}(d_1(t), \ldots, d_n(t))$, and $d_i(t)$ are polynomials of degree at most $r - 1$.

PROOF. By the Poincaré–Dulac theorem all the terms $t^k z_j e_i$, $i \neq j$, can be canceled (because they are non-resonant). There remain only the terms $t^k z_i e_i$ which are diagonal. Next one applies Proposition 2.3 to each component. □

If we write $D(t)t^{-r} = D_r t^{-r} + D_{r-1} t^{-r+1} + \cdots + D_2 t^{-2} + E t^{-1}$, where D_j and E are diagonal matrices, then the formal fundamental matrix of the system from Theorem 2.17 takes the form $\mathcal{F}(t) = \widehat{G}(t) t^E \exp[\sum D_j t^{j+1}/(j+1)]$, where $\widehat{G}$ is a formal power series.

In the resonant case there exists also a diagonal formal normal form, but in another category. One should change the local ring $\mathbb{C}[[t]]$ (of formal power series in t) by the ring $\mathbb{C}[[t^{1/b}]]$, where b is some positive integer.

EXAMPLE 2.18. (See [35,82].) The system

$$\dot{z}_1 = (1/t) z_3, \quad \dot{z}_2 = (1/t^2) z_1, \quad \dot{z}_3 = (1/t^2) z_2$$

is of the form $t^2 \dot{z} = B(t)z$, where the matrix $B(0)$ is nilpotent. The substitution $y_1 = t^{-1/3} z_1$, $y_2 = z_2$, $y_3 = t^{1/3} z_3$ gives

$$\dot{y}_1 = t^{-5/3} y_3 - (1/3t) y_1, \quad \dot{y}_2 = t^{-5/3} y_1, \quad \dot{y}_3 = t^{-5/3} y_2 + (1/3t) y_3.$$

Thus we get $\dot{y} = (t^{-5/3} F + t^{-1} E) y$, where

$$F = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad E = \operatorname{diag}(-1/3, 0, 1/3).$$

We see that the leading matrix F has three distinct eigenvalues, the cubic roots of -1. Transforming it to the diagonal form and applying the series of transformations from the proof of Theorem 2.17, which are power series in $t^{1/3}$, we can diagonalize also the remaining part of the system for y.

A general result concerning normal forms in the case of resonant irregular singular point belongs to M. Hukuhara H. Turrittin and A. Levelt and is formulated in the next theorem.

THEOREM 2.19. (See [39,81,46].) *If $t = 0$ is an irregular singular point of the system $\dot{z} = A(t)z$ then there exist a positive integer b and a change $y = H(t^{1/b})z$ in the class of formal series in $s = t^{1/b}$, such that H transforms the system for z to a system of the form*

$$\dot{y} = (D_m t^{-m/b} + D_{m-1} t^{-(m-1)/b} + \cdots + D_{b+1} t^{-(b+1)/b} + E t^{-1})y \tag{2.7}$$

where the matrices D_j and E are diagonal.

Two forms (2.7) *with D_j, E and D'_j, E' are formally equivalent* (*over* $\mathbb{C}[[t^{1/b}]]$) *iff the matrices D_j and D'_j and E and E' are mutually spontaneously conjugated by means of some matrix from* $\mathrm{GL}(n, \mathbb{C})$.

REMARK 2.20. In the variable $s = t^{1/b}$ system (2.7) takes the form

$$\mathrm{d}y/\mathrm{d}s = b(D_m s^{b-m-1} + \cdots + D_{b+1} s^{-2} + E s^{-1})y.$$

The number $r_1 = \frac{m}{b}$ is called the *Katz invariant* and the smallest integer b is called the *ramification index*.

PROOF OF THEOREM 2.19. (We follow the papers [72] of Y. Sibuya and [82] of V. Varadarajan and the book [89].)

Assume that we have $t^r \dot{z} = B(t)z$.

1. Repeating the proof of Theorem 2.17 we show that the division of the matrix $B(0)$ into diagonal blocks corresponding to different eigenvalues λ_i can be prolonged to a formal splitting of the system into several independent systems, which are characterized by having only one eigenvalue.

2. Let $\lambda_1 = \cdots = \lambda_n = \lambda$. The change $z = \mathrm{e}^{\lambda/(1-r)t^{r-1}} y$ leads to a system $t^r \dot{y} = B(t)y$ (with new $B(t)$) such that

$$B(0) = \begin{pmatrix} 0 & d_1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & d_{n-1} \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

where $d_i = 0, 1$

3. Assume that all $d_j = 1$. We introduce the variables $u_1 = y_1$, $u_2 = t^r \dot{u}_1, \dots, u_n = t^r \dot{u}_{n-1}$. Then we obtain a system $t^r \dot{u} = B(t)u$ such that

$$B(t) = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ b_1(t) & b_2(t) & \dots & b_n(t) \end{pmatrix},$$

where each $b_j(t) = \mathrm{O}(t)$.

Now the substitution $u_j = t^{a_j} v_j$, with suitable rational exponents a_j, leads to the equation $\dot{v} = (t^{-\alpha} C + \cdots)v$ where the matrix C has the form like the matrix $B(0)$ but with at least one constant $b_j(0) \neq 0$. This means that C has at least one non-zero eigenvalue and we can apply the induction with respect to the dimension of the eigenspace.

4. If some of the d_i's in the matrix $B(0)$ (from the point 2) are equal to zero then, using the change $y \to u$ analogous as in the point 3, we reduce the matrix $B(t)$ to a matrix

$$\begin{pmatrix} B_1 & E_{12} & \dots \\ E_{21} & B_2 & \dots \\ \dots & \dots & \dots \end{pmatrix},$$

such that $B_j = B_j(t)$ are of the form from the point 3 and

$$E_{ij} = \begin{pmatrix} 0 & \dots & 0 \\ e_1(t) & \dots & e_s(t) \end{pmatrix},$$

where $e_k(t)$ are functions. It means that most of the rows of $B(t)$ consist of zeroes and the other rows are composed of functions $b_k(t)$ and $e_l(t)$.

One can show that, after applying a suitable change $u \to v$ of the same form as in the point 3, we obtain $\dot{v} = (t^{-\alpha} C + \cdots)v$ where either:

(i) the matrix C has a non-zero eigenvalue (if some $b_k(0)$ becomes $\neq 0$ or some $e_l(0) \neq 0$ in some E_{ij}, $i > j$), or

(ii) the matrix C is nilpotent but its Jordan normal form contains more non-zero d_j's than in the matrix $B(0)$ from the point 2 (here some $e_l(0) \neq 0$ in a $E_{i,j}$, $i < j$), or

(iii) none of the above.

In the cases (i) and (ii) we apply the induction with respect to the dimension of the eigenspace and to the number of non-zero d_j's. (See also [82].)

5. The case (iii) occurs when some supposedly non-trivial row of $B(t)$ from the point 4 vanishes identically; (such phenomenon has infinite codimension). Note that all the changes used in the above point are polynomial in $s = t^{1/b}$. Therefore the system becomes analytically separated, although the first diagonal matrix D_m from the thesis of Theorem 2.19 has coinciding eigenvalues. □

Having solved the problem of formal classification of meromorphic systems with irregular singularity we pass to the analytic classification. It turns out that formal solutions of such systems (or higher order equations) can be divergent. The series from the formal normal forms are only asymptotic.

EXAMPLE 2.21. (See [34].) The series $\sum k!t^k$ formally satisfies the equation $t^2\ddot{x} + (3t-1)\dot{x} + x = 0$ but is divergent.

In what follows we assume that the eigenvalues of the matrix D_m from the thesis of Theorem 2.19 are different. From the point 5 of the above proof it follows that it is sufficient for the analytic classification.

We apply the change of time $t \to t^{1/b}$, and the new time we again denote by t. We apply a polynomial change $z \to \sum_{j=0}^N H_j(t)z$, which reduces the polar part of the matrix $A(t) = B/t^r$ to the diagonal form. Thus we get a system $t^r\dot{z} = B(t)z$, where $B(t) = D(t) + O(t^r)$ (we do not change notation of B) and whose formal normal form is (2.6), i.e. $t^r\dot{y} = D(t)y$ with diagonal and polynomial $D(t)$.

Recall that the eigenvalues of $B(0) = D(0)$ satisfy the condition (see the remark above)

$$\lambda_i \neq \lambda_j, \quad i \neq j.$$

If $i \neq j$ then the rays in the complex t plane defined by

$$\mathrm{Re}\big[(\lambda_i - \lambda_j)t^{1-r}\big] = 0$$

are called the *rays of division corresponding to the pair* (λ_i, λ_j). The following theorem of H. Poincaré is principal in this subsection. Its proof is sketched in Appendix.

THEOREM 2.22. (See [67].) *Let $S = \{|t| < \epsilon, \alpha < \arg t < \beta\}$ be a sector in the t-plane not containing two rays of division* (*corresponding to any pair* (λ_i, λ_j)). *Then there exists a unique matrix function $H(t) = I + O(t)$ analytic in S which transforms the system $t^r\dot{z} = B(t)z$ to the formal normal form system* (2.6).

Let S_1 and S_2 be two adjacent sectors (i.e. with non-empty intersection) satisfying the assumptions of Theorem 2.22. Let H_{S_1} and H_{S_2} be the corresponding operators guaranteed by this theorem. From Theorem 2.17 it follows that these two matrix-functions have the same Taylor expansions. Because these Taylor expansions are only asymptotic, then H_{S_i} are generally different in the intersection of the two sectors.

One defines the matrix-function

$$G_S(t) = H_{S_1}H_{S_2}^{-1}, \quad t \in S = S_1 \cap S_2.$$

Let $\mathcal{L}_S$ be the space of solutions of the normalized system. The transformation induced by G_S transforms solutions from $\mathcal{L}_S$ to solutions from the same space. Therefore it defines certain automorphism $\mathcal{C}_S : \mathcal{L}_S \to \mathcal{L}_S$.

If we choose a basis of $\mathcal{L}_S$ in the form

$$\phi_j = a_j(t)e_j, \quad a_j(t) = \exp\big[\lambda_j t^{1-r}/(1-r) + \cdots\big],$$

where (e_j) is the standard basis of $\mathbb{C}^n$, then the operator $\mathcal{C}_S$ expressed in this basis is a constant matrix $C_S = (c_{ij})$.

DEFINITION 2.23. The operator $\mathcal{C}_S$ which is called the *Stokes operator* and the matrix C_S is called the *Stokes matrix*.

If $\psi_i = G_S(t)\phi_i(t) = C_S\phi_i = \sum_j c_{ij}\phi_j$ then the matrix G_S acts as follows: $G_S e_i = \psi_i/a_i(t)$. We have

$$(G_S)_{ij} = a_i^{-1}(t) \cdot c_{ij} \cdot a_j(t) = c_{ij} \cdot \exp\bigl[(\lambda_j - \lambda_i)t^{1-r}/(1-r) + \cdots\bigr].$$

PROPOSITION 2.24. *The matrix $G_S(t)$ preserves the normalized system* (2.6) *and has the property $G_S(t) = I + O(t^N)$ as $t \to 0$ for any N.*

This implies that: $c_{ii} = 1$ *and* $c_{ij} = 0$ *when* $\mathrm{Re}(\lambda_i - \lambda_j)t^{1-r} \to -\infty$ *as* $t \to 0$. *Therefore, after suitable ordering of the basis (e_i), we obtain that the matrix C_S is unipotent upper triangular.*

The system of Stokes operators can be naturally described using the cohomology language. Take the circle S^1, which we treat as the circle $\{r = 0\}$ in $\mathbb{R}_+ \times S^1 = \{(r, \theta)\}$ where r, θ are the polar coordinates of the $t = r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C}$. If $U \subset S^1$ is an open arc then we associate with a germ of sector $S = S_U$ in $(\mathbb{C}, 0)$ with base at U.

DEFINITION 2.25. The *Stokes sheaf St* is defined by the presheaf of groups $\mathrm{St}(U)$, $U \subset S^1$ (as above), where $\mathrm{St}(U)$ consists of matrix-functions $S = S_U \ni t \to G_U(t)$ satisfying the following properties:

- $G_U(t) \sim I$,
- $G_U(t)$ preserve the normalized system.

REMARK 2.26. St is not a standard sheaf with which we are acquainted. The groups $\mathrm{St}(U)$ are (generally) non-Abelian groups and the Čech coboundary operator, Čech cocycles and Čech cohomologies must take proper meaning. Here we recall the definition of the first cohomology group of sheaves $\mathcal{F}$ of non-Abelian groups.

The cochains are the same as in the Abelian theory. In particular, a 0-cochain associated with a covering $\mathcal{U} = \{U\}$ is a system $(G_U)_{U \in \mathcal{U}}$ with $G_U \in \mathcal{F}(U)$. 1-cochains are systems (G_{UV}) with $G_{UV} \in \mathcal{F}(U \cap V)$ such that $G_{VU} = G_{UV}^{-1}$. This 1-cochain is a cocycle iff $G_{UV}G_{VW}G_{WU} = e$. Two 1-cochains (G_{UV}) and (G'_{UV}) are equivalent iff there is 0-cochain (K_U) such that

$$G'_{UV} = K_U G_{UV} K_V^{-1}.$$

The first cohomology group $H^1(\mathcal{F}, \mathcal{U})$ is defined as the set of classes of 1-cocycles with respect to the this equivalence. Such cocycles with $G_{UV} \in \mathrm{St}(U \cap V)$ are called the *Stokes cochains*.

Next one should takes the direct limit with respect to the coverings, but usually fine finite coverings are sufficient.

A Stokes cochain can be interpreted as a cocycle of Stokes matrices. If $\mathcal{F}_0$ is the fundamental matrix of the normalized system, then

$$C_{UV} = \mathcal{F}_0 G_{UV} \mathcal{F}_0^{-1}$$

is a cocycle of constant matrices. Two Stokes cocycles (C_{UV}) and (C'_{UV}) are called equivalent iff there exists a system (C_U) of constant matrices such that $C'_{UV} = C_U^{-1} C_{UV} C_V$.

If we calculate the cohomology group of the Stokes sheaf using C_{UV} then we see that it is finite-dimensional.

Now we pass to the description of moduli of analytic classification of linear systems with irregular singular point. We fix the normalized system (2.6), which we denote by $\mathcal{S}_0$, and consider the space $\mathcal{M} = \mathcal{M}_D$ of systems $\mathcal{S}: t^r \dot{z} = B(t)$ which are formally equivalent to $\mathcal{S}_0$. On the space $\mathcal{M}$ we introduce the following equivalence relation (compare Definition 2.2):

$\mathcal{S} \sim \mathcal{S}'$ iff there is a matrix function $H(t)$ holomorphic in a (*whole*) *neighborhood of $t = 0$ and realizing equivalence between $\mathcal{S}$ and $\mathcal{S}'$.*

The following fundamental in this theory result was proved by B. Malgrange and Y. Sibuya and is called also the *Malgrange–Sibuya Theorem*.

THEOREM 2.27. (See [52,73].) *The space of equivalence classes defined above coincides with the first cohomology group of the circle with coefficients in the Stokes sheaf. In other words, the group $H^1(S^1, St)$ parameterizes the orbits of the action of the group of analytic equivalences on the space of germs of meromorphic linear systems with fixed formal normal form.*

This property holds in the non-resonant case as well as in the resonant case with the formal normal form the thesis of Theorem 2.19.

PROOF. Let $\mathcal{M}$ be the above space of the equivalence classes. We define a natural map from $\mathcal{M}$ to $H^1(S^1, St)$.

Take a system $\mathcal{S} \in \mathcal{M}$. By the sectorial normalization theorem there exists a covering of a neighborhood of the point $t = 0$ by sectors S_i such that $\mathcal{S}$ is transformed to $\mathcal{S}_0$ by means of holomorphic maps $H_i(t)$ in S_i. Let U_i be the bases of the sectors S_i. Then the family of operators $H_i H_j^{-1}$ defines a Stokes cocycle associated with the covering $\{U_i\}$.

If two systems $\mathcal{S}$ (defining a cocycle $H_i H_j^{-1}$) and $\mathcal{S}'$ (defining a cocycle $H'_i (H'_i)^{-1}$) are equivalent by means of a holomorphic matrix $H(t)$, then $H'_i = H_i H$ (by the uniqueness in Theorem 2.22) and the cocycles define the same cohomology class. This gives the map $\Phi: \mathcal{M} \to H^1(S^1, St)$.

The injectivity of the map Φ is simple. If $H'_i (H'_j)^{-1} = G_i^{-1} H_i H_j^{-1} G_j$ then $H_i^{-1} G_i H'_i = H_j^{-1} G_j H'_j$. The latter matrix is thus univalent and defines a holomorphic conjugation between the systems $\mathcal{S}'$ and $\mathcal{S}$.

The surjectivity of the map Φ is the most difficult part of the proof. Here we follow [6], but we omit the details. Probably the first proof of this surjectivity was given by G.D. Birkhoff [11].

Let $(G_{ij}) = (G_{U_i U_j})$ be a cocycle with values in the Stokes sheaf. If we could find a system $H_i = I + O(t)$ of holomorphic matrices in sectors S_i (with bases U_i) and such that $G_{ij} = H_j^{-1} H_i$, then this would give us a corresponding meromorphic linear equation $\dot{z} = Az$.

Indeed, let $\mathcal{F}_0$ be a fundamental matrix for the normalized system $\dot{z} = A_0 z$, $A_0 = t^{-r}C(t)$. The matrices $\mathcal{F}_i = H_i\mathcal{F}_0$ are the fundamental matrices for systems with matrices $A_i = \dot{\mathcal{F}}_i\mathcal{F}_i^{-1}$ in the sectors S_i. We have $A_i = \dot{H}_i H_i^{-1} + H_i A_0 H_i^{-1}$, $A_0 = \dot{\mathcal{F}}_0\mathcal{F}_0^{-1}$. Calculations which use the fact that G_{ij} preserve A_0 show that $A_i = A_j$ at the intersections of sectors. Thus A_j's define one univalent matrix $A(t)$. Because $H_i = I + O(t)$ the matrix $A(t)$ has the same order of pole as A_0, $A(t) = t^{-r}B(t)$ with analytic B.

To prove the existence of such H_j's we use apparatus of the sheaf theory. It turns out that the fact that the Stokes sheaf (and the sheaf $\mathcal{E}$ defined below) is not Abelian is not a serious obstacle to apply the results from the Abelian theory to our purposes.

Firstly we solve the system of equations $G_{ij} = G_j^{-1}G_i$ in the class of smooth (C^∞) matrix-valued functions $G_i = I + O(t)$ in sectors S_i. To do this one introduces the sheaf $\mathcal{E}$ (on S^1) of germs of smooth matrix-functions in sectors, $\mathcal{E}(U) = C^\infty(S_U, GL(n))$. It is a flabby sheaf, admitting partitions of unity, and therefore has trivial higher cohomology groups, $H^i(S^1, \mathcal{E}) = 0$, $i > 0$. In the Abelian case it is given in [36], in the non-Abelian case it is proved in [6].

Assume that we have such G_i's. Because $\bar{\partial}G_{ij} = 0$, we have $\bar{\partial}G_i \cdot G_i^{-1} = \bar{\partial}G_j \cdot G_j^{-1}$. The latter expression defines a univalent in a punctured neighborhood of $t = 0$ smooth function F.

Now, applying a variant of the Poincaré $\bar{\partial}$-lemma, we find a matrix function $G = I + O(t)$ such that $\bar{\partial}G \cdot G^{-1} = F$.

The functions $H_i = G^{-1}G_i$ are holomorphic and satisfy the relation $H_j^{-1}H_i = G_{ij}$. $\square$

Theorem 2.27, together with Theorems 2.17 and 2.19, gives a complete solution of the problem of analytic classification of systems with irregular singularity. The formal normal form is determined in analytic way (using finite number of polynomial transformations). This formal normal form gives the moduli space. A separate problem constitutes the task of computations of the Malgrange–Sibuya moduli, i.e. how to calculate the elements from $H^1(S^1, St)$ from data of the initial analytic system. In the below examples we show how it is done in special cases.

EXAMPLE 2.28. This example comes from the work [88] (see also [89]).

Consider the following function

$$F(t) = \int_{0^+}^{t} s^a \mathrm{e}^{-1/s^k}\varphi(s)\,\mathrm{d}s = \int f(s)\,\mathrm{d}s$$

where φ is a germ of analytic function. We ask when this function is of the local Darboux type

$$F(t) = t^{a+k+1}\mathrm{e}^{-1/t^k}\psi(t), \tag{2.8}$$

with analytic ψ.

To get the answer we introduce the loops σ_j (called also *asymptotic cycles*) as in Fig. 1. The punctured neighborhood of $t = 0$ is divided into sectors of fall and of jump of e^{-1/s^k}

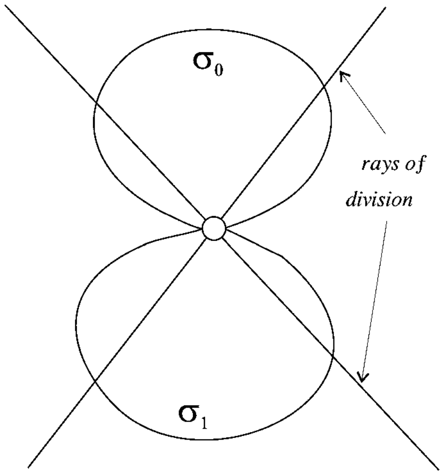

Fig. 1.

as $|t| \to 0$, $\arg t = \text{const}$. Each loop starts and ends at successive sectors of fall (at $t = 0$) and 'surrounds' one sector of jump.

We define the quantities

$$A_j = \int_{\sigma_j} f(s)\,\mathrm{d}s.$$

We have the following property:

The values of $F(t)$ *in neighboring sectors* (*from Fig.* 1) *differ by the constants* A_j. *Therefore the function* F *has the form* (2.8) *iff all* $A_j = 0$.

Consider the linear system

$$t^{k+1}\dot{z} = \begin{pmatrix} -k & t^{k+1}\varphi(t) \\ 0 & at^k \end{pmatrix} z. \tag{2.9}$$

Its general solution is $z_1(t) = D\mathrm{e}^{1/t^k} + C\mathrm{e}^{1/t^k}\int^t f(s)$, $z_2(t) = Ct^a$. Because the singular part of the matrix $A(t)$ is in the diagonal form $A_0(t) = \operatorname{diag}(-kt^{-k-1}, at^{-1})$ then such is also the normalized system.

We choose the system of fundamental solutions of the normalized system in the form $w_1 = \mathrm{e}^{1/t^k}$, $w_2 = t^a$. Therefore $z_1 = w_1 + \eta(t)w_2$, $z_2 = w_2$ where $\eta(t) = t^{-a}\mathrm{e}^{1/t^k}F(t)$.

In different sectors S_j (see Fig. 1) we have different branches of the function F and different branches η_j of η. We have $\eta_{j+1} = \eta_j + A_j t^{-a}\mathrm{e}^{1/t^k}$.

Therefore the normalizing matrices take the form $H_j = \left(\begin{smallmatrix} 1 & \eta_j \\ 0 & 1 \end{smallmatrix}\right)$ and the Stokes cocycle $C_{j,j+1}$ is given by the matrices $\left(\begin{smallmatrix} 1 & A_j \\ 0 & 1 \end{smallmatrix}\right)$.

We have $H^1(S^1, St) \simeq \mathbb{C}^k$ and the Stokes cocycle associated with the system (2.9) is equal to $G_{j,j+1} = \left(\begin{smallmatrix} 1 & A_j t^{-a}\exp(1/t^k) \\ 0 & 1 \end{smallmatrix}\right)$.

EXAMPLE 2.29. (We follow [6] and [37].) Recall the *Bessel equation*

$$t^2\ddot{x} + t\dot{x} + (t^2 - \nu^2)x = 0.$$

We see that the point $t = 0$ is regular (see Definition 2.1 and Theorem 2.7). The point $t = \infty$ turns out irregular.

We introduce the notations $\tau = 1/t$, $u(\tau) = (x(1/\tau), \dot{x}(1/\tau))^\top$ and below by the dot we denote the derivative with respect to τ. Then we get the system $\tau^2\dot{u} = B(\tau)u$, where

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\tau + \begin{pmatrix} 0 & 0 \\ -\nu^2 & 0 \end{pmatrix}\tau^2.$$

In order to get the formal normal form we have to transform only the first two terms of $B(\tau)$ to diagonal form. The matrix B_0 has distinct eigenvalues $\pm\mathrm{i}$, i.e. we have the non-resonant case. The transformation matrix is chosen in the form $H(t) \sim H_0(I + H_1\tau + \cdots)$, where only the first two terms are important.

We have $H_0 = \left(\begin{smallmatrix} -1 & -\mathrm{i} \\ -\mathrm{i} & -1 \end{smallmatrix}\right)$ and $H_0 B_0 H_0^{-1} = D_0 = \operatorname{diag}(\mathrm{i}, -\mathrm{i})$. Next $D_1 = H_0(B_1 + [H_1, B_0])H_0^{-1} = H_0 B_1 H_0^{-1} + [\tilde{H}_1, D_0]$, $\tilde{H}_1 = H_0 H_1 H_0^{-1}$. The commutator in the latter expression acts onto the non-diagonal entries (because D_0 is diagonal) and, choosing suitable $\tilde{H}_1$, we can obtain D_1 in the diagonal form. Because $H_0 B_1 H_0^{-1} = \frac{1}{2}\left(\begin{smallmatrix} 1 & * \\ * & 1 \end{smallmatrix}\right)$, then $D_1 = \frac{1}{2}I$. Thus the basis of solutions of the normalized system is $u_{1,2} = \tau^{1/2}\mathrm{e}^{\pm\mathrm{i}/\tau}e_{1,2} = t^{-1/2}\mathrm{e}^{\pm\mathrm{i}t}e_{1,2}$.

There are two rays of division, both in the real axis, and we can take the covering of $\mathbb{C}\setminus 0$ by two sectors with angles $\frac{3\pi}{2}$: S_1 around the real half-line $\{t > 0\}$ and S_2 around $\{t < 0\}$. The intersection $S_1 \cap S_2$ consists of two sectors around $\{\operatorname{Im} t > 0\}$ and $\{\operatorname{Im} t < 0\}$ respectively.

Recall that the Stokes sheaf consists of such matrix functions $G(t)$ that $G \sim I$ and $GDG^{-1} + \tau^2\dot{G}G^{-1} = D$ (preservation of the normalized system). In view of the fact that $2D_1 = I$ this is equivalent to the following condition:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\chi^{-1}G\chi) = 0 \quad \text{where } \chi(\tau) = \operatorname{diag}(\mathrm{e}^{-\mathrm{i}/\tau}, \mathrm{e}^{\mathrm{i}/\tau}).$$

Therefore the map $G \to G = \chi^{-1}G\chi$ defines an isomorphism of the sheaf St with some sheaf of subgroups of $\mathrm{GL}(2,\mathbb{C})$. The first condition for the Stokes sheaf $\chi F\chi^{-1} \sim I$ implies that the matrix elements of F fulfill the conditions $F_{11} = F_{22} = 1$, $F_{21}\mathrm{e}^{-2\mathrm{i}/\tau} \sim 0$, $F_{12}\mathrm{e}^{2\mathrm{i}/\tau} \sim 0$. Therefore, depending in which part of the τ-plane we are, the matrix F takes one of the two forms

$$\begin{pmatrix} 1 & f_+ \\ 0 & 1 \end{pmatrix}, \quad \operatorname{Im} t > 0; \qquad \begin{pmatrix} 1 & 0 \\ f_- & 1 \end{pmatrix}, \quad \operatorname{Im} t < 0. \tag{2.10}$$

The parameters $f_\pm$ parameterize the space $H^1(S^1, St)$ and constitute the moduli of analytic classification of equations which are formally equivalent to the Bessel equation.

We calculate the constants f_+, f_- for the very Bessel equation following the book of J. Heading [37]. One uses the basis of solutions of the Bessel equation consisting of the *Bessel functions*

$$J_{\pm\nu}(t) = \sum_{j=0}^{\infty} \frac{(-1)^j (t/2)^{2j\pm\nu}}{j!\Gamma(\pm\nu + j + 1)} = t^{\pm\nu} P_{\pm}(t^2). \tag{2.11}$$

Note that the function $P_\pm$ are integer. Assume that a solution $AJ_\nu(t) + BJ_{-\nu}(t)$ has the asymptotic expansion $\sim t^{-1/2}\mathrm{e}^{\mathrm{i}t}$ for $\arg t = 0$ and $t \to \infty$. After analytic prolongation of this solution to the rays $\arg t = \pi$ and $\arg t = 2\pi$, with the use of formulas (2.10) and (2.11), we obtain

$$\begin{aligned}
&A\mathrm{e}^{\mathrm{i}\pi\nu} J_\nu(t) + B\mathrm{e}^{-\mathrm{i}\pi\nu} J_{-\nu}(t) \sim \mathrm{e}^{-\mathrm{i}\pi/2} t^{-1/2} \mathrm{e}^{-\mathrm{i}t},\\
&A\mathrm{e}^{2\mathrm{i}\pi\nu} J_\nu(t) + B\mathrm{e}^{-2\mathrm{i}\pi\nu} J_{-\nu}(t) \sim \mathrm{e}^{-\mathrm{i}\pi} t^{-1/2} \mathrm{e}^{\mathrm{i}t} + f_- \mathrm{e}^{-\mathrm{i}\pi} t^{-1/2} \mathrm{e}^{-\mathrm{i}t},
\end{aligned}$$

as $t \to +\infty$. Together with $AJ_\nu(t) + BJ_{-\nu}(t) \sim t^{-1/2}\mathrm{e}^{\mathrm{i}t}$ this gives the value

$$f_- = 2\mathrm{i}\cos(\pi\nu).$$

Analogous calculations give $f_+ = 2\mathrm{i}\cos(\pi\nu)$.

2.3. *Appendix. Proof of the normalization theorems*

Here we prove Theorems 2.13 and 2.22. These proof are conceptually similar, since they are reduced to the contraction principle for a suitable map in a Banach space. We begin with the proof of Theorem 2.22, where we follow the book of W. Wasow [85]. Another proof of Theorem 2.22 uses the Gevrey expansions and multi-summability (see next Subsection 3.6 below) and is given in the paper of B. Braaksma [14].

PROOF OF THEOREM 2.22. Recall that this theorem states that:

there exists a unique matrix function $H(t) = I + O(t)$ analytic in a sector S, not containing two rays of division, which transforms the system $t^r\dot{z} = B(t)z$ to the formal normal form system (2.6).

The essential fact which has to be proved is the following

PROPOSITION. *Let the eigenvalues of $B(0)$ be divided into two groups $\lambda_1, \dots, \lambda_p$ and $\lambda_{p+1}, \dots, \lambda_n$ such that $\lambda_i \neq \lambda_j$ for $i \leqslant p < j$. Then there exists a matrix $H(t)$, holomorphic in S, which transforms the system to an analogous system with the matrix $C(t)$ of the block-diagonal form*

$$\begin{pmatrix} C^{11}(t) & 0 \\ 0 & C^{22}(t) \end{pmatrix}.$$

Moreover, we can assume that the sector S is symmetric with respect to the real positive semi-axis and has the magnitude $< \pi/(r-1)$.

PROOF OF PROPOSITION. If $z = H(t)y$ then y satisfies the equation $t^r H\dot{y} = (BH - t^r \dot{H})y$ and we have

$$HC = BH - t^r \dot{H}.$$

So we assume that $B(0)$ is in the block-diagonal form $B(0) = \operatorname{diag}(B^{11}(0), B^{22}(0))$ and we seek $H(t)$ in the form

$$\begin{pmatrix} I & H^{12}(t) \\ H^{21}(t) & I \end{pmatrix}.$$

In what follows we will deal only with H^{12}. The analysis for H^{21} is quite analogous.

The matrix $X(t) = H^{12}(t)$ satisfies the following non-linear differential equation

$$t^r \dot{X} = B^{11}X - XB^{22} + B^{12} - XB^{21}X$$

where $B^{ij}(t)$ are the corresponding block-matrices.

The latter equation is a particular case of the following non-linear equation

$$t^r \dot{u} = \Lambda u + p(t, u), \quad u \in \mathbb{C}^m, \tag{2.12}$$

where Λ has non-zero eigenvalues Λ_k (equal to $\lambda_i - \lambda_j$ in our case) and the term p is non-linear.

Equation (2.12) is equivalent to the following integral equation

$$u(t) = V(t)D + \int_{t_0}^{t} V(t)V^{-1}(s)s^{-r} p\bigl(s, u(s)\bigr)\,\mathrm{d}s \tag{2.13}$$

where $V(t) = \mathrm{e}^{\Lambda/(1-r)t^{r-1}}$ and U is some constant vector.

In fact, the constant vector D and the limit t_0 of integration are in some way connected one with another. One can replace the integration in (2.13) by the integration along some contour $\Gamma(t)$ in the complex s-plane in such a way that it ends at t. In this case the constant vector can be put equal to zero. Moreover, for each component of Eq. (2.13) we can choose the integration path independently.

The main trick of the proof is to choose the path $\Gamma(t)$ in such a way that the expression $V(t)V^{-1}(s)$ is not too big along the path of integration; then we will be able to apply the principle of contracting maps. In particular, we want that the expressions

$$\exp\bigl[\operatorname{Re}\bigl(\Lambda_k(t^{1-r} - s^{1-r})/(1-r)\bigr)\bigr],$$

associated with the k-th components, are not big.

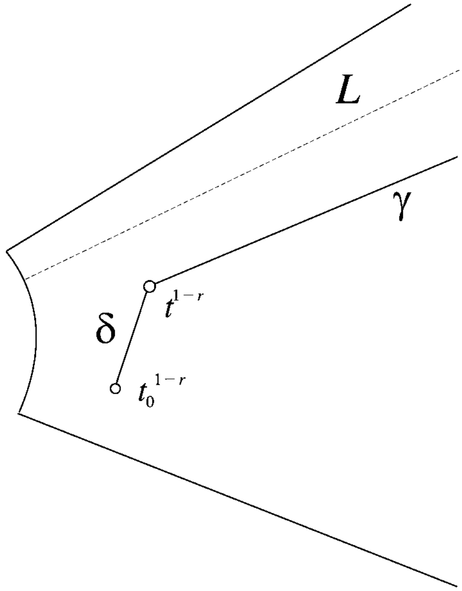

Fig. 2.

We fix for a while the index k and some point $t_0 \in \mathbb{R} \cap S$. We shall consider only such t's that $|t| << t_0$. Moreover it is useful to pass to the chart $\zeta = s^{1-r}$; then the sector S is replaced by a sector with vertex at infinity.

Assume that S does not contain any ray of division corresponding to Λ_k. It means that $\operatorname{Re} \Lambda_k \zeta/(1-r)$ has definite sign in the whole S. Depending on this sign we choose either:

- the straight semi-line γ from $\zeta = \infty$ to $\zeta = t^{1-r}$ with constant $\arg \zeta$, if the sign is positive, or
- the interval δ joining t_0^{1-r} with t^{1-r}, otherwise; (see Fig. 2).

In both cases we have $\operatorname{Re} \Lambda_k (t^{1-r} - \zeta)/(1-r) < 0$.

If S contains a ray of division L (only one by the assumption) then the choice of the path δ remains unchanged but the path γ passes along a straight half-line parallel to L (from infinity to t^{1-r}, see Fig. 2). Here we have $\operatorname{Re} \Lambda_k (t^{1-r} - \zeta)/(1-r) = 0$ along γ.

The further proof uses the standard analytic methods and we describe it only shortly, without detailed estimates.

We solve the fixed point equation

$$u = T(u),$$

where $T(u)$ is the non-linear integral operator $\int V(t) V^{-1}(s) s^{-r} p(s, u(s))\, \mathrm{d}s$ and the integration runs along the paths described above.

In order to apply the method of contracting maps, we must define some Banach space of holomorphic functions u and get an estimate for $\|T(u_1) - T(u_2)\|$ (i.e. the Lipschitz continuity of $p(t, u)$ with respect to u).

The class of u's consists of those which satisfy the estimate $|u(t)| < c|t|^m$ with certain fixed m. This m is defined as the order of $p(t,0) = \text{const} \cdot t^m + \cdots$. One shows that, if u satisfies this estimate, then $|T(u)(t)| < Kc|t|^m$ with a constant K not depending on u (see Lemma 14.2 in [85]).

The Lipschitz estimate $|p(t,u_1) - p(t,u_2)| < \mu|u_1 - u_2|$, with μ arbitrarily small, follows from the fact that p contains only terms of the form $\mathrm{O}(t^m)u^0$, $\mathrm{O}(t)u$ and $\mathrm{O}(|u|^2)$.

Next, one chooses the sequence $u_0 = 0$, $u_1 = T(u_0)$, $u_2 = T(u_1), \ldots$ of successive approximations. One gets $|u_1| < c|t|^m$ and $|u_{n+1} - u_n| < \mu K|u_n - u_{n-1}|$. If $\mu < K^{-1}$ then this series converges to a function holomorphic in some sector S_0 of small radius. □

PROOF OF THEOREM 2.13. Recall that this theorem states that:

for a system $t\dot{z} = B(t)z$ with Fuchsian and resonant singularity there exists an analytic matrix $H(t)$ which transforms it to a system of the form,

$$t\dot{z}_i = \lambda_i z_i + \sum_{\lambda_i - \lambda_j \in \mathbb{Z}_+} a_{ij} t^{\lambda_i - \lambda_j} z_j, \quad i = 1, \ldots, n,$$

with constant coefficients a_{ij}.

Note also that the method of the proof of Theorem 2.12 does not work in this situation.

There are finitely many resonant relations $\lambda_i - \lambda_j = k_{ij} \in \mathbb{Z}_+$. Let an integer $m > 0$ be such that $m > \mathrm{Re}(\lambda_i - \lambda_j)$ for all i, j. We can assume that the system takes the form

$$t\dot{z} = Dz + tB_0(t)z + t^m B_1(t)z,$$

where the matrix D is constant in the Jordan form and $tB_0(t)z$ contains only the resonant terms (like in thesis of the theorem).

We seek the matrix H in the form

$$H(t) = I + t^m G(t)$$

and obtain the equation

$$t\dot{G} = \big([D,G] - mG\big) + \big(\big[tB_0(t), G\big] + B_1(t) + t^m B_1(t)G\big).$$

It has the same type as Eq. (2.13) but with $r = 1$ and with the eigenvalues of Λ equal $\lambda_i - \lambda_j - m$, which are all non-zero. The fundamental matrix $V(t)$ has the eigenvalues $t^{\lambda_i - \lambda_j - m}$.

The corresponding integral equation is following

$$u(t) = \big(T(u)\big)(t) = V(t)\int_{t_0}^{t} V(s)^{-1} s^{-1} p\big(s, u(s)\big)\,\mathrm{d}s,$$

where the initial point t_0 can be chosen equal to 0. Indeed, since $p(s,u(s))$ is bounded and the entries of the matrix $V(s)^{-1}s^{-1}$ vanish at $s = 0$, the above integral is well defined.

It is easy to check that the Lipschitz condition for $p(t, \cdot)$ is satisfied and we can apply the contraction principle. Therefore there exists a unique fixed point for the map T which is an analytic solution to our equation. $\square$

3. Holomorphic vector fields

The previous section was devoted to linear analytic ODEs. In this section we develop the theory of normal forms of non-linear analytic differential equations. These are the equations of the form

$$\dot{x} = V(x), \quad x \in \mathbb{C}^n, \tag{3.1}$$

where $V(x)$ is a holomorphic vector-function, i.e. *holomorphic vector fields*. Here the time t is also complex. Therefore the phase curves become Riemann surfaces and the phase portrait of such a vector field defines a foliation $\mathcal{F}$ into complex phase curves. Such foliations are called *holomorphic foliations*.

The zeroes x_0 of the vector field (3.1) are the *singular points* of the vector field $V(x)$ and of the foliation $\mathcal{F}$. In this (and next) section we study the holomorphic foliations near a singular point, which we assume $x_0 = 0$.

DEFINITION 3.1. Two germs of holomorphic vector fields V and W are called (analytically) *equivalent* if there exists a local analytic diffeomorphism $H : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ transforming V to W, i.e.

$$W = H^* V := (H_*^{-1} V) \circ H.$$

These germs are *formally equivalent* if $H(x)$ is defined by means a formal invertible series and the above equality holds at the level of formal power series.

The germs V and W are called (analytically) *orbitally equivalent* if there exists a local analytic diffeomorphism $H : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ transforming the phase curves of V to phase curves of V. This means that

$$W = F \cdot H^* V$$

where $F \neq 0$ is some analytic function. These germs are *formally orbitally equivalent* if H and F are formal series and the above equality holds at the formal level.

3.1. *Resolution for planar vector fields and the center–focus problem*

This section is not devoted to classification of vector fields with respect to equivalence or orbital equivalence. We present here a result about reduction of a complex singular point of a vector field to a collection of more simple singularities. Moreover we restrict the analysis to the case $n = 2$, i.e. the plane vector fields.

DEFINITION 3.2. A singular point of a germ of planar holomorphic planar vector field is called *elementary* (or *reduced*) iff at least one of its eigenvalues is non-zero.

Thus we can assume that we have

$$\dot{x} = \lambda_1 x + \cdots, \quad \dot{y} = \lambda_2 y + \cdots, \quad \lambda_1 \neq 0.$$

Important invariant of the singular point is the *ratio of eigenvalues*

$$\lambda = \lambda_2/\lambda_1.$$

If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ then we say that the singular point is a *focus*; if $\lambda > 0$ then it is a *node*; if $\lambda < 0$ then it is a *saddle*; if $\lambda = 0$ then it is *saddle–node*.

The singular point is called *resonant* if λ is a rational number. The resonant node is called *dicritical* if it is formally equivalent to its linear part.

DEFINITION 3.3. The *elementary blowing-up* is the map

$$\pi : (M, E) \to (\mathbb{C}^2, 0),$$

where M is a 2-dimensional complex manifold equal (as a set) to $(\mathbb{C}^2 \setminus 0) \sqcup E$, where $E \simeq \mathbb{C}P^1$ is the set of lines passing through the origin. E is called the *exceptional divisor*. If $(u_1 : u_2)$ are the homogeneous coordinates in $\mathbb{C}P^1$ then the blowing-up map is locally given by the formulas

$$(x, u) \to (x, xu), \quad u = u_2/u_1, \quad (u_1 \neq 0),$$
$$(y, z) \to (yv, y), \quad v = u_1/u_2, \quad (u_2 \neq 0).$$

If $V : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a germ of holomorphic vector field, then after the elementary blowing-up we obtain usually not a vector field in M but rather a field of directions $\tilde{V}$ near the distinguished divisor E. In the local (x, u)-charts (or (y, v)-charts) in M this field of directions is defined by means of an analytic vector field.

Let $V : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a germ of holomorphic vector field. We say that it admits a *good resolution* if after a *finite* number of elementary blowing-ups we obtain a field of directions in a 2-dimensional complex manifold with only elementary and non-dicritical singular points.

EXAMPLE 3.4. *A general homogeneous vector field of degree n*

$$\dot{x} = P_n(x, y), \qquad \dot{y} = Q_n(x, y)$$

has $n + 1$ *invariant lines*.

Indeed, in the blowing-up coordinates $x, u = y/x$ we get the system $\dot{x} = x^n P_n(1, u)$, $\dot{u} = x^{n-1}[Q_n(1, u) - u P_n(1, u)] = x^{n-1} H_{n-1}(u)$, where the polynomial $H_{n+1}(u)$ in the square brackets is (generally) non-zero. In order to obtain a field of directions one has

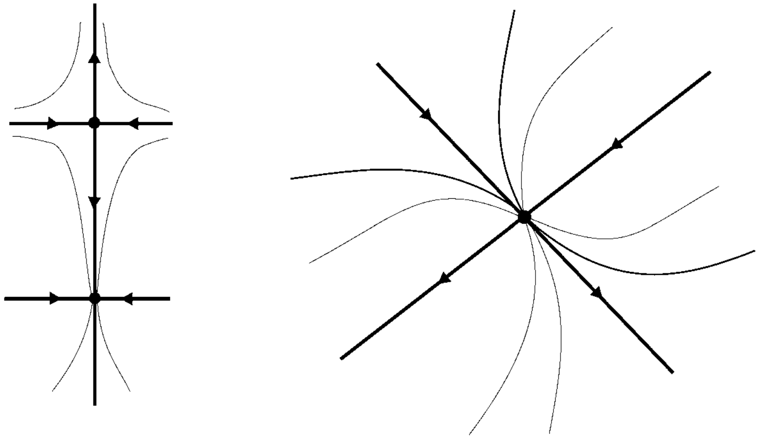

Fig. 3.

to divide the latter system by x^{n-1}. The zeroes u_i of $H_{n+1}(u)$ define the invariant lines $y = u_i x$, see Fig. 3.

If the zeroes u_i are isolated then we can perturb this vector field by adding terms of degree $> n$ and the above blowing-up gives the resolution of the perturbed vector field; the singular points $(u_i, 0)$ remain elementary.

If $P_n = xR_{n-1}$, $Q_n = yR_{n-1}$ then $H_{n+1} \equiv 0$ and the foliation in the (x,u)-coordinates is the flow-box foliation. If additionally we consider perturbation, with terms of higher degree of general type, then in the resolution we must divide the blown-up vector field by x^n. The blowing-up defines the resolution of the singularity but the exceptional divisor $x = 0$ is not invariant for the corresponding foliation. In such a case we say that the divisor E is *dicritical*.

EXAMPLE 3.5. Consider the *Bogdanov–Takens singularity*

$$\dot{x} = 2y, \quad \dot{y} = ax^2 + bxy + cy^2 + \cdots,$$

where we assume that $a \neq 0$ (and we can put $a = 3$). This system can be treated as a small perturbation of the Hamiltonian system X_H with the Hamilton function $H = y^2 - x^3$ with the cusp singularity. Indeed, in the quasi-homogeneous filtration with the exponents $d(x) = 2$, $d(y) = 3$, the Hamiltonian part has degree 1 and the remaining terms have greater degrees.

The blowing-up of the cusp curve is presented at Fig. 4. It is easy to see that the same elementary blowing-up maps can be applied to the Bogdanov–Takens singularity. The resolved field of directions has the same singularities as X_H which are also hyperbolic saddles. The curve Γ is an invariant curve of the vector field (separatrix) and has the cusp singularity. We shall return to this singularity in the next section.

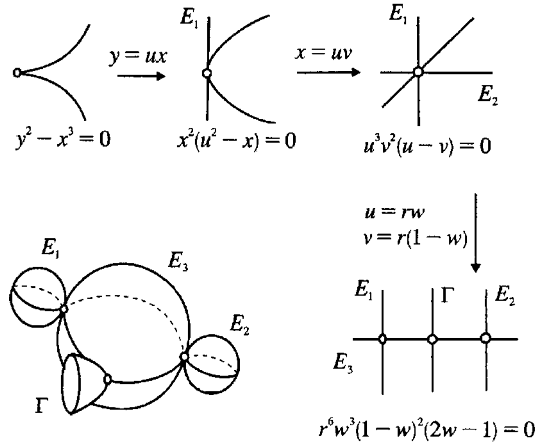

Fig. 4.

The following theorem is principal in this section. It was firstly formulated by I. Bendixson under the assumption that the vector field is real and analytic. F. Dumortier also considered the real case with only elementary singularities at the end. Moreover, he assumed that the initial vector field is only smooth. The assumption that the singular point is isolated is replaced by the following *Łojasiewicz condition* $|V(x)| > C|x|^\alpha$. Our version of the desingularization theorem was firstly formulated by A. Seidenberg. However its final proof, which is the just proof, was given by A. van den Essen.

THEOREM 3.6. (See [10,70,27,32].) *Any germ of analytic planar vector field with isolated singularity admits a good resolution.*

REMARK 3.7. There exists an analogous theorem (about desingularization) for germs of holomorphic foliations of codimension 1 in $(\mathbb{C}^3, 0)$ (see [22]). But no analogous result about the codimension 1 foliations is known in dimensions >3. Also there is no desingularization theorem for holomorphic vector fields in dimensions >2.

PROOF OF THEOREM 3.6. (We follow the article of J.-F. Mattei and R. Moussu [56] and [89].)

1. If we have a $p:q$ resonant node $\dot{x} = px + \cdots$, $\dot{y} = qy + \cdots$, $p < q$, then the resolved vector field has two singular points in the divisor $E = \{(u_1 : u_2)\}$: the $p:(q-p)$ resonant node $(1:0)$ (i.e. $\dot{x} \approx px$, $\dot{u} \approx (q-p)u$) and $p:(p-q)$ resonant saddle $(0:1)$. If the node $x = y = 0$ is dicritical then the node $(1:0)$ is also dicritical.

Repeating this several times, we arrive to a $1:1$ resonant node which after elementary blowing-up is transformed to the flow-box (see Example 3.4).

2. We work with the vector fields $V = b(x, y)\partial_x - a(x, y)\partial_y$ and with the equivalent to them 1-forms

$$\omega = a\,\mathrm{d}x + b\,\mathrm{d}y.$$

We recall the definition of *index* of the vector field (or of the 1-form) $i_0(V) = i_0(\omega) = i(a, b; 0)$. Let $b(x, y) = 0$ be an irreducible (analytic) curve passing through the origin. It can be locally parametrized by an analytic map $(\mathbb{C}, 0) \ni t \to \gamma(t) \in (\mathbb{C}^2, 0)$ such that $b \circ \gamma \equiv 0$. There is the so called *primitive parametrization* such that any other parametrization of the same curve is induced from the primitive parametrization by a change of parameters. (Example: the primitive parametrization of the cusp $y^2 = x^3$ is $x = t^2$, $y = t^3$.) Then we have $a \circ \gamma(t) = \alpha t^d + \cdots$, $\alpha \neq 0$. We get $i(a, b; 0) = d$. If the function b has the representation $b = b_1^{l_1} \cdots b_r^{l_r}$ with irreducible b_j's then $i(a, b; 0) = \sum l_j \cdot i(a, b_j; 0)$.

Note that $i_0(V) = 1$ iff 0 is a non-degenerate elementary singular point, i.e. not a saddle-node.

The *rank* of the vector field (or of the 1-form) $\nu(V) = \nu(\omega) = \nu_0(\omega)$ equals to the order of the first non-zero term in the Taylor expansion of V at 0. Analogously one defines the rank of a function.

3. LEMMA. *Let $\pi : (x, u) \to (x, xu)$ be the blowing-up map with the exceptional divisor $E = \pi^{-1}(0)$. We define the functions $\tilde{a}, \tilde{b}$ by the formulas*

$$a \circ \pi = x^{\nu(a)} \tilde{a}, \qquad b \circ \pi = x^{\nu(b)} \tilde{b}.$$

Then we have

$$i(a, b; 0) = \nu(a)\nu(b) + \sum_{z \in E} i(\tilde{a}, \tilde{b}; z).$$

PROOF. When the polynomial $\tilde{b}(0, u)$ has only simple zeroes $z \in E$ (there is $\nu(b)$ of them) then each such zero gives the contribution $\nu(a) + i(\tilde{a}, \tilde{b}; z)$ to the index $i(a, b; 0)$. The general case is a consequence of the additivity of the index; for a family V_ε of vector fields with singular points $x_i(\varepsilon)$, $i = 1, \ldots, k$, degenerating to 0 as $\varepsilon \to 0$ we have $i_0 V_0 = \sum i_{x_j}(V_\varepsilon)$. □

4. PROPOSITION. *There exists a sequence of elementary blowing-ups such that at the end we obtain only singularities with rank $\leqslant 1$.*

PROOF. Let $\omega_\nu = a_\nu \, \mathrm{d}x + b_\nu \, \mathrm{d}y$ be the homogeneous part of ω of order $\nu = \nu(\omega) = \min(\nu(a), \nu(b))$. We put

$$P_{\nu+1} = x a_\nu + y b_\nu.$$

Then we have

$$\begin{aligned} \pi^* \omega &= a(x, ux) \, \mathrm{d}x + b(x, ux) \, \mathrm{d}(ux) \\ &= (a + ub) \, \mathrm{d}x + x b \, \mathrm{d}u \\ &= x^\nu \bigl\{ \bigl[P_{\nu+1}(1, u) + \cdots \bigr] \mathrm{d}x + x \bigl[b_\nu(1, u) + \cdots \bigr] \mathrm{d}u \bigr\}. \end{aligned}$$

Here the case with $P_{\nu+1} \equiv 0$ is dicritical.

For any $z \in E$ we define the germ $\tilde{\omega} = \tilde{\omega}_z$ as the germ of $\pi^*\omega$ at z, divided by the greatest common multiplier of the germs its components. Thus if $\pi^*\omega = a_1\,\mathrm{d}x + b_1\,\mathrm{d}u$, then $\tilde{\omega} = \pi^*\omega/\gcd(a_1, b_1)$. The Pfaff equation $\tilde{\omega} = 0$ defines the blown-up foliation near z.

Proposition 4 follows from the following result. □

5. LEMMA. *If $\nu > 1$ then for any $z \in E$ we have*

$$i_z(\tilde{\omega}) < i_0(\omega).$$

PROOF. Consider firstly the dicritical case, i.e. $P_{\nu+1} \equiv 0$. We shall show the identity

$$i_0(\omega) = \nu^2 + \nu - 1 + \sum i_z(\tilde{\omega}).$$

We can choose a local system of coordinates $(x, y) \to (x + \cdots, y + \cdots)$ such that the following conditions hold:

(i) $P(x, y) = xa + yb$ has rank exactly $\nu + 2$;

(ii) $b(0, y) = y^{\nu+1}$ (then $P(0, y) = y^{\nu+2}$).

We have

$$i(P, b; 0) = i(x, b; 0) + i(a, b; 0)$$

where

$$i(P, b; 0) = \nu(\nu + 2) + \sum i(\tilde{P}, \tilde{b}; z)$$

(by Lemma 3) and $i(x, b; 0) = \nu + 1$ (by (ii)). Now one must only notice that $\tilde{\omega} = \tilde{P}\,\mathrm{d}x + \tilde{b}\,\mathrm{d}u$ and thus $i(\tilde{P}, \tilde{b}; z) = i_z(\tilde{\omega})$.

Consider now the non-dicritical case $P_{\nu+1} \not\equiv 0$. We shall prove that

$$i_0(\omega) = \nu^2 - \nu - 1 + \sum i_z(\tilde{\omega}).$$

We have (by Lemma 3)

$$i(a, b; 0) = \nu^2 + \sum i(\tilde{a}, \tilde{b}; z). \tag{3.2}$$

Next

$$i_z(\tilde{\omega}) = i(\tilde{a} + u\tilde{b}, x\tilde{b}; z) = i(\tilde{a} + u\tilde{b}, x; z) + i(\tilde{a}, \tilde{b}; z)$$

where $i(\tilde{a} + u\tilde{b}, x; z)$ is the multiplicity of z as zero of the polynomial $P_{\nu+1}$; the sum of these multiplicities is $\nu + 1$.

Summing up the latter equality over z's and taking into account (3.2) we get the result. □

6. Let $\nu = 1$. The only case with non-elementary singular point of rank one is the case with nilpotent linear part $\dot{x} = y + \cdots$, $\dot{y} = \cdots$. The corresponding 1-form is following

$$\omega = \big[y + A_1(x, y)\big]\,\mathrm{d}y + B_1(x, y)\,\mathrm{d}x,$$

with $\nu(A_1), \nu(B_1) \geqslant 2$.

After blowing-up in the x-direction, i.e. with $y = xu$, we get $\tilde{\omega} = x(u + xA_2)\,\mathrm{d}u + (u^2 + xB_2)\,\mathrm{d}x$ where A_2, B_2 are analytic functions (without restrictions onto ranks). The latter form can be included into the following series of 1-forms

$$\eta = x(y + xA)\,\mathrm{d}y + (ny^2 + xB)\,\mathrm{d}x \tag{3.3}$$

where n is a natural number. In what follows we deal with (3.3).

Note also that $\nu(\eta) = 2$.

7. Let $B(0) = b_0 \neq 0$. Here we blow-up η in the y-direction by putting $x = vy$. We obtain

$$\tilde{\eta} = v\big[(n+1)y + b_0 v + A'\big]\,\mathrm{d}y + y[ny + b_0 v + B']\,\mathrm{d}v.$$

This form has good quadratic part. Note that the polynomial

$$P_3(\tilde{\eta}) = vy\big[(2n+1)y + 2b_0 v\big]$$

has distinct factors. After the next blowing up we obtain three elementary singular points (see Example 3.4).

8. Let $B(0) = 0$. From (3.3) we get

$$P_3(\eta) = x\big[(n+1)y^2 + \alpha xy + \beta y^2\big] = (n+1)x(y - c_1 x)(y - c_2 x).$$

If $c_1 \neq c_2$ then the next blowing-up solves the problem (like in the point 7).

Let $c_1 = c_2$. Eventually applying the change $y \to y - cx$ we can assume that $c_1 = c_2 = 0$, i.e. $P_3(\eta) = (n+1)xy^2$. The singular point $(0:1) \in E$, corresponding to the line $x = 0$ in $P_3 = 0$, is elementary; the point $(1:0)$ is non-elementary. Then calculations of the blowing up with $y = ux$ give

$$\tilde{\eta} = x(u + A')\,\mathrm{d}u + \big[(n+1)u^2 + xB'\big]\,\mathrm{d}x$$

where $A' = A(x, ux)$ and $B' = B(x, ux)/x + uA(x, ux)$.

If $A(0) \neq 0$ then the point $x = u = 0$ is elementary.

9. Let $A(0) = 0$. Then $A' = xA''$ and $\tilde{\eta}$ has the same form as η in (3.3) but with n replaced by $n+1$. So we can repeat the analysis from the points 7 and 8. We have two possibilities: either after finitely many steps we obtain elementary singular point or infinitely many times we obtain the form (3.3).

However the second possibility contradicts Proposition 4, because we would get infinitely many times $\nu > 1$. This contradiction proves Theorem 3.6. $\square$

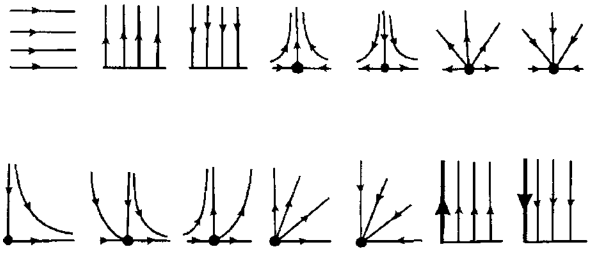

Fig. 5.

Theorem 3.6 has important consequences in the complex domain and in the real domain. Below we present its application in the study of phase portraits of real analytic vector field about a singular point.

Theorem 3.6 implies that after blowing-ups the phase portrait near the distinguished divisors can be composed from the finite collection of standard portraits (presented at Fig. 5).

COROLLARY 3.8. *The resolution of singular point of a real vector field allows to determine its topological type modulo solution of the center–focus problem. If there is a characteristic trajectory* (*see Definition* 3.9) *of the resolved field* (*tending to a point in the distinguished divisors*) *then the topological type of the vector field is completely determined*; *if there is no characteristic trajectory then the topological type is determined by the stability type of the singularity.*

Therefore we arrive to the center–focus problem for germs of analytic vector fields.

DEFINITION 3.9. The *characteristic trajectory* of a real planar vector field V at the singular point $x = 0$ is a phase curve which approaches the singularity with definite limit direction. It means that after resolution some local phase portrait different from the first and eighth portraits in Fig. 5 appears.

If there is no characteristic trajectories then the singular point is called *monodromic*.

The *center–focus problem* is the problem of describing conditions for stability or unstability of a monodromic singular point for analytic vector field.

V. Arnold [3] formulated this problem as follows. The space J^k of $k-$jets $j^k V$ at $x = 0$ of germs $V : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ of analytic vector fields is divided into three subspaces J_s^k, J_u^k and J_n^k. Here the jets $j^k V$ from J_s^k (respectively from J_u^k) are such that any germ W with $j^k W = j^k V$ has 0 as an asymptotically stable point (respectively unstable point). The subspace J_n^k consists of 'neutral' jets. Arnold asked whether the sets $J_{s,u,n}^k$ are semi-algebraic, or semi-analytic, and conjectured that the codimension of J_n^k grows to infinity with k.

In practice one divides the center–focus problem into two problems. Firstly one asks whether the singular point is monodromic; then one can define the Poincaré return map from an arc ending at the singularity to itself. Next one calculates the asymptotic expansion of the Poincaré map.

Theorem 3.6 implies that the monodromicity problem is algebraically solvable, i.e. that the division of the space J^k into 'stably' monodromic, non-monodromic and neutral jets is defined by algebraic equations and inequalities onto the coefficients of the Taylor expansion of V.

Therefore there appears the problem of calculating the leading term in the asymptotic expansion of the Poincaré map for germs from a given class. Such class may consists, for example, of germs with a fixed resolution of the singularity.

The first case with elementary singular point, i.e. with imaginary eigenvalues, was solved by H. Poincaré and A. Lyapunov.

THEOREM 3.10. (See [67,51].) *The problem of distinguishing between center and focus for germs of the form*

$$\dot{x} = -y + \cdots, \qquad \dot{y} = x + \cdots$$

is algebraically solvable. More precisely, the Poincaré map $\mathcal{P}$: $\{x = r \geqslant 0, y = 0\} \to \{x \geqslant 0, y = 0\}$ *has the expansion*

$$\mathcal{P}(r) = r + g_3 r^3(1 + \cdots) + g_5 r^5(1 + \cdots) + \cdots$$

where the Poincaré–Lyapunov focus quantities g_{2j+1} *are polynomials with respect to* (*finite number*) *of coefficients of the Taylor expansion of the vector field.*

Moreover, if all $g_j = 0$ *for a given germ* V *then there exists a local analytic first integral*

$$F = x^2 + y^2 + \cdots$$

and the singular point $x = y = 0$ *is a center for* V.

PROOF. It is useful to pass to the complex variables $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$. Then we have

$$\dot{z} = \mathrm{i}z + \sum a_{kl} z^k \bar{z}^l.$$

We look for a Lyapunov function (in the stable or unstable case) or for a first integral $F = z\bar{z} + \sum f_{kl} z^k \bar{z}^l$. It should satisfy the equation

$$\dot{F} = \frac{1}{\pi}\big(g_3|z|^4 + g_5|z|^6 + \cdots\big),$$

where the coefficients f_{kl} and $\frac{1}{\pi} g_j$ are defined recursively. We get a system of equations $\sqrt{-1}\,(k-l) f_{kl} = -(a_{k,l-1} + \bar{a}_{k-1,l}) +$ (terms depending on previous f_{ij}'s). We see that we can calculate f_{kl} always when $k \neq l$, when $k = l$ we find the obstacle which equals $\frac{1}{\pi} g_k$.

To relate the expansion of $\dot{F}$ with the expansion of the Poincaré map we parametrize the section by $F = r^2 + \cdots$. Then $\mathcal{P}(r) - r = \int_0^T \dot{r}\,\mathrm{d}t \approx \frac{1}{2r}\int_0^{2\pi} \dot{F}\,\mathrm{d}t \approx \frac{1}{2r} \cdot 2\pi \cdot \frac{g_{2j-1}}{\pi} r^{2j}$, where $T \approx 2\pi$ is the return time to the section and g_{2j-1} is the first non-zero coefficients in $\dot{F}$.

The proof of the analyticity of the first integral in the center case can be obtained by direct estimations of the coefficients (like in [67] and [51]). There exists a geometrical proof, due to R. Moussu [63], which uses the monodromy transformation Δ_γ associated with a loop γ in one of the separatrices $\Gamma_+ = \{x + iy + \cdots = 0\} \subset (\mathbb{C}^2, 0)$ (see Theorem 3.20 and Definition 3.42 below). Δ_γ is a holomorphic map from a small disc $D \simeq (\mathbb{C}, 0)$ transversal to Γ_+ at the base point of the loop γ (which surrounds the point $x = y = 0$ in Γ_+) to itself. Due to the vanishing of the Poincaré–Lyapunov quantities it is an identity map. So, we can define the first integral F; firstly on $(\mathbb{C}^2, 0)\backslash$separatrices by the intersection of the leaves of the complex foliation $\mathcal{F}$ with the disc D (parametrized by $F|_D : D \to (\mathbb{C}, 0)$), next on the separatrices by putting $F = 0$. This function F turns out holomorphic in $(\mathbb{C}^2, 0)$. For more details we refer the reader to [63]. □

The next is the case of nilpotent singular points:

$$\dot{x} = y + \cdots, \quad \dot{y} = \cdots.$$

Here we have so-called Takens–Bogdanov normal form

$$\dot{x} = y + a(x), \qquad \dot{y} = \varepsilon x^{s-1}, \tag{3.4}$$

where $a(x) = a_r x^r + a_{r+1} x^{r+1} + \cdots$, $a_r \neq 0$ and $\varepsilon = \pm 1$ (see Theorem 4.2 and Proposition 4.5 in Section 4). The coefficients a_j in the expansion of $a(x)$ are algebraic functions of the coefficients in the Taylor expansion of the vector field V and this normal form is analytic. The following result belongs to A. Sadovski and R. Moussu.

THEOREM 3.11. (See [68,62].) *The singularity $x = y = 0$ of a vector field V with the Takens–Bogdanov form* (3.4) *is monodromic if and only if*: s *is even*, $\varepsilon = -1$ *and either* $r > \frac{s}{2}$ *or* $r = \frac{s}{2} = 2\tilde{r}$ *is even and* $ra_r^2 < 4$.

In this case the singular point is a center iff all the coefficients a_{2j} with odd indices vanish. If a_{2j+1} is the first non-zero odd coefficient, then the singular point is stable (*respectively unstable*) *if* $a_{2j+1} < 0$ (*respectively* $a_{2j+1} > 0$).

PROOF. The conditions: s even, $\varepsilon = -1$, $s \leqslant 2r$ and $s = 2r$ but r even follow from qualitative analysis of the phase portrait of the system (3.4).

If $s = 2m$ is even then the leading part of this system, with respect to the quasi-homogeneous gradation such that degree of y equals m times degree of x, is

$$\dot{x} = y + a_{2\tilde{m}} x^{2\tilde{m}}, \qquad \dot{y} = -x^{2m-1} \tag{3.5}$$

(compare Section 4.1). Here the term $a_{2\tilde{m}} x^{2\tilde{m}} = a_m x^m$, $m = 2\tilde{m}$, vanishes when $s < 2r$ and when $s = 2r$ but r is odd. This system is invariant with respect to the reflection $x \to -x$ and reversion of time; we say that the system is *time–reversible*. Hence system (3.4) is monodromic iff system (3.5) has center at the origin. Moreover, the change $z = x^m$ and division by x^{m-1} gives a linear system with eigenvalues $\lambda_{1,2} = \frac{1}{2}(ma_m \pm \sqrt{m^2 a_m^2 - 4m})$.

In the case with real eigenvalues the linear system has an invariant line (eigenspace) which gives a characteristic trajectory for system (3.4). Therefore $m^2a_m^2 - 4m < 0$.

To prove the second statement we assume that a_{2j+1} is the first non-zero odd coefficient. Then the initial part

$$\dot{x} = y + a_{2\tilde{m}}x^{2\tilde{m}} + \cdots + a_{2j}x^{2j}, \qquad \dot{y} = -x^{2m-1}$$

of the (3.4) is time-reversible and hence with center at the origin. It has first integral $F(x, y)$, which can be not analytic at the origin, but its first approximation is a first integral $F_0(x, y)$ of the quasi-homogeneous vector field (3.5). F_0 is of the form $(y - \mu_1 x^m)^{\lambda_1}(y - \mu_2 x^m)^{\lambda_2}$. The term $a_{2j+1}x^{2j+1}\partial_x$ is the first perturbation of the latter system and we can calculate the increments of the first integral F along trajectories of the perturbed system. Near the origin it is approximately the increment of F_0 and can be expressed in terms of the Melnikov type integral $\int_{F_0=h} \frac{\partial F_0}{\partial x} x^{2j+1}\,\mathrm{d}t$, where $\mathrm{d}t = -\mathrm{d}y/x^{2m+1}$.

The result is that the latter integral is positive. $\square$

The following example demonstrates that in the case of more degenerate non-elementary monodromic singular point the center focus problem is not analytically solvable. (Another example of similar type was given by Y. Il'yashenko in [41].)

EXAMPLE 3.12. Consider the following family of vector fields

$$\dot{x} = -y\big[(2\mu + 1)x^2 + y^2\big] + x^2\big[\lambda_1 x^2 + \lambda_2(x^2 + y^2)\big],$$
$$\dot{y} = x\big[x^2 + (1 - 2\mu)y^2\big] + xy\big[\lambda_1 x^2 + \lambda_2(x^2 + y^2)\big].$$

Each vector field from this family is of the type

$$V = V_d + f \cdot E,$$

where V_d is a homogeneous vector field of degree d, $E = x\partial_x + y\partial_y$ is the *Euler vector field* and $f(x, y)$ is a homogeneous polynomial of degree $d + k$. It has first integral of the so called *Darboux–Schwarz–Christoffel type*

$$H = x^{-k}M(u) + k\int^u M(v)P(v)B(v)^{-1}\,\mathrm{d}v, \quad u = y/x,$$

where $M(u) = \mathrm{e}^{g(u)}\prod(u - u_j)^{\alpha_j}$ is of generalized Darboux type and $B(u) = \prod(u - u_j)^{m_j}$ and $P(u)$ are polynomials. Indeed, in the variables x, u we get the Bernoulli equation $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{A(u)}{B(u)}x + \frac{P(u)}{B(u)}x^{k+1}$, $P(u) = f(1, u)$, with the first integral as above, where $M = \exp\int A/B$. The center conditions mean that the function $H(x, y)$ is single–valued in $\mathbb{R}^2\setminus 0$. This implies that the homogeneous function $x^{-k}M(y/x)$ is single valued and that $\int_{-\infty}^{\infty} M(v)P(v)B(v)^{-1}\,\mathrm{d}v = 0$.

In our case the first integral equals $H = (x^2 + y^2)^{-1/2} \mathrm{e}^{\mu x^2/(x^2+y^2)} + F(y/x)$, where

$$F(u) = \int_{-\infty}^{u} \mathrm{e}^{\mu/(v^2+1)}(v^2+1)^{-3/2}\big[\lambda_1 + \lambda_2(v^2+1)\big]\,\mathrm{d}v.$$

The center condition means that $F(-\infty) = F(\infty) = 0$. It can be rewritten in the form

$$\lambda_1 \Psi'(\mu) + \lambda_2 \Psi(\mu) = 0,$$

where

$$\begin{aligned}\Psi(\mu) &= \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mu/(v^2+1)}}{\sqrt{v^2+1}}\,\mathrm{d}v = \frac{2}{\sqrt{\mu}}\mathrm{e}^{\mu} \int_0^{\sqrt{\mu}} \mathrm{e}^{-z^2}\,\mathrm{d}z \\ &= \frac{2}{\sqrt{\mu}}\mathrm{e}^{\mu}\,\mathrm{Erf}(\sqrt{\mu}) = 2\mathrm{e}^{\mu}\Phi(1/2, 3/2; -\mu)\end{aligned}$$

where Erf is the probability integral and $\Phi(a, c; x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \cdots$ is the confluent hypergeometric function. By the Kummer identity $\Phi(a, c; x) = \mathrm{e}^x \Phi(c-a, c; -x)$ (see [9]) we have $\Psi(\mu) = \Phi(1, 3/2; \mu)$.

The function Ψ has the following properties (see [9]):

- $\Psi(\mu)$ is an entire function,
- $\Psi(\mu) \sim \Gamma(\frac{3}{2})\mathrm{e}^{\mu}\mu^{-1/2}$ as $\operatorname{Re}\mu \to \infty$,
- $\Psi(\mu) \sim -\frac{1}{2\mu}$ as $\operatorname{Re}\mu \to -\infty$.

Therefore Ψ is not of the form polynomial$\times$ exp(polynomial). The same holds for the function $\psi(\mu) = \lambda_1\Psi'(\mu) + \lambda_2\Psi(\mu)$. Therefore the function $\psi(\mu)$ has infinitely many zeroes in $\mathbb{C}$ and hence the set $\{(\lambda_1, \lambda_2, \mu)\colon \lambda_1\Psi'(\mu) + \lambda_2\Psi(\mu) = 0\}$ is not semi-algebraic.

The above shows that the problem of center is not algebraically solvable.

On the other hand Y. Il'yashenko proved the following almost algebraic solvability of the center–focus problem.

THEOREM 3.13. (See [41].) *Consider a class of germs of vector fields of the form*

$$\dot{x} = P_n + \cdots, \qquad \dot{y} = Q_n + \cdots \tag{3.6}$$

where $P_n(x, y)$ and $Q_n(x, y)$ are fixed homogeneous polynomials such that the polynomial

$$R_{n+1}(x, y) = x Q_n - y P_n \tag{3.7}$$

is non-zero in $\mathbb{R}^2 \setminus 0$. *Then the problem of stability of the singularity* 0 *is algebraically solvable with respect to the coefficients of higher order terms in* (3.6).

PROOF. The phase curves of the system (3.6) in the polar coordinates r, φ satisfy the equation

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{A_{n+1}(\varphi)r + A_{n+2}(\varphi)r^2 + \cdots}{B_{n+1}(\varphi) + B_{n+2}(\varphi)r + \cdots}$$

where

$$A_{n+1}(\varphi) = S_{n+1}(\cos\varphi, \sin\varphi),\ A_{n+2}(\varphi), \ldots,$$
$$B_{n+1}(\varphi) = R_{n+1}(\cos\varphi, \sin\varphi),\ B_{n+2}(\varphi), \ldots, S_{n+1}(x, y) = xP_n + yQ_n, \ldots,$$

are trigonometric polynomials. Moreover, $B_{n+1}(\varphi) \neq 0$.

Therefore, the solutions $r = r(\varphi; r_0)$ which satisfy the initial condition $r(0; r_0)$ can be expanded into power series $\sum a_j(\varphi) r_0^j$, where the coefficients are calculated recursively. Hence the Poincaré map is also expanded into power series $\mathcal{P}(r_0) = c_1 r_0 + c_2 r_0^2 + \cdots$. The first coefficient equals

$$c_1 = \exp\left(\int_0^{2\pi} \frac{A_{n+1}(\psi)}{B_{n+1}(\psi)} \,\mathrm{d}\psi\right) = \exp\left(2\int_{-\infty}^{\infty} \frac{P_n(1,u)}{R_{n+1}(1,u)} \,\mathrm{d}u\right) \tag{3.8}$$

and the sequel coefficients depend polynomially on the coefficients in the polynomials $A_{n+2}, B_{n+2}, A_{n+3}, \ldots$. □

The quantity

$$g_1 = \ln c_1$$

is called the *first focus quantity*.

In formula (3.8) for the first focus quantity only one blowing-up of the singularity was used. N. Medvedeva generalized this formula to a more general case.

THEOREM 3.14. (See [57].) *Assume that we have a germ of the form* (3.6) *with monodromic singularity and such that the resolution of this singularity involves two blowing-ups* (*so the polynomial* $R_{n+1}(x, y)$ *has a zero*). *Then the first focus value is defined, i.e.* $c_1 = \mathcal{P}'(0) \neq 0$, *and equals*

$$g_1 = 2 \cdot \left(\text{V.P.} \int_{-\infty}^{\infty} \frac{P_n(1,u)}{R_{n+1}(1,u)} \,\mathrm{d}u\right),$$

where V.P. denotes the principal value of an integral from a rational function.

The proof of this theorem relies on decomposing the Poincaré map into composition of correspondence maps (defined by phase curves of the resolved foliation) near the saddle point (which is the intersection point of two exceptional divisors) and near the remaining parts of the divisor. We do not present this proof.

We finish this section by presenting (without proof) the following recent result of N. Medvedeva which solves the problem of analytical solvability of the Arnold's problem about center and focus. She restricts the problem to a fixed *class of germs with monodromic singular point*; such class consists of germs with fixed scheme of resolution of the singularity.

THEOREM 3.15. (See [58,59].) *The center–focus problem is analytically solvable in any class of germs with monodromic singularity. More precisely, the Poincaré map admits asymptotic expansion*

$$\mathcal{P}(\rho) = c_1\rho + \sum P_k(\ln\rho)\rho^{\nu_k},$$

where $c_1 > 0$, *the exponents* ν_j *grow to infinity and the coefficients* $P_k(\ln\rho)$ *are polynomials whose coefficients depend analytically on the coefficients of the jets* $j^{s_k}V$ *and* $s_k \to \infty$ *with* k.

3.2. *Poincaré–Dulac normal form*

Consider a germ of holomorphic vector field in $(\mathbb{C}^n, 0)$ of the form

$$\dot{x} = V(x) = Ax + \cdots \tag{3.9}$$

where A is the linearization matrix at the singular point $x = 0$. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the matrix A.

DEFINITION 3.16. We say that the eigenvalues satisfy the *resonant relation of the type* $(i; k)$ if

$$\lambda_i = \lambda_1 k_1 + \cdots + \lambda_n k_n$$

where $k = (k_1, \dots, k_n)$ and k_j are non-negative integers.

We say that the germ (3.9) is of the *Poincaré type* if the convex hull of the set $\{\lambda_1, \dots, \lambda_n\}$ in the complex plane does not contain the origin.

The following result is known as the Poincaré–Dulac normal form.

THEOREM 3.17. (See [4,67,26].) *There exists a change* $x \to y = G(x)$ *given by formal power series which transforms the system* (3.9) *to the system*

$$\dot{y}_i = \lambda_i y_i + \sum_{(i;k)\ \text{resonant}} c_{i;k} y^k, \quad i = 1, \dots, n.$$

PROOF. We apply a series of changes of the type $x \to x' = x + \phi(x)$, where $\phi = (\phi_1, \dots, \phi_n)$ is homogeneous transformation of degree m,

$$\phi_i(x) = \sum_{|k|=m} b_{i,k} x^k.$$

The inverse map has the form $x' \to x = x' - \phi(x') + \cdots$. We strive to cancel all possible terms in the vector field of homogeneous degree m.

We have $\dot{x}_i' = (\text{old part of degree} < m) + (\text{old part of degree } m) + (L_A\phi)_i(x') + \cdots$ where

$$(L_A\phi)_i = \partial\phi_i(x)/\partial x \cdot Ax' - (A\phi)_i$$

is called the *homological operator*.

If the matrix A is diagonal then L is also diagonal:

$$L_A(x^k e_i) = \left(\sum k_j\lambda_j - \lambda_i\right)(x^k e_i)$$

where e_i form the standard basis in $\mathbb{C}^n$. Because the eigenvalues of L_A vanish only at the resonant terms all the non-resonant terms in the vector field of degree m can be canceled in this way.

If A is upper-triangular then, taking the suitable ordering of the basis $x^k e_i$, we get the triangular form of the operator L_A

$$L_A(x^k e_i) = \Lambda_{i,k} x^k e_i + \sum_{j<i,l} \text{const} \cdot x^l e_j.$$

□

EXAMPLE 3.18. Consider the case $n = 2$. The resonance means

(i) $\lambda_1 = k_1\lambda_1 + k_2\lambda_2$ and/or (ii) $\lambda_2 = k_1\lambda_1 + k_2\lambda_2$.

Next we divide the problem into several subcases.

(a) If $\lambda_1 = \lambda_2 = 0$ then all terms in the expansion of the vector field are resonant and nothing can be reduced (by means of Theorem 3.17).

(b) If $\lambda_1 = 0 \neq \lambda_2$ (the s*addle–node* case) of then in (i) we must have $k_2 = 0$ and in (ii) we must have $k_2 = 1$. Thus the Poincaré–Dulac formal normal form is

$$\dot{x} = x^2 f(x), \qquad \dot{y} = \lambda_2 y\bigl(1 + g(x)\bigr).$$

(c) If $\lambda_2/\lambda_1 \geqslant 1$ is a rational number (the case of *resonant node*) then (i) cannot hold and (ii) can hold with $k_1 = \lambda_2/\lambda_1$, $k_2 = 0$ provided that $\lambda_2/\lambda_1 = m$ is integer. The normal form is either linear system or

$$\dot{x} = \lambda_1 x, \qquad \dot{y} = m\lambda_1 y + ax^m.$$

(d) If $\lambda_2/\lambda_1 = -p/q < 0$ is rational (the case of *resonant saddle*) then the normal form is

$$\dot{x} = \lambda_1 x\bigl(1 + f(x^q y^p)\bigr), \qquad \dot{y} = \lambda_2 y\bigl(1 + g(x^q y^p)\bigr).$$

(e) If λ_2/λ_1 is irrational (the non-resonant case) then the normal form is linear.

In some cases the Poincaré–Dulac normal form is analytic.

THEOREM 3.19. (See [67,17].) *If the germ $V(x)$ is of the Poincaré type (see Definition 3.16) then the change $y = G(x)$ which reduces it to the polynomial Poincaré–Dulac normal form can be chosen analytic.*

PROOF. (We follow the paper [17] by N. Brushlinskaya.) After changing (if necessary) the time $t \to ct$ we can assume that the eigenvalues lie in the right half-plane, $\operatorname{Re}\lambda_j < 0$.

Note also that there can exist only finitely many resonant relations; for instance in the case (c) of Example 3.18 there is only one such relation. Let $0 < m < -\operatorname{Re}\lambda_j < M$ and let an integer r be such that $\frac{M}{n} < r$. Then all resonant terms $x^k e_i$ have degree $|k| = k_1 + \cdots + k_n < r$. We assume that the part $W(x) = Ax + W_2(x)$ of order $< r$ of the vector field $V(x)$ is in the normal form, $V_1 = V - W = \mathrm{O}(|x|^r)$.

Choose so-called Lyapunov metric $\|\cdot\|$ in $\mathbb{C}^n$, such that for $t > 0$ we have

$$\left\|g^t(x_0)\right\| \leqslant \mathrm{e}^{-mt}\|x_0\|, \qquad \left\|(g_*^t)^{-1}\right\| \leqslant \mathrm{e}^{Mt},$$

where $g^t(x_0)$ is the phase flow map generated by the vector field W (solution $x(t)$ after the time t with initial condition $x(0) = x_0$) and $g_*^t = \frac{\partial g^t}{\partial x_0} = \Psi(t, x_0)$ is the solution of the corresponding variation equation $\dot{\Psi} = \frac{\partial W}{\partial x}\Psi$, $\Psi(0) = I$. If we transform the matrix A to the form $\operatorname{diag}(\lambda_1, \ldots, \lambda_n) + \epsilon A_1$, where A_1 is nilpotent, then the Lyapunov metric is the Hermitian metric associated with this new basis of $\mathbb{C}^n$.

As in other similar situations we reduce the problem to a fixed point problem for some operator which is contracting in suitable Banach space. We consider changes of the form $x = y + G(y)$, where $\|G(y)\| = \mathrm{O}(\|y\|^r)$. As the Banach space we choose the space $E_{r,\rho}$ of vector fields $G(y)$, analytic in a ball $\|y\| \leqslant \rho$ and having zero at $y = 0$ of order r, and with the norm $\|G\|_{r,\rho} = \sup_{\|y\|\leqslant\rho} \|G(y)\|/\|y\|^r$.

Let $L_W G = \frac{\partial G}{\partial y}W - \frac{\partial W}{\partial y}G$ be the Poisson bracket of the vector fields V and G. Then the equation for conjugation of V and W_0 by means of the diffeomorphism $y + G(y)$ leads to the equation

$$L_V G = \Phi(G), \tag{3.10}$$

where $\Phi(G) = V_1(y + G(y)) + W_2(y + G(y)) - W_2(y) - \frac{\partial W_2}{\partial y}G(y)$.

The operator L_W has the right inverse defined by means of the method of characteristics for the non-homogeneous PDE $L_W G = H$. If we put $g(t) = G(g^t(y_0))$ and $h(t) = H(g^t(y_0))$, then we get the equation $\dot{g} - a(t)g = h$, $a(t) = \frac{\partial W}{\partial y}(g^t(y_0))$. Its solution is $g(t) = -\Psi(t, y_0)\int_t^\infty \Psi^{-1}(s, y_0)h(s)\,\mathrm{d}s$, where the latter integral is absolutely convergent. Putting $t = 0$ and $y_0 = y = g^0(y)$ we find

$$G(y) = (L_W^{-1}H)(y) = -\int_0^\infty \Psi^{-1}(t, y)H\big(g^t(y)\big)\,\mathrm{d}t.$$

Due to the properties of the Lyapunov metric we have $\|H(g^t(y))\|_{r,\rho} \leqslant \|H\|_{r,\rho}\|y\|^r \mathrm{e}^{-rmt}$ and $\|\Psi^{-1}\| \leqslant \mathrm{e}^{Mt}$ and hence the operator L_W^{-1} is bounded.

Finally, standard estimated for the non-linear part $\Phi(G)$ in (3.10) imply that the operator $L_W^{-1} \circ \Phi$ is a contraction on some ball $\{\|G\|_{r,\rho} \leqslant R\}$ in the space $E_{r,\rho}$. □

The next theorem, which concerns the cases of saddle–node and of saddle, was firstly proved by C. Briot and J. Bouquet [15] and can be called the *analytic Hadamard–Perron theorem*. It will be used in further sections.

THEOREM 3.20. *Consider an analytic planar system*

$$\dot{x} = \lambda_1 x + \cdots, \qquad \dot{y} = \lambda_2 y + \cdots$$

such that $\lambda_2 \neq 0$ *and* $\lambda_1/\lambda_2 \leqslant 0$. *Then there exists an invariant analytic curve tangent to* $x = 0$ *at the origin.*

PROOF. In the case of a saddle–node we can assume that $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{f(x,y)}{\lambda_2 y + g(x,y)}$ where f, g have second order zero at the origin and $f(0, y) = \mathrm{O}(y^3)$. In the case of a saddle we can assume that $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\lambda_1 x + f(x,y)}{\lambda_2 y + g(x,y)}$ with f, g having trird order zero at the origin (because the resonant terms in the Poincaré–Dulac normal form start from cubic monomials).

We put the equation of the invariant curve in the form $x = y^2\phi(y)$. In the case of saddle–node we get the equation $y^2\phi' + 2y\phi = f(y^2\phi, y)/(\lambda_2 y + g(y^2\phi, y))$. After division by y^2 it takes the form $\phi' + 2\phi/y = F(y, \phi)$ (F – analytic) which can be written in the integral form

$$\phi(y) = y^{-2} \int_0^y s^2 F\bigl(s, \phi(s)\bigr)\,\mathrm{d}s.$$

The latter equation is solved using the contraction principle in some ball in the Banach space of analytic functions ϕ defined in a disc $|y| < \epsilon$ with the sup-norm.

In the saddle case we arrive to the equation $\phi' = (\lambda - 2)\phi/y + F(y, \phi)$, $\lambda = \lambda_1/\lambda_2 < 0$, or to the integral equation $\phi = y^{\lambda-2} \int_0^y s^{2-\lambda} F(s, \phi(s))\,\mathrm{d}s$. □

3.3. *Ecalle–Voronin moduli*

After the resolution theorem our next task is to describe the analytic classifications of planar holomorphic foliations near elementary singular points. As we have said this theory is practically completed. However before classifying the two-dimensional vector fields we shall present the classification of analogous one-dimensional objects.

These objects are the germs of analytic diffeomorphisms $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$:

$$z \to f(z) = \mu z + a_2 z^2 + \cdots. \tag{3.11}$$

DEFINITION 3.21. Two germs f and $\tilde{f}$ of conformal diffeomorphisms of $(\mathbb{C},0)$ are *analytically equivalent* (respectively, *formally equivalent, topologically equivalent)* if there is a germ $h(z)$ of analytic diffeomorphism (respectively, a formal power series $h(z) \sim h_1 z + h_2 z^2 + \cdots$, a homeomorphism $h(z)$) conjugating f with $\tilde{f}$. It means that

$$\tilde{f} = h^{-1} \circ f \circ h.$$

Analogous definition takes place in the case of multi-dimensional diffeomorphisms of $(\mathbb{C}^n, 0)$.

The diffeomorphism h transforms the orbits of $\tilde{f}$, i.e. the sets $\{\ldots, (\tilde{f})^{-1}(z), z, \tilde{f}(z), (\tilde{f})^2(z) = \tilde{f} \circ \tilde{f}(z), \ldots\}$, to the orbits of f. Thus f and $\tilde{f}$ have diffeomorphic sets of fixed points and of periodic points of given period.

The following result is an analogue of the Poincaré–Dulac normal form (Theorem 3.17).

THEOREM 3.22. *If the multiplicator μ in* (3.11) *is not a root of unity then the map f is formally linearizable, i.e. it is formally equivalent to the linear map μz.*

If $|\mu| \neq 1$ then f is analytically linearizable.

PROOF. The analogue to the homological equation for changes of the form $z \to z + g_k z^k$ applied to the maps $\mu z + a_k z^k + \cdots$ is following

$$(\mu - \mu^k) g_k = a_k.$$

This gives the first statement of the theorem.

The case $|\mu| \neq 1$ for a diffeomorphism corresponds to the Poincaré type for a vector field. The proof of analyticity of the linearizing transformation repeats the Brushlinskaya's proof of Theorem 3.19. Instead of the continuous flow $\{g^t\}_{t\in\mathbb{R}}$ one should use the discrete cascade $\{f^n\}_{n\in\mathbb{Z}}$. We omit the details. □

We focus our attention on *germs of analytic diffeomorphisms tangent to identity*

$$f(z) = z + a z^{p+1} + \cdots, \quad a \neq 0.$$

Note that, after suitable normalization, we can assume that $a = 1$. Following Il'yashenko in [42] we denote the space of such germs by $\mathcal{A}_p$.

PROPOSITION 3.23. *Any germ from $\mathcal{A}_p$ is formally equivalent to the germ*

$$f_{p,\lambda} = g_w^1,$$

where g_w^t is the phase flow map after time t of the vector field

$$w = w_{p,\lambda} = \frac{z^{p+1}}{1 + \lambda z^p} \frac{\partial}{\partial z}$$

and $\lambda \in \mathbb{C}$ is the invariant of the formal classification.

PROOF. We cancel the suitable terms in f by means of the conjugating maps $z \to z + b_l z^l$. Simple calculations show that the leading contribution from this change is the term

$$(l - p - 1)b_l z^{l+p}.$$

If $l \neq p + 1$ then the corresponding power of z in f can be canceled. There remains only the power z^{2p+1}.

Thus the formal normal form is $z + z^{p+1} + \nu z^{2p+1}$. Expanding $f_{p,\lambda}$ into powers of z we see that the latter form is formally equivalent to $f_{p,\lambda}$ for $\lambda = p + 1 - \nu$. $\square$

The space of germs f which are formally equivalent to $f_{p,\lambda}$ is denoted by $\mathcal{A}_{p,\lambda}$.

PROPOSITION 3.24. (See [18,71].) *Any germ from $\mathcal{A}_p$ is topologically equivalent to $g = f_{p,0} = z(1 - z^p)^{-1/p}$.*

PROOF. This proof relies on drawing the phase portrait of the vector field $\dot{z} = z^{p+1}/p$, whose phase flow (with real time) is $g^t(z) = z(1 - tz^p)^{-1/p}$. It consists of $2p$ elliptic sectors and $2p$ parabolic sectors. Next one shows that if we perturb the map g^1 within the class $\mathcal{A}_p$ then the qualitative behavior of the orbits remain the same. We omit the details (which can be found also in [89]). $\square$

The functional moduli of the analytical classification of germs of conformal mappings are connected with the structure of the space of orbits of the action of such a mapping on a neighborhood of the fixed point. It means that we look at the space U^*/f, where $U^* = U \setminus 0$ is a punctured neighborhood of $z = 0$ and points z and $f(z)$ are treated equivalent. (The space U^*/f is Hausdorff, but the analogous space U/f space is not Hausdorff.)

The structure of the orbit space can be better seen in the t-chart where

$$t = t(z) = -1/pz^p + \lambda \ln z$$

is the (complex) time in the vector field $w_{p,\lambda}$. Because the solution of the equation $\dot{z} = w_{p,\lambda}$ is $t = t(z) + \text{const}$, the map $f_{p,\lambda} = g_w^1$ takes the simple form

$$t \to t + 1.$$

Assume, for example, that $\lambda = 0$ (then $t(z)$ is single-valued) and that $p = 1$. Then the neighborhood $U^* = \{|z| < r\}$ is replaced by $W = \{|t| > R\}$. The set W is divided into the vertical strips of width 1. Identifying two sides of such strip we get a cylinder which can be identified with the doubly punctured sphere $\mathbb{C}^*$: $-\mathrm{i}\infty \to 0$, $+\mathrm{i}\infty \to \infty$. The space $W/(\mathrm{id} + 1)$ consists of two copies of $\bar{\mathbb{C}}$ with identifications near 0 and ∞, like at Fig. 6. The vertical strips from the t-chart correspond to the crescents in the z-chart.

In the general case (λ and p arbitrary and f close to $f_{p,\lambda}$) the chart $t(z)$ is a well defined diffeomorphism in sectors (of width $> 2\pi/p$)

$$S_j = \left\{ \frac{j-2}{2p}\pi - \delta < \arg z < \frac{j}{2p}\pi + \delta \right\},$$

$j = 1, \dots, 2p$, where $\delta > 0$ is small.

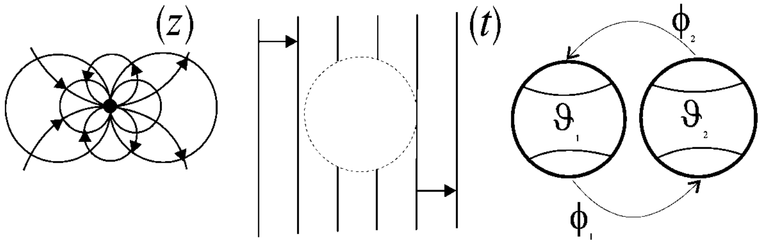

Fig. 6.

The function $t(z)$ sends S_j to $\tilde{S}_j$. In $\tilde{S}_j$ we denote by $t_j = t(z)|_{S_j}$ the local charts, where the branches of $\ln z$ are such that they are real in $S_1 \cap \mathbb{R}$, $t_j = t_{j+1}$ in the intersections $S_j \cap S_{j+1}$ for $j < 2p$ and $t_{2p} = t_1 + 2\pi \mathrm{i}\lambda$.

The maps $f_{p,\lambda}$ and f are replaced by the mappings

$$\tilde{f}_{p,\lambda} : t_j \to t_j + 1,$$
$$\tilde{f} : t_j \to t_j + 1 + R_j(t_j),$$

in $\tilde{S}_j$. Here we can assume that

$$R_j(t) = \mathrm{O}(t^{-M})$$

for some large M. The vertical strips from Fig. 5 and orbit spaces will be only slightly perturbed. The vertical lines are replaced by the iterations of one line, $f^n(l_0)$ where, due to the fact that $R(t)$ is small, there are no intersections between them.

LEMMA 3.25. *Each quotient space $\tilde{S}_j / \tilde{f}$ is conformally equivalent to $\mathbb{C}^*$.*

PROOF. This quotient space, which we denote by T, is obtained from the (non-straight) strip between the lines $l_0 = \{\operatorname{Re} t = c\}$ and $\tilde{f}(l_0)$ by identifying its sides. Here $\tilde{f}(c + \mathrm{i}\tau) = c + \mathrm{i}\tau + 1 + R_j(c + \mathrm{i}\tau)$ where $R(t)$ is small.

We treat $\mathbb{C}^*$ as the straight strip $c \leqslant \operatorname{Re} t \leqslant c + 1$ with identified sides. It is parametrized by $\theta \in [0, 1]$ and $\tau \in \mathbb{R}$: $t = c + \theta + \mathrm{i}\tau$.

We construct a map from $\mathbb{C}^*$ to T

$$F(c + \theta + \mathrm{i}\tau) = c + \theta + \mathrm{i}\tau + \theta R_j(t).$$

Because $F'_t = 1 + R(t)/2 + \theta R'_j(t)$, $F'_{\bar{t}} = R_j(t)/2$, the map F is quasi-conformal and satisfies the condition of regularity from Proposition 3.74 in Appendix 1. The modulus of quasi-conformality $\mu_F = F'_{\bar{t}}/F'_t$ is small and tends to zero at the ends of the strip $\{c < \operatorname{Re} t < c + 1\}$; (they correspond to 0 and ∞ in $\bar{\mathbb{C}}$).

Applying Proposition 3.40 from Appendix 1 we get the assertion of the lemma. $\square$

Therefore, U^*/f is topologically equivalent to $2p$ copies of $\bar{\mathbb{C}}$ with suitable identifications of the neighborhoods of 0 and ∞. More precisely, we have the spheres $\bar{\mathbb{C}} \times \{j\}$, $j = 1, \ldots, 2p$, and identifying maps of the form:

$$\phi_j : (\bar{\mathbb{C}}, 0) \times \{j\} \to (\bar{\mathbb{C}}, 0) \times \{j+1\}, \quad j \text{ even},$$
$$\phi_j : (\bar{\mathbb{C}}, \infty) \times \{j\} \to (\bar{\mathbb{C}}, \infty) \times \{j+1\}, \quad j \text{ odd}.$$

The functional invariants of the classification of germs of maps are the invariants of the holomorphic structure on the space U^*/f.

In the spheres one uses conformal coordinates ϑ_j, e.g. $\vartheta_j = \mathrm{e}^{2\pi \mathrm{i} t_j}$ if $f = f_{p,\lambda}$. Generally, $\vartheta_j = \mathrm{e}^{2\pi \mathrm{i} \tau_j}$ for suitable 'times' $\tau_j = t_j + \mathrm{o}(1)$.

ϑ_j are defined uniquely modulo multiplication by a constant. If we have chosen ϑ_1, then ϑ_2 can be taken in such a way that $\vartheta_2 = \phi_1(\vartheta_1) = \vartheta_1 + \mathrm{o}(\vartheta_1)$ as $\vartheta_1 \to \infty$, $\vartheta_3 = \phi_2(\vartheta_2) = \vartheta_2 + o(\vartheta_2)$ is such that $\phi_2'(0) = 1$, etc. However the charts ϑ_{2p} and ϑ_1 near ∞ cannot be correlated in this way. We have $\vartheta_{2p} = \mathrm{e}^{2\pi \mathrm{i} t_{2p}} = \mathrm{e}^{2\pi \mathrm{i} t_1 - 4\pi^2 \lambda} = \nu \vartheta_1$. Thus $\phi_{2p}(\vartheta_{2p}) = \nu \vartheta_{2p} + \mathrm{o}(\vartheta_{2p})$, $\nu = \exp(-4\pi^2 \lambda)$.

REMARK 3.26. Changing the chart $\vartheta_1 \to C\vartheta_1$ induces the same changes in the spheres $\bar{\mathbb{C}}_j$ and the maps ϕ_j become conjugated by means of C. This non-uniqueness of the charts ϑ_j is equivalent to the non-uniqueness of the 'times', $\tau_j \to \tau_j + \theta$ where $C = \mathrm{e}^{2\pi \mathrm{i}\theta}$, and the non-uniqueness of the coordinate z, $z \to g_w^\theta(z)$.

The above suggests the following definition.

DEFINITION 3.27. Consider the space of collections $\phi = (\phi_1, \ldots, \phi_{2p})$ of germs of maps $\phi_j : (\bar{\mathbb{C}}, 0(\infty)) \times \{j\} \to (\bar{\mathbb{C}}, 0(\infty)) \times \{j+1\}$ with identity linear parts for $j < 2p$ and $\phi_{2p}'(\infty) = \nu$. Two such collections ϕ, ϕ' are called *equivalent*, $\phi \sim \phi'$, iff there exists $C \in \mathbb{C}^*$ such that

$$\phi_i \circ C = C\phi_i'.$$

The set of equivalence classes of such ϕ's is *the moduli space* and is denoted by $\mathcal{M}_{p,\lambda}$.

If $f \in \mathcal{A}_{p,\lambda}$, then the construction (with the strips in $\tilde{S}_j$'s and identifications of U^*/f with a collection of glued together Riemann spheres) defines a collection ϕ. Thus we have a mapping from $\mathcal{A}_{p,\lambda}$ to $\mathcal{M}_{p,\lambda}$. It is the first statements of the following *Ecalle–Voronin theorem*.

THEOREM 3.28. (See [28,83].) *There is a map $f \to \mu_f$ from $\mathcal{A}_{p,\lambda}$ to $\mathcal{M}_{p,\lambda}$ with the following properties*:

- *if f and $\tilde{f}$ are holomorphically equivalent then $\mu_f = \mu_{\tilde{f}}$ (invariance)*;
- *if $\mu_f = \mu_{\tilde{f}}$ then f and $\tilde{f}$ are holomorphically equivalent (equimodality)*;
- *for any class $[\phi] \in \mathcal{M}_{p,\lambda}$ there exists $f \in \mathcal{A}_{p,\lambda}$ such that $[\phi] = \mu_f$ (realization).*

J. Ecalle proved a variant of this result in 1975. His proof relies on exploiting the Borel transform. S. Voronin's proof appeared in 1981 and it relies on application of analytic methods (almost complex structures and quasi-conformal mappings). Below we present the Voronin's proof (following the Il'yashenko's paper [42]).

However it must be said that the Ecalle–Voronin moduli were already described by G. Birkhoff in 1939 in [12] and partial results appeared in the woks of L. Leau [45] and other mathematicians. Very interesting review of the history of the problem of analytic classification is presented in the lectures of F. Loray [49].

The essential part of the proof of the Ecalle–Voronin theorem is the fact that the dynamics defined by the map f and by the model map are conjugated holomorphically in sectors.

THEOREM 3.29. *In every sector S_j there is an unique analytic diffeomorphism of the form*

$$H_j(z) = z + h_j(z), \quad h_j = \mathrm{o}(z^{p+1}),$$

conjugating f with $f_{p,\lambda}$.

If we skip the assumption $h_j = \mathrm{o}(z^{p+1})$ then we can replace H_j by $g_w^\theta \circ H_j$ for some $\theta \in C$, where g_w^θ is the phase flow generated by the vector field $w = [z^{p+1}/(1+\lambda z^p)]\partial_z$.

PROOF. It is enough to construct the conjugating diffeomorphisms in the t-charts, i.e. $\tilde{H}_j(t) = t + \tilde{h}_j(t)$, $t \in \tilde{S}_j$. Then $H_i = t^{-1} \circ \tilde{H}_j \circ t$.

The conjugation condition $\tilde{H}_j \circ \tilde{f} = \tilde{f}_{p,\lambda} \circ \tilde{H}_j$, i.e. $t + 1 + R_j + \tilde{h}_j \circ \tilde{f} = t + \tilde{h}_j + 1$, means that

$$\tilde{h}_j = \tilde{h}_j \circ \tilde{f} + R_j.$$

Iterating this equation (with respect to $\tilde{h}_j$) infinitely many times, we find the solution

$$h(t) = \sum_0^\infty R_j \circ \tilde{f}^n(t).$$

The condition $|R_j| < |t|^{-M}$ and the property $\tilde{f}^n(t) \sim \mathrm{const} \cdot n$ ensures that this series is convergent and represents an analytic function of t.

To prove the second statement we compose f with suitable g_w^θ such that $f - f_{p,\lambda} = \mathrm{o}(z^{p+1})$ and apply the first statement. □

Take the system $(H_1, \ldots, H_{2p})$ of germs diffeomorphisms from Theorem 3.29. It can be interpreted as certain Čech cochain. Namely we consider the (non-Abelian) sheaf on S^1 of germs of analytic diffeomorphisms strongly tangent to identity. If U is an open arc in S^1 then the group of sections of this sheaf at U consists of diffeomorphism $H_{S(U)} = \mathrm{id} + h_{S(U)}$, $h_{S(U)}(z) = \mathrm{O}(z^{p+1})$ defines on a sector $S(U)$ with base at U (in the polar coordinates $S(U) = \{z = r\mathrm{e}^{\mathrm{i}\theta}, \theta \in U\}$, see [54]).

The system

$$H = (H_1, \dots, H_{2p})$$

defines the Čech cochain associated with the covering of S^1 induced by the sectors S_i. We consider the coboundary of this cochain, i.e. $\delta H = (\Phi_1, \dots, \Phi_{2p})$

$$\Phi_j = H_{j+1} \circ H_j^{-1}$$

in $S_j \cap S_{j+1}$.

PROPOSITION 3.30.
(a) *The maps Φ_j commute with the standard map $f_{p,\lambda}$.*
(b) *They are exponentially close to identity*

$$\left|\Phi_j(z) - z\right| < \mathrm{e}^{-c/|z|^p};$$

(*equivalently, $H_j - H_{j+1}$ are exponentially small*).
(c) *They are defined uniquely up to conjugation with the map g_w^θ for some $\theta \in \mathbb{C}$.*
(d) *In suitable charts ϑ_j in the spheres $\bar{\mathbb{C}} \times \{j\}$ they define the element μ_f of the moduli space $\mathcal{M}_{p,\lambda}$.*

PROOF. (a) The first statement follows from the identity $H_j \circ f = f_{p,\lambda} \circ H_j$.

(b) The second property follows from the first one. Indeed, in the t-chart the commutativity of the induced map

$$\tilde{\Phi}_j(t) = t + \tilde{\phi}_j(t)$$

with the standard map $t \to t+1$ means that the functions $\tilde{\phi}_j(t)$ are periodic, $\tilde{\phi}_j(t+1) = \tilde{\phi}_j(t)$. Thus $\tilde{\phi}_j$ are expanded into the Fourier series $\sum c_{jl} \mathrm{e}^{2\pi \mathrm{i} l t}$.

The function Φ_j is analytic in $S_j \cap S_{j+1}$, which corresponds to the domain $\operatorname{Im} t > t_0 > 0$ in the $t = t_j$-chart in $\tilde{S}_j$ if j is odd, and to the domain $\operatorname{Im} t < -t_0$ if j is even. Moreover, $\tilde{\Phi}_j(t) = t + \mathrm{o}(t)$ as $t \to \infty$ (because $\tilde{H}_j$ have this property).

This implies that

$$\begin{aligned}
\tilde{\Phi}_j(t_j) &= t_j + \sum_{l>0} c_{jl} \mathrm{e}^{2\pi \mathrm{i} l t_j}, && j \text{ odd}, \\
\tilde{\Phi}_j(t_j) &= t_j + \sum_{l<0} c_{jl} \mathrm{e}^{2\pi \mathrm{i} l t_j}, && j \text{ even}, \\
\tilde{\Phi}_{2p}(t_{2p}) &= -2\pi \mathrm{i}\lambda + t_{2p} + \sum_{l<0} c_{jl} \mathrm{e}^{2\pi \mathrm{i} l t_j}. &&
\end{aligned} \tag{3.12}$$

The constant in the latter formula follows from the difference between the local charts $t_{2p} = t_1 + 2\pi \mathrm{i}\lambda$ and t_1.

Now we have $\Phi_j = t_{j+1}^{-1} \circ \tilde{\Phi}_j \circ t_j$ with $t_j = -1/(pz^p) + \lambda \ln z$. If j is odd then we see that $\Phi_j(z) = z + \mathrm{O}(|c_{j1} z^{\alpha} \mathrm{e}^{-2\pi \mathrm{i}/(pz^p)}|)$. Analogous estimates hold in the case of even j.

(c) This point follows from the second statement of Theorem 3.29.

(d) Recall that the moduli space $\mathcal{M}_{p,\lambda}$ is the space of conformal structures on the orbit space U^*/f. Thus we should associate with any system $\tilde{\Phi}$ a system of conformal charts on the spaces S_j/f (or $\tilde{S}_j/\tilde{f}$) and a system of gluing maps between them.

We introduce new charts $\tau_j := \tilde{H}_j(t_j) = t_j + \cdots$ in $\tilde{S}_j$ having the property that they conjugate f with the translation $\mathrm{id} + 1$. They are also defined by the formulas

$$\tau_j = t_j \circ H_j.$$

One easily checks that $\tau_j \circ f \circ \tau_j^{-1} = \mathrm{id} + 1$. It is also clear that the space $\tilde{S}_j/\tilde{f}$ is conformally equivalent to the τ_j-plane divided by the action of the translation $\mathrm{id} + 1$.

The identification of $\tilde{S}_j/\tilde{f}$ with $\mathbb{C}^* = \mathbb{C}^* \times \{j\}$ is given by the formula

$$\tau_j \to \vartheta_j = \mathrm{e}^{2\pi \mathrm{i} \tau_j}.$$

The gluing maps $\vartheta_j \to \vartheta_{j+1} = \phi_j(\vartheta_j)$ (see Definition 3.27) arise from the differences between τ_j and τ_{j+1} in $\tilde{S}_j \cap \tilde{S}_{j+1}$. We have $\phi_j = [\exp 2\pi \mathrm{i}(\cdot)] \circ (\tau_{j+1} \circ \tau_j^{-1}) \circ [(1/2\pi \mathrm{i}) \ln(\cdot)]$. Therefore we should find an expression for $\tau_{j+1} \circ \tau_j^{-1}$. We have

$$\tau_{j+1} \circ \tau_j^{-1} = t_{j+1} \circ H_{j+1} \circ H_j^{-1} \circ t_j^{-1} = \tilde{\Phi}_j.$$

If $\tilde{\Phi}_j$ are defined by the formulas (3.12) then $\phi_j(\vartheta_j) = \exp\{2\pi \mathrm{i}[\tau_j + \sum c_{jl} \vartheta_j^l]\}$, or

$$\phi_j = \vartheta_j + \sum d_{j,l} \vartheta_j^l,$$

$$\phi_{2p} = \nu \vartheta_{2p} + \sum d_{jl} \vartheta_{2p}^l.$$

(Above the sums run either over positive or over negative l's.)

Like t_j, the variables τ_j are defined uniquely modulo translation by a constant θ. This means that the charts ϑ_j in $\mathbb{C}^* \times \{j\}$ are defined uniquely modulo multiplication by the constant $C = \mathrm{e}^{2\pi \mathrm{i} \theta}$. □

PROOF OF ECALLE–VORONIN THEOREM 3.28. (a) *Invariance of the class* μ_f. Let f, g be two analytically equivalent germs from the class $\mathcal{A}_{p,\lambda}$, h being the conjugating holomorphism: $h \circ f = g \circ h$. Let H, G be the cochains normalizing f, g respectively. Then the cochain $G \circ h$ normalizes f. Moreover, we find that $g \circ h = g_w^{\theta} \circ H$ (as H and $G \circ h$ conjugate the same f with $f_{p\lambda}$). So

$$G_{j+1} \circ G_j^{-1} = g_w^{\theta} \circ (H_{j+1} \circ H_j^{-1}) \circ g_w^{-\theta}.$$

This means that, if $\tilde{\Phi}$, $\tilde{\Psi}$ are the coboundaries associated with f, g (expressed in the t-charts), then they are equivalent by means of the translation map $\mathrm{id} + \theta$. Hence the maps

$\phi_j = [\exp 2\pi \mathrm{i}(\cdot)] \circ \tilde{\Phi}_j \circ [(1/2\pi \mathrm{i}) \ln(\cdot)]$ and $[\exp 2\pi \mathrm{i}(\cdot)] \circ \tilde{\Psi}_j \circ [(1/2\pi \mathrm{i}) \ln(\cdot)]$ are equivalent by means of a rescaling $C \times \mathrm{id}$.

(b) *Equimodality*. Let $\tilde{\Phi}$, $\tilde{\Psi}$ be two collections associated with maps f, g respectively. Assume that they are equivalent by means of translation. Therefore choosing another normalizing cochain we can obtain the coincidence of the cochains $\tilde{\Phi}$ and $\tilde{\Psi}$.

But the identity $G_{j+1} \circ G_j^{-1} = H_{j+1} \circ H_j^{-1}$ is equivalent to

$$H_j^{-1} \circ G_j = H_{j+1}^{-1} \circ G_{j+1}.$$

This means that the maps $H_j^{-1} \circ G_j$ are prolonged to a punctured neighborhood of $z = 0$ as a single valued mapping. By the Riemann removability of singularities theorem we prolong it to an analytic map h in a whole neighborhood of 0.

Moreover, because G_j conjugates g with $f_{p,\lambda}$ and H_j^{-1} conjugates $f_{p,\lambda}$ with f then h conjugates g with f.

(c) *Realization*. Let $\mu \in \mathcal{M}_{p,\lambda}$ be an element of the moduli space. We have to show that $\mu = \mu_f$ for some map f.

We choose a representative $\tilde{\Phi}$ of μ and associated with it the 1-cocycle $\Phi = (\Phi_j)$ of germs of holomorphic maps in the sectors $S_j \cap S_{j+1}$, all but one tangent to the identity.

Consider the disjoint collection of the sectors, i.e. $S_j \times \{j\}$. Denote by z_j the charts induced by (z, j). We glue these sectors using the mappings $\Phi_j : z_{j+1} \sim \Phi_j(z_j)$. We obtain certain topological space $\mathcal{S}$ homeomorphic to a punctured disc. This space has a conformal structure arisen from the coordinates z_j. The space $\mathcal{S}$ (and its closure $\bar{\mathcal{S}}$) admits a homeomorphism f_0 induced from $f_{p,\lambda}$ at each $S_j \times \{j\}$. f_0 is analytic with respect to the conformal structure. Our task is to show that the conformal structure of $\bar{\mathcal{S}}$ coincides with the conformal structure of the punctured disc and that f_0 belongs to $\mathcal{A}_{p,\lambda}$.

Let $\chi_j(z)$ be an infinitely smooth partition of unity associated with the covering S_j. We need that the functions χ_j and their gradients have at worst power type singularities as $z \to 0$.

We construct the map

$$H_0 = \sum \chi_j(z_j) z_j$$

from $\mathcal{S}$ to a punctured disc $D^* \subset \mathbb{C}$.

LEMMA 3.31. *The mapping H_0 is quasi-conformal.*

PROOF. (See Appendix 1.) In $S_j \cap S_{j+1}$ we have $H_0 = \chi_j z_j + \chi_{j+1} z_{j+1} = z_j + (\Phi_j(z_j) - z_j)\chi_{j+1}$ which implies

$$\partial H_0/\partial z_j = 1 + \mathrm{o}(1), \qquad \partial H_0/\partial \bar{z}_j = (\Phi_j - z_j)\partial \chi_{j+1}/\partial \bar{z}_j = \mathrm{o}(1).$$

This means that the almost complex structure, induced on D^* by H_0, is defined by the Beltrami differential $\mu(\mathrm{d}\bar{z}/\mathrm{d}z)$, where $\mu = \mu(H_0^{-1})$ is exponentially small together with its derivatives. $\square$

Let us prolong μ to the whole disc D. By Theorem 3.72 from Appendix 1 the almost complex structure on D is integrable and there exists a diffeomorphic mapping $G : D \to \mathbb{C}$ such that $\mu_G = \mu$. This means that the composition $H = G \circ H_0 : \mathcal{S} \to \mathbb{C}$ is conformal and $\mathcal{S}$ is conformally equivalent to D^*.

The mapping f_0 in $\mathcal{S}$, induced from $f_{p,\lambda}$, is prolonged to a mapping of D. We denote it by f. Of course, it is analytic.

Because the functions μ, G, H are smooth and f is conjugated by means of H with $f_{p,\lambda}$ at the sectors S_j, then their linear parts are also conjugated. So $f'(0) = 1$ and $f \in \mathcal{A}_l$ for some l.

But the index l is a topological invariant (see Proposition 3.24), it is the number of 'elliptic sectors' (divided by 2). So $l = p$.

Finally, the formal classification invariant λ is included into the functional modulus $\tilde{\Phi}_{2p} = -2\pi \mathrm{i}\lambda + t_{2p} + \cdots$.

Note that the maps $H_j = z_j \circ H^{-1}$ form the normalizing maps for f (the normalizing cochain). Its coboundary maps are $z_{j+1} \circ z_j{}^{-1} = \Phi_j$. This means that the collection $\tilde{\Phi}$ is the functional modulus for f.

Theorem 3.28 is complete. $\square$

Now we consider conformal maps of the form

$$f(z) = \mathrm{e}^{2\pi \mathrm{i} m/n} z + \cdots, \tag{3.13}$$

i.e. $f'(0)$ is a primitive root of unity of order n. We call them $m : n$ *resonant maps*.

PROPOSITION 3.32. *Any germ of this form is formally equivalent to*

$$\mathrm{e}^{2\pi \mathrm{i} m/n} g_w^1, \quad w = \frac{z^{np+1}}{1 + \lambda z^{np}} \frac{\partial}{\partial z}.$$

PROOF. The Poincaré–Dulac theorem for diffeomorphisms allows to reduce f to the form $\mathrm{e}^{2\pi \mathrm{i} m/n} z(1 + \sum a_l z^{nl})$. Let p be the first index l with non-zero a_l. We can assume that the coefficient $a_p = 1$.

Next, repeating the proof of Proposition 3.21, we successively cancel the terms $a_l z^{nl+1}$ using the changes $z \to z + b_l z^{n(l-p)}$. It is possible to do it in all cases but $l = 2p$. $\square$

One denotes by $\mathcal{A}_{m,n,p,\lambda}$ the space of germs with the formal normal form having the indicated indices.

Let us pass to analytic classification of the germs from $\mathcal{A}_{m,n,p,\lambda}$. Note firstly that f^n is tangent to identity, it belongs to $\mathcal{A}_{q,\beta}$, $q = np$ with some β (determined by λ).

Consider the orbit space U^*/f^n, consisting of $2np$ Riemann spheres $\mathbb{C}^* \times \{j\}$. The map f transforms the orbits of f^n to orbits of f^n and induces the map $\pi_* f : U^*/f^n \to U^*/f^n$. We describe the action of this map on the Riemann spheres.

Note that the sphere $\mathbb{C}^* \times \{j\}$ is the quotient of the sector S_j with the bisectrix $\arg z = \pi(j-1)/(2pn)$ and the transformation f is approximately the rotation by the angle $2\pi m/n$. Therefore $f(S_j) \approx S_{j+2pm}$ and $\pi_* f : \mathbb{C}^* \times \{j\} \to \mathbb{C}^* \times \{j + 2pm\}$.

Moreover, $\pi_* f$ is a holomorphic diffeomorphism and takes the form $\pi_* f(\vartheta_j) = C_j \vartheta_{j+2pm}$. It means also that $\vartheta_{j+2pm} \circ f = C_j^{-1} \vartheta_j$. One can see that $C_j = \mathrm{e}^{2\pi \mathrm{i} r/n}$ is a root of unity.

Indeed, because $\phi_j = \vartheta_{l+1} \circ \vartheta_l^{-1}$ are close to the identity (or to the multiplication by the constant ν for $l = 2pn$) then $\vartheta_{j+2pm+1} \circ \vartheta_{j+2pm}^{-1} = \vartheta_{j+2pm+1} \circ f \circ f^{-1} \circ \vartheta_{j+2pm}^{-1} = C_{j+1}^{-1} \vartheta_{j+1} \circ \vartheta_j^{-1} \circ C_j = C_{j+1}^{-1} \phi_j \circ C_j = \mathrm{id} + \cdots$. So $C_j = C_{j+1} = \cdots = C$. Because $\pi_* f^n = \mathrm{id}$ we get $C^n = 1$.

From the above it follows that only $2p$ first gluing maps are needed to determine the functional modulus $\mu^*_{f^n}$. Indeed, we have $\phi_{j+2pm} = \mathrm{e}^{-2\pi \mathrm{i} r/n} \phi_j \circ \mathrm{e}^{2\pi \mathrm{i} r/n}$. For the last gluing map we get $\phi_{2pn} = \nu \mathrm{e}^{-2\pi \mathrm{i} r/n} \phi_{2p(n-m)} \circ \mathrm{e}^{2\pi \mathrm{i} r/n}$, where $\nu = \phi'_{2pn}(\infty)$ is the constant expressed by the formal modulus λ.

All this leads to the following result.

THEOREM 3.33. (See [42].) *The space $\mathcal{A}_{m,n,p,\lambda}$ of germs of conformal diffeomorphism modulo analytic equivalence is in one-to-one correspondence with the space $\mathcal{M}_{m,n,p,\lambda}$ of systems*

$$\phi = (\phi_1, \ldots, \phi_{2p})$$

(*of germs of analytic diffeomorphisms $\phi_j(\bar{\mathbb{C}}, 0) \to (\bar{\mathbb{C}}, 0)$ $(j$ odd$)$ or $\phi_j(\bar{\mathbb{C}}, \infty) \to (\bar{\mathbb{C}}, \infty)$ $(j$ even$)$ with identity linear parts*) *modulo the equivalence $\phi \sim D\phi \circ D^{-1}$.*

3.4. *Martinet–Ramis moduli*

The saddle–node singularity of analytic planar vector field is such that one of its eigenvalues vanishes and the other eigenvalue is non-zero. Firstly we present the formal classification of saddle–nodes and next we describe the moduli of their analytic classification.

PROPOSITION 3.34. *Any complex analytic saddle–node is formally orbitally equivalent to*

$$\dot{x} = x^{p+1}, \qquad \dot{y} = -y(1 + \lambda x^p).$$

PROOF. The Poincaré–Dulac theorem 3.17 and Example 3.18(b) give the following (non-orbital) normal form $\dot{x} = a_{p+1} x^{p+1} + a_{p+2} x^{p+2} + \cdots$, $\dot{y} = \lambda_2 y(1 + b_1 x + b_2 x^2 + \cdots)$, where $a_{p+1} \neq 0$; if all $a_j = 0$ then the singular point would have infinite multiplicity (non-isolated in the complex analytic case). By linear changes of variables and time we can assume that $a_{p+1} = 1$, $\lambda_2 = -1$.

Now we divide the field by $1 + b_1 x + \cdots$ what gives $\dot{y} = y$ and

$$\dot{x} = x^{p+1}(1 + c_1 x + \cdots). \tag{3.14}$$

Therefore the problem is reduced to formal *non-orbital* classification of 1-dimensional saddle–nodes (3.14).

The latter problem is solved in the same way as in the proof of Theorem 3.23 (about formal classification of diffeomorphisms $z+z^{p+1}+\cdots$). Moreover, this special change can be made analytic. Using the changes $x \to x+d_l x^l$, we obtain the (leading) term $(l-p-1)d_l x^{l+p}\partial_x$ in the transformed vector field. If $l \neq p+1$, then the corresponding monomial vector field can be canceled. There remains only the field $w_{p,\nu} = x^{p+1}(1+\nu x^p)\partial_x$ where ν is the formal invariant.

Of course, one can fix also other terms in ∂_x, like $x^{p+1}(1-\lambda x^p+(\lambda x^p)^2-\cdots) = x^{p+1}(1+\lambda x^p)^{-1}$, $\lambda = -\nu$. Multiplication of this vector field by $1+\lambda x^p$ gives the vector field from thesis of the proposition. $\square$

The integer p is called the *codimension* of saddle–node and $\lambda \in \mathbb{C}$ is the *modulus of formal classification.* The space of germs with the formal normal form as in Proposition 3.34 will be denoted by $\mathcal{E}_{p,\lambda}$.

As we shall see, the normal form from Proposition 3.34 is not analytic. However some of its statements have analytic equivalents. Note that it has two formal invariant curves: the *strong manifold* $x=0$ and the formal *center manifold* $y=0$. By Theorem 3.20 the strong manifold is always analytic but usually the center manifold is not analytic.

EXAMPLE 3.35. (See [33].) The equation $\mathrm{d}y/\mathrm{d}x = (y-x)/x^2$ has the following unique formal solution $y = \sum_1^\infty (n-1)!x^n$ passing through 0.

LEMMA 3.36. *There exist local analytic coordinates and an analytic change of time which give the following initial analytic form*

$$\dot{x} = x^{p+1}, \qquad \dot{y} = -y(1+\lambda x^p) + g_0(x) + y^2 g_2(x,y),$$

where g_0, $y^2 g_2$ *have zero of order* $\geqslant p+2$ *at the origin.*

PROOF. Since the strong manifold is analytic we can assume that the vector field has the form (in some analytic coordinates) $\dot{x} = xf(x,y)$, $\dot{y} = -y+g(x,y)$. This preliminary form is not satisfactory to our purposes. We want to get the factor x^{p+1} in the first component.

This is done using series of the analytic changes $x \to x + x^j\psi_j(y)$, $j=1,\ldots,p$. Let $\dot{x} = x[\phi_1(y)+\mathrm{O}(x^2)]$ where $\phi_1(0)=0$. Then we use the change $x = x_1(1+\psi_1(y))$ such that $-y\psi_1' + \phi_1(y) = 0$. Similarly, further terms $x^j\phi_j(y)$, $\phi_j(0)=0$, for $j=2,\ldots,p$, are removed from $\dot{x}$. Thus we can put $\dot{x} = x^{p+1}$ and $\dot{y} = g_0(x)+yg_1(x)+y^2g_2(x,y)$.

Moreover, using polynomial changes in y we can cancel some initial terms from g_0 and g_2. The function $g_1(x)$ is reduced to $-1-\lambda x^p$ by a change like in the proof of Proposition 3.34; it turns out to be analytic. $\square$

We define the *extended sectors*

$$T_j = S_j' \times \{|y| < \epsilon\}$$

where $S_j' = \{|x| < \epsilon, \frac{(2j-1)\pi}{2p} - \alpha < \arg x < \frac{(2j-1)\pi}{2p} + \alpha\} = \mathrm{e}^{\pi \mathrm{i}/2p} S_j$, $\alpha \in (\frac{\pi}{2p}, \frac{\pi}{p})$ are sectors rotated with respect to the sectors S_i from the previous subsection. The following

important result belongs to M. Hukuhara, T. Kimura and T. Matuda. Its proof is postponed to Appendix 2.

THEOREM 3.37. (See [40].) *In each extended sector* T_j *there exists a unique analytic diffeomorphism*

$$H_j(x,y) = \left(x, y + \tilde{H}_j(x,y)\right), \quad \tilde{H}_j = \mathrm{O}(x^{p+1}),$$

conjugating the system from Lemma 3.36 *with its formal normal form from Proposition* 3.34.

We pass to description of the Martinet–Ramis moduli of analytical orbital equivalence of germs from $\mathcal{E}_{p,\lambda}$.

The reader can guess that these moduli are constructed using the differences between the normalizing diffeomorphisms,

$$\Phi_j = H_{j+1} \circ H_j^{-1} \quad \text{in } T_j \cap T_{j+1}.$$

However, the description of the Φ_j's can be simplified.

Notice that the diffeomorphisms Φ_j keep the first coordinate fixed and preserve the formal normal form $\omega_0 = x^{p+1}\,\mathrm{d}y + y(1+\lambda x^p)\,\mathrm{d}x = 0$ (of the corresponding Pfaff equation). The latter fact gives series of restrictions onto possible forms of Φ_j's.

Let us look at the behavior of the leaves of the foliation defined by $\omega_0 = 0$ in the sectors $T_j \cap T_{j+1}$. These leaves are given by

$$y = C\mathrm{e}^{-t(x)}, \quad t(x) = t_j(x) = -1/(px^p) + \lambda \ln x, \quad C \in \mathbb{C},$$

and the first integral takes the form

$$h(x,y) = h_j(x,y) = y\mathrm{e}^{t(x)} = y\mathrm{e}^{t_j(x)}.$$

(Because of $\ln x$ in the definition of $t(x)$ we index the 'times' and first integrals in different sectors. Thus $t_{2p} = t_1 + 2\pi\mathrm{i}\lambda$, $h_{2p} = \mathrm{e}^{-2\pi\mathrm{i}\lambda}h_1$.)

The neighborhood of $x = y = 0$ is divided into sectors of two types:

- the sectors of *jump* where $\mathrm{Re}\,x^{-p} \to +\infty$ and $\mathrm{e}^{-t(x)} \to \infty$ (in particular, $\{x > 0\}$ lies in a sector of jump $T_{2p} \cap T_1$),
- the sectors of *fall* where $\mathrm{Re}\,x^{-p} \to -\infty$ and $\mathrm{e}^{-t(x)} \to 0$.

In the sectors of jump only one leaf approaches the singularity, it is the *local center manifold*. Other leaves diverge as $x \to 0$.

In the sectors of fall all the leaves are indistinguishable. All approach the singularity. We see also that the extended sectors $T_{j+1} \cap T_j$ lie wholly either in the sectors of jump or in the sectors of fall.

PROPOSITION 3.38. *The transition diffeomorphisms* Φ_j *take the forms*

$$\Phi_j(x,y) = \left(x, \phi_j(h_j)\mathrm{e}^{-t_{j+1}}\right),$$

where

$$\phi_j(h) = a_j + h, \quad a_j \in \mathbb{C},$$

if $T_{j+1} \cap T_j$ lies in a sector of fall,

$$\phi_j(h) = h + \alpha_{j,2}h^2 + \cdots,$$

if $T_{j+1} \cap T_j$ lies in a sector of jump and $j \neq 2p$, and

$$\phi_{2p}(h) = \mathrm{e}^{-2\pi\mathrm{i}\lambda}h + \alpha_{2p,2}h^2 + \cdots.$$

Moreover Φ_j tend to id *exponentially fast as $x \to 0$.*

PROOF. The first statement follows from the fact that Φ_j preserve the formal normal form. So if h_{j+1} is the first integral in T_{j+1} then the function $h_{j+1} \circ \Phi_j$ is the first integral in $T_j \cap T_{j+1}$. But also h_j is a first integral. Thus the first integral is a function of the second first integral, $h_{j+1} \circ \Phi_j = \phi_j \circ h_j$.

In the coordinates (x, h) the map Φ_j is equal to $(x, \phi_j(h))$. Passing to the (x, y) coordinates we get the formula for Φ_j.

The proof of the further statements relies on the fact that $\Phi_j - \mathrm{id} = \mathrm{O}(x^{p+1})$.

Let $T_j \cap T_{j+1}$ be a sector of fall. We have $t_j = t_{j+1}$ (because $j \neq 2p$) and $\operatorname{Re} t(x) \to \infty$. Let the Taylor expansion of ϕ_j be $\sum_{n=0}^{\infty} \alpha_{j,n}h^n$. Then

$$\Phi_j = \Bigl(x, \alpha_{j,0}\mathrm{e}^{-t(x)} + \alpha_{j,1}y + \sum \alpha_{j,n}y^n\mathrm{e}^{(n-1)t(x)}\Bigr).$$

The condition $\Phi_j \to id$ implies that $\alpha_{j,1} = 1$ and $\alpha_{j,2} = \alpha_{j,3} = \cdots = 0$. This means that ϕ_j is a translation. Note also that in this case $\Phi_j - \mathrm{id} = (0, \mathrm{O}(\mathrm{e}^{-t(x)}))$ is exponentially small.

Let $T_j \cap T_{j+1}$ be a sector of jump and $j \neq 2p$. Then, acting exactly in the same manner as in the previous case, we get $\alpha_{j,0} = 0$ and $\alpha_{j,1} = 1$. It means that ϕ_j is a germ of diffeomorphism of $(\mathbb{C}, 0)$ with $\phi'(0) = 1$. In this case $\Phi_j - \mathrm{id}$ is exponentially small too.

Let $j = 2p$. We are in the sector of jump with $\mathrm{e}^{t_{2p}} = \mathrm{e}^{2\pi\mathrm{i}\lambda}\mathrm{e}^{t_1}$. We have $\Phi_{2p} = (x, \phi_{2p}(h_{2p})\mathrm{e}^{-t_1}) = (x, \mathrm{e}^{2\pi\mathrm{i}\lambda}\phi_{2p}(h_{2p})\mathrm{e}^{-t_{2p}})$. Analogously as in the previous case we get the formula for ϕ_j and the exponential closeness to the identity. $\square$

The fact that the diffeomorphisms Φ_j have different forms can be explained geometrically. In the sector of jump the values of the first integral are bounded, $h \in (\mathbb{C}, 0)$. In the sectors of fall the first integral takes values in the Riemann sphere $\bar{\mathbb{C}}$ and $h = \infty$ along the strong manifold $x = 0$; the corresponding map ϕ_j is an automorphism of this sphere satisfying $\phi_j(\infty) = \infty$.

The collection of ϕ_j's satisfying the properties of Proposition 3.38 do not yet form a complete set of invariants. The reader can notice that the coordinates in the normalized Pfaff equation $\omega_0 = 0$ are not unique. Its non-uniqueness lies in the possibility to multiplication of y by a constant. This gives the non-uniqueness of the choice of the local first integral: $h_j \to Ch_j$, where C is the same in each sector.

DEFINITION 3.39. Two collections $\phi = (\phi_1, \ldots, \phi_{2p})$ and $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_{2p})$, satisfying the conditions from Proposition 3.38, are called *equivalent* iff they are conjugated by a linear transformation: $\phi_j \circ C = C\tilde{\phi}_j$.

The equivalence class $[\phi]$ of such a collection is called the *Martinet–Ramis modulus*. The space of Martinet–Ramis moduli is denoted by $\mathcal{N}_{p,\lambda}$.

The above construction of the Martinet–Ramis modulus, associated to saddle–node, defines a map from $\mathcal{E}_{p,\lambda} \to \mathcal{N}_{p,\lambda}$

$$V \to \epsilon_V.$$

The following result is known as the *Martinet–Ramis theorem*.

THEOREM 3.40. (See [54].) *The map $V \to \epsilon_V$ has the following properties*:

- *it sends orbitally analytically equivalent germs of vector fields to the same point* (*invariance*);
- *if $\epsilon_V = \epsilon_{V'}$ then V and V' are orbitally analytically equivalent* (*equimodality*);
- *for any $\epsilon \in \mathcal{N}_{p,\lambda}$ there exists $V \in \mathcal{E}_{p,\lambda}$ such that $\epsilon = \epsilon_V$* (*realization*).

PROOF. The first two points are proved in the same way as the analogous points of the Ecalle–Voronin theorem 3.28. Only the realization property needs separate arguments.

Assume that we have an element from $\mathcal{N}_{p,\lambda}$ which is an equivalence class with some representative $\phi = (\phi_j)$. These ϕ_j define the diffeomorphisms Φ_j in the extended sectors $T_j \cap T_{j+1}$ (see Proposition 3.38).

Take disjoint union of the extended sectors $\bigsqcup T_j \times \{j\}$ and glue them using the Φ_j as the gluing maps. We obtain certain (real) 4-dimensional manifold $\mathcal{S}$. In each $T_j \times \{j\}$ we have the 'coordinates' $z_j = (x, y_j)$ and $z_{j+1} = \Phi_j(z_j)$ at the glued part.

Note also that $\mathcal{S}$ is equipped with a foliation $\mathcal{F}_0$ such that in each T_j it is the foliation of the vector field in the formal normal form. We have to define certain complex structure on $\mathcal{S}$ such that the foliation $\mathcal{F}_0$ arises from a holomorphic vector field with saddle–node singularity.

Firstly one defines the map $H_0 : \mathcal{S} \to W \subset \mathbb{C}^2 \setminus \{x = 0\}$ as

$$H_0 = \sum \chi_j z_j$$

where $\{\chi_j\}$ is a partition of unity of $\mathcal{S}$ associated with the covering (T_j) with regular properties as $x \to 0$.

H_0 defines an almost complex structure on W, the image of the complex structures on T_j. By definition this almost complex structure is integrable (see Definition 3.67 in Appendix 1). By Theorem of Newlander–Nirenberg 3.70 (in Appendix 1) the torsion of this almost complex structure vanishes identically.

It turns out that the almost complex structure from W can be prolonged smoothly to an almost complex structure on a full neighborhood of 0 in $\mathbb{C}^2$. Indeed, the almost complex structure is defined by $(1, 0)$-forms $\mathrm{d}x$ and $\mathrm{d}y$, where $\mathrm{d}x$ is good (with zero derivative). Since $y = \sum \chi_j y_j$. In $T_j \cap T_{j+1}$, we have $y = \chi_j y_j + \chi_{j+1} y_{j+1} = y_j + h_j(x, y_j)\chi_j$

(see Theorem 3.37). Thus $\mathrm{d}y = \mathrm{d}y_j + \mathrm{o}(1)$, where the term $\mathrm{o}(1)$ is exponentially small as $x \to 0$.

The torsion of the latter almost complex structure is also equal to zero. Applying again the Newlander–Nirenberg theorem, we show that this almost complex structure is integrable. It means that it is equal to the transportation of the standard complex structure by means of some differentiable map $G : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$

The composition $H = G^{-1} \circ H_0 : \mathcal{S} \to \mathbb{C}^2$ is holomorphic on $\mathcal{S}$ and prolongs to the closure of $\mathcal{S}$. It transforms the foliation $\mathcal{F}_0$ to a foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$.

The foliation $\mathcal{F}$ is a saddle–node foliation with the Martinet–Ramis invariant equal to $[\phi]$. □

The Martinet–Ramis theorem allows to solve the problem of analyticity of the center manifold and to explain the reason of non-analyticity of the Euler's solution. Note that in each sector of jump the local center manifold is defined uniquely, using topological criteria. It is the only leaf of the holomorphic foliation containing the singular point in its closure. This local center manifold prolongs itself (uniquely) to the adjacent sectors of fall. The gluing map Φ_j in the sectors of jump leave this local center manifold invariant, $\phi_j(0) = 0$.

The problem is whether the local center manifolds from different sectors of jump lie in one leaf of the foliation. It means, whether the gluing maps in the sectors of fall preserve the two prolongations of the local center manifolds (from the adjacent sectors of jump). This condition reads as $\phi_j(0) = 0$, where $\phi_j(h) = h + a_j$.

We have proved the following result.

COROLLARY 3.41. *A saddle–node singularity has analytic center manifold iff all $a_j = 0$ in its Martinet–Ramis modulus. In other words this modulus has the form* $(\mathrm{id}, \phi_2, \mathrm{id}, \phi_4, \ldots, \phi_{2p})$.

3.5. *Holonomy maps and normal forms for resonant saddles*

Here we finish the description of analytic normal forms for germs of planar vector fields with elementary singular points.

The singular points from the *Poincaré domain* have analytic Poincaré–Dulac normal form. This means that the normalizing transformation can be chosen analytic (see Theorem 3.19).

Therefore, it remains to investigate the saddles, i.e. singularities with negative ratio of eigenvalues $-\lambda < 0$. The saddles are of two kinds: *resonant* (i.e. with rational $\lambda = m/n$) and *non-resonant*.

The description of moduli of resonant saddles relies on the classification of germs of resonant 1-dimensional diffeomorphisms and was completed firstly by J. Martinet and J.-P. Ramis [55].

The problem of analytic conjugation of non-resonant saddles is analogous to the problem of analytic linearization of germs of non-resonant 1-dimensional diffeomorphisms and was solved by J.-C. Yoccoz [87] (see Subsection 3.7).

In classification of saddle singularities the following notion is useful.

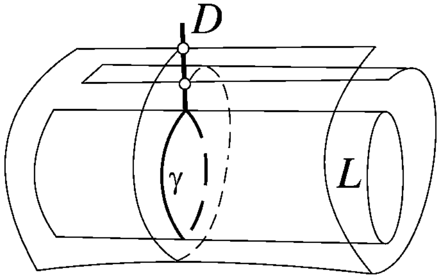

Fig. 7.

DEFINITION 3.42. Let $\mathcal{F}$ be a holomorphic foliation in a 2-dimensional complex manifold, L be a non-singular leaf of $\mathcal{F}$ and $\gamma \subset L$ be a closed loop with beginning at a point x_0. Take a small holomorphic disc D transversal to L at x_0 and parameterized by $z \in (\mathbb{C}, 0)$, $z(x_0) = 0$. We can cover the loop γ by open sets U_i, where the foliation is the flow-box foliation: $\mathcal{F}|_{U_i} = \{dy = 0\}$.

Let $z \in D$. Take a leaf $L(z)$ passing through z. In each U_i we can lift the path $\gamma \cap U_i$ to the leaf $L(z) \cap U_i$. After returning of γ to the point x_0, the lifted curve intersects the disc D again. However the intersection point can differ from z (see Fig. 7). We denote it by $\Delta_\gamma(z)$. We define the *monodromy transformation* (or the *holonomy transformation)* associated with the loop γ as the map

$$z \to \Delta_\gamma(z).$$

The holonomy map Δ_γ is a local holomorphic diffeomorphism (by analytic dependence of solutions on initial conditions). It does not depend on the homotopy class of the loop γ (for fixed D); this is a consequence of the theorem about monodromy. The change of the section D and of the initial point x_0 results in internal conjugations of Δ_γ in the group $\mathrm{Diff}(\mathbb{C}, 0)$ of germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$.

The connection between *saddles* and one-dimensional maps lies in the holonomy transformation associated with any of its separatrices. (Recall that, by the analytic Hadamard–Perron theorem, Theorem 3.20, the saddle has both separatrices analytic.) We can assume that

$$\dot{x} = \lambda x(1 + \cdots), \qquad \dot{y} = -y(1 + \cdots).$$

Consider the monodromy transformation $\Delta(x)$ corresponding to a loop in the separatrix $x = 0$. One can easily check that

$$\Delta(x) = \mathrm{e}^{-2\pi \mathrm{i}\lambda} x + \cdots.$$

The following result was firstly proved by J. Mattei and R. Moussu.

PROPOSITION 3.43. (See [56].) *Two germs with saddle singularity with the same ratio $-\lambda$ are analytically orbitally equivalent iff the corresponding germs of monodromy maps are analytically equivalent.*

PROOF. If two germs of vector fields are orbitally equivalent then restriction of the conjugating map H to a disc D transversal to $\{x=0\}$ defines a conjugation of monodromies (in D and in $H(D)$).

Let the monodromies be conjugated. Consider two copies of $(\mathbb{C}^2, 0)$ with foliations $\mathcal{F}$ and $\mathcal{F}'$ (by the phase curves L, L' of the two vector fields V and V'). In each copy we consider the fibration $\pi:(x, y)\to y$. Their fibers outside $y=0$ are transversal to the leaves of the foliations $\mathcal{F}$ and $\mathcal{F}'$. We denote by $\mathcal{F}_0$ the foliation by the fibers $y=\text{const}$.

Take two discs $D=\{y=\epsilon, |x|<\epsilon\}$ and $D'=\{y=\epsilon, |x|<2\epsilon\}$. Let $h: D\to D'$ be the map conjugating the monodromies.

We extend the map h to a map $H_0: A\to A'$, where $A=\{|x|<\epsilon, \epsilon/2<|y|<\epsilon\}$, $A'=\{|x|<2\epsilon, \epsilon/2<|y|<\epsilon\}$ are ring-like domains. H_0 preserves the foliation $\mathcal{F}_0$, i.e. $H_0(x, y)=(F(x, y), y)$ and sends leaves L of the foliation $\mathcal{F}$ to leaves L' of the foliation $\mathcal{F}'$. Because of the transversality of the fibers $y=\text{const}$ to the leaves L, L' in the domains A and A' the prolongation H_0 is locally analytic and unique.

H_0 is single-valued in the whole domain A. Indeed, the change of H_0, as the argument turns around the axis $y=0$, is equal to the composition of the monodromy map (defined by $\mathcal{F}$) and of H_0. Because $h=H_0|_D$ conjugates the monodromies the map H_0 is univalent, i.e. $H_0(x, y)$ is meromorphic in y.

Some additional arguments allow to eliminate the possibility that H_0 would have a pole along $y=0$. We refer the reader to [56,54] and [75]. □

Consider firstly the case when $\lambda=-m/n$, $\gcd(m, n)=1$, is a rational number. We call such singular point a $m:n$ *resonant saddle*. Then the monodromy map $\Delta=\mathrm{e}^{2\pi \mathrm{i}m/n}x+\cdots$ is a resonant one-dimensional map considered in Subsection 3.3. Its formal normal form is of one of the following types $\mathrm{e}^{2\pi\mathrm{i}m/n}g_w^1$, $w=[z^{np+1}/(1+\alpha z^{np})]\frac{\partial}{\partial z}$ (see Proposition 3.32). The space of such maps was denoted by $\mathcal{A}_{m,n,p,\alpha}$.

The above form for diffeomorphism corresponds to the following formal orbital normal form for vector fields

$$\dot{x}=x,\quad \dot{y}=y\bigl(-m/n+u^p(1+\lambda u^p)^{-1}\bigr),\quad u=x^m y^n, \tag{3.15}$$

where $\lambda=2\pi \mathrm{i}n\alpha$. (The proof relies on passing to the variables x, u in the Poincaré–Dulac normal form and further applying the reduction as in the case of saddle–node.) We denote the class of analytic saddles with the formal orbital normal form (3.15) by $\mathcal{B}_{m,n,p,\lambda}$.

THEOREM 3.44. (See [55].) *The space of classes of analytical orbital equivalence of vector fields from the class $\mathcal{B}_{m,n,p,\lambda}$ is the same as the space of classes of analytical equivalence of diffeomorphisms from the class $\mathcal{A}_{m,n,p,\alpha}$ described in Theorem* 3.33.

REMARKS ABOUT THE PROOF. In view of Proposition 3.41 the proof of this theorem is reduced to demonstration of the surjectivity of the map $V\to\Delta$ from $\mathcal{B}_{m,n,p,\lambda}$ to $\mathcal{A}_{m,n,p,\alpha}$.

One has to show that any resonant map arises from some resonant saddle as its holonomy. The construction of V is analogous to the construction of the realization parts of the Ecalle–Voronin and Martinet–Ramis classification theorems. We refer the reader to [42]. $\square$

Also with a saddle–node singularity some holonomy transformation can be associated and its relation with the Martinet–Ramis moduli can be studied.

Indeed, the strong invariant manifold of a saddle–node singularity V consists of two leaves of the holomorphic foliation defined by V: the singular point and the leaf L diffeomorphic to a punctured disc. Take a loop $\gamma \subset L$ generating its fundamental group. Using Definition 3.42, we associate with γ the monodromy transformation $\Delta = \Delta_\gamma$.

Recall that $\mathcal{A}_{p,\alpha}$ is the space of germs from $\mathrm{Diff}(\mathbb{C}, 0)$ formally equivalent to the time 1 flow map generated by the vector field $w = [z^{p+1}/(1 + \alpha z^p)]\partial_z$ (see Proposition 3.23).

THEOREM 3.45.

(a) *If* $V \in \mathcal{E}_{p,\lambda}$ *then* $\Delta \in \mathcal{A}_{p,\alpha}$, $\alpha = 2\pi\mathrm{i}\lambda$.

(b) *The Martinet–Ramis modulus of* V *coincides with the Ecalle–Voronin modulus of* Δ. *It means that two vector fields are orbitally analytically equivalent iff their monodromy maps are analytically conjugated. In particular, we have an embedding of the set of equivalence classes of vector fields* $\mathcal{E}_{p,\lambda}/(anal)$ *into the set of equivalence classes of diffeomorphisms* $\mathcal{A}_{p,\alpha}/(anal)$ (*i.e. embedding of the moduli space* $\mathcal{N}_{p,\lambda}$ *into the moduli space* $\mathcal{M}^*_{p,\alpha}$).

PROOF. The first statement can be checked for the vector field in its formal normal form. Then we have $L = \{x = 0 \neq y\}$, $\gamma = \{(0, \mathrm{e}^{\mathrm{i}t})\}$ and the foliation is given by $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\ln y} = x^{p+1}/(1 + \lambda x^p) = w_{p,\lambda}$. The monodromy map Δ is given by the time $2\pi\mathrm{i}$ flow map $\Delta = \exp[2\pi\mathrm{i} w_{p,\lambda}] = \exp[w_{p,\alpha}]$.

To prove the second statement let us keep Δ formally equivalent to $g_w^{2\pi\mathrm{i}}$. Then the sectors S_j (from Theorem 3.29) are turned by the angle $\pi/2p$; we denote them by S_j'. The extended sectors T_j are of the form $S_j' \times \{|y| < \epsilon\}$. We fix also the disc $D = \{y = y_0\}$ transversal to the axis $\{x = 0\}$.

The extended sectors contain complete lifts (to the leaves of the foliation) of the loop in the strong manifold appearing in the definition of the holonomy map.

The normalization diffeomorphisms $H_j(x, y)$ in T_j (from Theorem 3.37) define altogether the normalization diffeomorphisms for the monodromy map Δ in $S_j' \times \{y_0\} \subset D$. They define the local parametrization of S_j by means of the first integral $h_j(x) = h \circ H_j|_{S_j' \times \{y_0\}}$. In the charts h_j the map Δ is equal to the formal normal form.

The transition diffeomorphisms $H_{j+1} \circ H_j^{-1}$ are expressed by means of the one-dimensional transition maps $h_{j+1} \circ h_j^{-1}$. The latter define the Martinet–Ramis moduli as well as the Ecalle–Voronin moduli. $\square$

REMARK 3.46. The image of the embedding from the point (b) of the last theorem forms a proper subset of the set $\mathcal{M}^*_{p,\lambda}$. It is because the diffeomorphisms ϕ_j are subject to the restrictions from Proposition 3.36, $\phi_j = h + a_j : (\bar{\mathbb{C}}, \infty) \to (\bar{\mathbb{C}}, \infty)$ (j odd).

It turns out that the correspondence $V \to \Delta$ is not one-to-one even at the level of topological equivalences; i.e. the sets $\mathcal{E}_{p,\lambda}/top$ and $\mathcal{A}_{p,\lambda}/top$ are really different. (At the level of formal equivalences it is one-to-one.)

As the first example we consider the case with the Martinet–Ramis modulus equal to $(\mathrm{id}+a_1, \mathrm{id}, \mathrm{id}+a_2, \mathrm{id}, \dots, \mathrm{id})$ with some $a_j \neq 0$. The monodromy map Δ is topologically equivalent to its formal normal form (see Proposition 3.22) but the vector field is not equivalent to its formal normal form because it does not have analytic center manifold.

Another example concerns the case with $\lambda = 0$ and with all $a_j = 0$, i.e. with analytic center manifold. We consider the monodromy map Δ_c associated with the leaf L_c containing the center manifold. The topology of the foliation determines the topological type of Δ_c, which is determined by the order of tangency of Δ_c to identity (see Proposition 3.24). So the class $[\Delta]_{top}$ is fixed, but there can be infinitely many classes $[\Delta_c]_{top}$.

3.6. *Geometric realization of Martinet–Ramis moduli*

Unlike the Ecalle–Voronin moduli the Martinet–Ramis moduli can be sometimes calculated explicitly. In this subsection we present a result of P. Elizarov about dependence of the Martinet–Ramis moduli on the vector field itself and we give examples (due to the author and L. Teyssier) of explicit calculation of these moduli.

We begin with definitions of the Borel transform and Gevrey series.

DEFINITION 3.47. Let $f(t) = \sum_{k=0}^{\infty} a_k t^k$ be a formal power series. Its *Borel transform* is the series

$$(\mathcal{B}f)(\tau) = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} a_k \tau^k.$$

By analog to the above standard Borel transform we introduce the following modified Borel transforms

$$(\mathcal{B}^l_{SN} f)(\tau) = \sum_{k=1}^{\infty} \frac{k}{\Gamma(1 + (k + \lambda l)/p)} a_k \tau^k$$

and

$$(\mathcal{B}^l_{S} f)(\tau) = \sum_{k=1}^{\infty} \frac{k}{\Gamma(1 + (km - \lambda l)/(pm^2))} a_k \tau^k.$$

DEFINITION 3.48. Let the series $f(t)$ as above is an asymptotic expansion of a function, denoted also by f, which is holomorphic in a sector S with vertex at $t = 0$. We say that f is of *Gevrey type of order s* if

$$|a_k| \leqslant C(k!)^{s-1} A^k$$

and

$$\left| f(t) - \sum_{j=1j}^{k-1} t^j \right| \cdot |t|^k \leqslant C(k!)^{s-1} A^k,$$

where the constants A, C depend on f and the sector S. The series f of Gevrey type of order

$$s = 1 + 1/p$$

is called *p-summable in the direction θ* if it is an asymptotic expansion of some function f_S holomorphic in a sector S with bisectrix $\arg t = \theta$ and with angle $> \pi/p$.

REMARK 3.49. If a series is of Gevrey type of order $s = 1 + 1/p$ then its Borel transforms $\mathcal{B}^l_{SN}$ or $\mathcal{B}^l_S$ define convergent series.

The transform $\mathcal{B}^l_{SN}$ is used in study of the saddle–node singularities from the class $\mathcal{E}_{p,\lambda}$ and the transform $\mathcal{B}^l_S$ is used in the study of the saddle singularities from the class $\mathcal{B}_{m,n,p,\lambda}$. The sectorial normalizing diffeomorphisms, which are used in the proofs of the Ecalle–Voronin and Martinet–Ramis theorems, have asymptotic expansion in the form of Gevrey series of order $s = 1 + 1/p$; moreover, they turn out p-summable in sectors S_j and T_j respectively (appearing in the proofs). The Gevrey series were used in the works of J. Ecalle [28,29] and J. Martinet and J.-P. Ramis [54,55], see also [53].

Recall that the Martinet–Ramis modulus of a saddle–node singularity from the class $\mathcal{E}_{p,\lambda}$ is a class $[\phi]$ of collections $(\phi_1, \dots, \phi_{2p})$ of germs of holomorphic maps such that $\phi_1(h) = a_1 + h$, $\phi_2(h) = h + a_{2,2}h^2 + a_{2,3}h^3 + \cdots$, $\phi_3(h) = a_3 + h, \dots \phi_{2p-1}(h) = a_{p-1} + h$, $\phi_{2p}(h) = \mathrm{e}^{-2\pi \mathrm{i}\lambda} h + a_{2p}h^2 + \cdots$, and the equivalence relation is defined by means of simultaneous mutual conjugation of the maps ϕ_j by means of a rescaling $h \to Ch$.

Let us pass to the logarithmic chart t, such that $h = \mathrm{e}^{2\pi \mathrm{i} t}$. Then the maps ϕ_j become maps $\Phi_j(t)$ expanded into Fourier series:

$$\Phi_1(t) = t + \sum_{l=-\infty}^{-1} b_{1,l} \mathrm{e}^{2\pi \mathrm{i} l t}, \qquad \Phi_2(t) = t + \sum_{l=1}^{\infty} b_{2,l} \mathrm{e}^{2\pi \mathrm{i} l t},$$

$$\vdots$$

$$\Phi_{2p}(t) = t - 2\pi \mathrm{i}\lambda + \sum_{l=1}^{\infty} b_{2p,l} \mathrm{e}^{2\pi \mathrm{i} l t},$$

where $b_{j,l} = (-a_i)^{-l}/(-l)$, $l = -1, -2, \dots$, for odd j are defined by $b_{j,-1} = -a_1$. In other words, the Martinet–Ramis moduli can be defined by the collection of the coefficients $b_{j,l}$, where $p \in \{1, \dots, 2p\}$ and $l \in \mathbb{N} \cup \{-1\}$.

P. Elizarov considered vector fields of the form

$$V_{p,\lambda} + \varepsilon \cdot \left(y \sum_{l=-1}^{\infty} y^l f_l(x) \right) \partial_y,$$

where $V_{p,\lambda}$ is the formal normal form vector field, f_l are holomorphic germs and ε is a small parameter. By Lemma 3.36 we can assume that $f_0(x) \equiv 0$ and that the other germs f_j have high order zero at $x = 0$. The following result an be regarded as calculation of the tangent to the moduli map.

THEOREM 3.50. (See [30].) *We have*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} b_{j,l}|_{\varepsilon=0} = \bigl(\operatorname{sign}(l) \mathrm{e}^{-2\pi j\lambda li/p} l^{\lambda l/p-1}/p\bigr) \cdot (\mathcal{B}^l_{SN} f_l)\bigl(\mathrm{e}^{-(2j+1)\pi \mathrm{i}/p}(l/p)^{1/p}\bigr).$$

EXAMPLE 3.51. As an application of this result we consider the following germ

$$V = V_{p,\lambda} + x^k y^{q+1} \partial_y,$$

where $k > p$ and $q \in \mathbb{N} \cup \{-1\}$ are such that $(k + q\lambda)/p$ is not a negative integer. Then the series normalizing it to the formal orbital normal form is divergent. Moreover, two such germs with different exponents q are not analytically equivalent.

IDEA OF THE PROOF. We do not prove the Elizarov theorem, but we can say few words about the proof. By Theorem 3.45 the Martinet–Ramis modulus for the vector field $V \in \mathcal{E}_{p,\lambda}$ coincides with the Ecalle–Voronin of the monodromy diffeomorphism $\Delta \in \mathcal{A}_{p,\alpha}$. The holonomy map Δ can be calculated modulo $\mathrm{O}(\varepsilon^2)$ by integration of suitable equation in variations. We get $\Delta = \Delta_0 + \varepsilon R$ where Δ_0 is the time 1 phase flow map g^1_w and R is expressed via the series $y \sum y^l f_l(x)$. Recall that the conjugating map from the sectorial normalization theorem (Theorem 3.29) in the t_j-chart was defined by the formula $\tilde{H}_j(t_j) = t_j + \varepsilon \sum_m \tilde{R} \circ \tilde{\Delta}^m$, where $\tilde{\Delta}(t_j) = t_j + 1 + \mathrm{O}(\varepsilon)$. So, it is enough to compute the sums $\sum_m \tilde{R}(t_j + m)$ or rather differences between such sums for pairs of consecutive j's. The result is the above Elizarov's formula. □

Elizarov calculated also the tangent to the moduli map in the case of resonant saddle. Recall that here the modulus associated to a germ of vector field V from the class $\mathcal{B}_{m,n,p,\lambda}$ is the Ecalle–Voronin modulus of the holonomy map Δ associated with a loop in one of the separatrices. This modulus, rewritten in the logarithmic chart $t = \log \vartheta$ is an equivalence class of collection of $2p$ functions

$$\Phi_j(t) = t - \pi \mathrm{i}\lambda/np + \sum_{l=1}^{\infty} b_{j,l} \mathrm{e}^{2\pi \mathrm{i} lt}, \quad j \text{ odd},$$

$$\Phi_j(t) = t - \pi \mathrm{i}\lambda/np + \sum_{l=-\infty}^{-1} b_{j,l} \mathrm{e}^{2\pi \mathrm{i} lt}, \quad j \text{ even}.$$

Elizarov considers following germs

$$V = V_0 - \varepsilon x \frac{u^p}{1+\lambda u^p} P(x,y)\partial_x,$$

where V_0 is in the formal orbital normal form, defined in the previous subsection, $u = x^m y^n$ and $P = \sum c_{k,s} x^k y^s = \sum_l f_l(\tilde{x}) z^l$ and $\tilde{x} = xz^n$, $y = z^m$ and $f_l(\tilde{x}) = \sum_{ms-nk=l} c_{k,s}\tilde{x}^k$. The next theorem is proved in the same way as Theorem 3.50.

THEOREM 3.52. (See [30].) *We have*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} b_{j,l}|_{\varepsilon=0} = \alpha_{j,l} \cdot (\mathcal{B}_S^l f_l)\big(\mathrm{e}^{-(2j+1)\pi \mathrm{i}/(pm)} (l/pm^2)^{1/pm}\big)$$

where $\alpha_{j,l} \neq 0$ are some concrete constants.

We promised to calculate explicitly the Martinet–Ramis moduli in some situations. The following example is a continuation of Example 2.28 from Subsection 2.2.

EXAMPLE 3.53. Consider of the system

$$\dot{x} = x^{p+1}, \qquad \dot{y} = -y(p + ax^p) - x^{p+1}\varphi(x),$$

where $\varphi(x)$ is a germ of analytic function and $a \in \mathbb{C}$. Of course, it is a saddle–node singularity with the first integral

$$H = yx^a \mathrm{e}^{-1/x^p} + H_1(x)$$

where $H_1(x) = \int_0^x s^a \mathrm{e}^{-1/s^p}\varphi(s)\,\mathrm{d}s = \int f$ is the function defined in Example 2.28. Recall that the function $H_1(x)$ is well defined only in some sectors which contain one sector of fall of the function f, where $\operatorname{Re} 1/x^p > 0$ (and is a sector of jump for the foliation), and do not contain whole sectors of jump of f, where $\operatorname{Re} 1/x^p < 0$. These sectors are equal $S_1' \cap S_{2p}'$, where $H = x^a \mathrm{e}^{-1/x^p}(y + \chi_1(x))$, $S_3' \cup S_4'$, where $H = x^a \mathrm{e}^{-1/x^p}(y + \chi_3(x)) + A_0$, etc. The functions $\chi_j(x)$ are analytic in the sectors and the constants A_j are given by integration of $x^a \mathrm{e}^{-1/x^p}\varphi(x)\,\mathrm{d}x$ along so-called *asymptotic cycles* σ_j which begin and end at the origin and surround exactly one sector of jump (like at Fig. 1 in Section 2).

It follows that the Martinet–Ramis modulus is the following

$$\phi = (\mathrm{id} + A_0, \mathrm{id}, \mathrm{id} + A_1, \mathrm{id}, \ldots, \mathrm{id} + A_{p-1}, \mathrm{e}^{-2\pi \mathrm{i}\lambda}\,\mathrm{id}).$$

Consider saddle–node singularities with *analytic center manifold*. By Lemma 3.36 we can write them in the following analytic form

$$\begin{aligned} V &= V_0 + y^2 g_2(x,y)\partial_y = V_0 + RY, \\ V_0 &= x^{p+1}\partial_x - y(1 + \lambda x^p)\partial_y, \end{aligned} \tag{3.16}$$

where $R = yg_2(x, y)$ and $Y = y\partial_y$. So the center manifold is $x = 0$. Note that the vector fields V_0 and Y commute, $[V_0, Y] = 0$. L. Teyssier [80] noticed that one can try to conjugate V with V_0 by means of a local diffeomorphism of the form

$$G = g_Y^{F(x,y)}$$

where $\{g_T^t\}$ is the flow generated by the vector field Y: $G(x, y) = (x, \mathrm{e}^{F(x,y)}y)$. Rather elementary calculations, which use the equivalence of the equation $G^*U = W$ to $U \circ G = \frac{\partial G}{\partial W}$, give the following formula.

LEMMA 3.54. *The condition $G^*V_0 = V$ is equivalent to the equation*

$$\frac{\partial F}{\partial V} = -R.$$

A natural way to solve the equation $V(F) = -R$ is to rewrite it as $\dot{f}(t) = -r(t)$, when restricted to a leaf L of the foliation $\mathcal{F}$ defined by the vector field V. On the leaf L the 1-form $\mathrm{d}t = \tau = \mathrm{d}x/x^{p+1}$ is the 'time' form and we can express the solution in the form

$$F(x, y) = \int_{\gamma(x,y)} (-R)\tau$$

where $\gamma(x, y)$ is a path in the leaf $L(x, y)$ which connects the origin $x = y = 0$ with the point (x, y). There arises the problem of correctness of this definition.

Let us numerate the leaves of the foliation $\mathcal{F}$ by a value h of the first integral $yx^{-\lambda}\mathrm{e}^{1/px^p}$ (of V_0) restricted to a transverse line $x = x_*$, $L = L_h$. Since the vector field V has analytic center manifold, the function $(y|_{L_h})(x)$ on a fixed leaf L_h should tend to zero as $x \to 0$ within a sector of fall. Let $\gamma_j^\infty(h)$ be the lifts to L_h of the loops σ_j in the x-plane which begin and end at the origin in adjacent sectors of fall and surround exactly one sector of jump (like in Fig. 1 in Section 2); they are called the *asymptotic cycles* (see [79,80]). The quantities

$$\chi_j(h) = \int_{\gamma_j^\infty(h)} (-R)\tau$$

are the obstacles to the single validity of the function F.

But in suitable sectors which contain just one sector of fall we can define corresponding functions $F_j(x, y) = \int_{\gamma_j(x,y)} (-R)\tau$, where $\gamma_j(x, y)$ are paths in $L(x, y)$ which begin at the origin in the j-th sector of fall and end at the point (x, y). We have $F_{j+1} - F_j = \chi_j$. These functions define corresponding conjugating sectorial diffeomorphisms $H_j = g_Y^{F_j}$ and the functional cochain $\Phi_j = H_{j+1} \circ H_j^{-1} = (x, y\mathrm{e}^{\chi_j(h)}) = (x, h\mathrm{e}^{\chi_j(h)}\mathrm{e}^{-t(x)})$.

We can generalize the above to the case of resonant saddle, i.e. for $V = V_0 + RY$, where $R = yg(x, y)$ and $Y = y\partial_y$ are as above and

$$V_0 = -\frac{m}{n}x\big(1 + (\lambda - 1)u^p\big)\partial_x + y(1 + \lambda u^p)\partial_u, \quad u = x^n y^m. \tag{3.17}$$

Here also one tries changes of the form $G = g_Y^{F(x,y)}$ and arrives to the functions $\chi_j(h)$ defined in the same way.

Let us summarize this in the following theorem of L. Teyssier. (In his work [80] he uses a version of the Martinet–Ramis moduli when the germs ϕ_j are such that $\phi'_{2j}(0) = \mathrm{e}^{2\pi\mathrm{i}\lambda/p}$, but it is a subject to different choices of first integrals in the extended sectors T_j.)

THEOREM 3.55. (See [80].) *For a saddle–node singularity with analytic center manifold the Martinet–Ramis modulus equals*

$$(\mathrm{id}, h\mathrm{e}^{\chi_1(h)}, \mathrm{id}, h\mathrm{e}^{\chi_2(h)}, \mathrm{id}, \mathrm{e}^{-2\pi\mathrm{i}\lambda}h\mathrm{e}^{\chi_p(h)}).$$

An analogous formula holds for functional moduli for resonant saddle singularity from Theorem 3.44.

We finish this subsection by description of some recent results (due to S. Voronin, A. Grintchy, Yu. Meshcheryakova and L. Teyssier) about classification of saddle–node and resonant saddle singularities with respect to conjugation by means of local diffeomorphisms, but without change of time.

PROPOSITION 3.56. (See [60,38].) *Any saddle–node singularity whose formal orbital normal form is like V_0 in* (3.16) *is formally equivalent to*

$$P(x)V_0,$$

where $P(x)$, $P(0) \neq 0$, is a polynomial of degree at most p. Two such vector fields with polynomials P and P' are equivalent if $P'(x) = P(\mathrm{e}^{2\pi\mathrm{i}k/p}x)$ for some integer k.

Any resonant saddle singularity with formal orbital normal form like V_0 in (3.17) *is formally equivalent to*

$$P(u)V_0,$$

where $P(u)$, $u = x^n y^m$,is a polynomial of degree at most p.

PROOF. Consider the saddle–node case. Of course, we can assume that the vector field is of the form $Q(x, y)V_0$ where Q is a series with non-zero constant term. We try to reduce Q by means of diffeomorphisms which preserve the foliation $\mathcal{F}_0$ defined by V_0 and which are close to identity. They are of the form $G = g_{V_0}^{F(x,y)}$, i.e. the phase flow maps after varying time $F(x, y)$. An analogue of Lemma 3.54 says that $G^*(QV_0) = Q'V_0$, where Q' satisfies the equation

$$V_0(F) = \frac{1}{Q} - \frac{1}{Q'}.$$

Since $V_0(F) = x^{p+1}F_x - (1 + \lambda x^p)yF_y$, it is easy to see that we can firstly reduce from $\frac{1}{Q}$ all terms divisible by y and next all terms divisible by x^{p+1}. From this the result easily follows.

In the resonant saddle case analogously we firstly reduce all terms from $\frac{1}{Q}$ different from a power of u and next use the fact that $V(R\circ u)=u^{p+1}\frac{\mathrm{d}R}{\mathrm{d}u}$. □

Let us describe the moduli, which generalize the Martinet–Ramis moduli for saddle–node singularities for fixed the polynomial $P(x)$ from the last proposition. Denote by $\mathcal{E}_{p,\lambda,P}$ the space of saddle–node germs with such formal normal form.

We consider collections $(\phi;\psi)=(\phi_1,\dots,\phi_{2p};\psi_1,\dots,\psi_p)$ where $\phi=(\phi_1,\dots,\phi_{2p})$ is the collection appearing in Definition 3.39 of the Martinet–Ramis moduli and $\psi_j(h)$ are germ of holomorphic functions in $(\mathbb{C},0)$ such that $\psi_j(0)=0$.

DEFINITION 3.57. Two such collections $(\phi;\psi)=(\phi_1,\dots,\phi_{2p};\psi_1,\dots,\psi_p)$ and $(\phi';\psi')$ $=(\phi_1',\dots,\phi_{2p}';\psi_1',\dots,\psi_p')$ are called *equivalent* iff ϕ_j and ϕ_j' are simultaneously conjugated by a linear transformation, $\phi_j\circ C=C\phi_j'$, and altogether we have $\psi_j\circ C=\psi_j'$.

The space of equivalence classes $[\phi;\psi]$ of such a collections is the *moduli space* denoted by $\mathcal{N}_{p,\lambda,P}$.

THEOREM 3.58. (See [60,79].) *There exists a map $W\to\varepsilon_W$ from $\mathcal{E}_{p,\lambda,P}$ to $\mathcal{N}_{p,\lambda,P}$ with the following properties*:

- *it sends analytically equivalent germs of vector fields to the same point* (*invariance*);
- *if $\varepsilon_W=\varepsilon_{W'}$ then W and W' are analytically equivalent* (*equimodality*);
- *for any $\varepsilon\in\mathcal{N}_{p,\lambda,P}$ there exists $W\in\mathcal{E}_{p,\lambda,P}$ such that $\varepsilon=\varepsilon_W$* (*realization*).

IDEA OF THE PROOF. We can assume that the vector field is of the form $W=QV$, where V is like in the thesis of Lemma 3.36, i.e. with $\dot{x}=x^{p+1}$. We try to reduce the germ QV to the germ $P(x)V$ using conjugation by means of the map $G=g_V^F$. Recall that this leads to the equation $V(F_j)=\frac{1}{Q}-\frac{1}{P}$. We can solve this equation in the extended sectors T_j. The obstacles to a solution in a whole neighborhood of the origin are given by the integrals

$$\psi_j(h)=\int_{\gamma_j^\infty}\left(\frac{1}{Q}-\frac{1}{P}\right)\frac{\mathrm{d}x}{Qx^{p+1}}$$

along the asymptotic cycles γ_j^∞ in the leaves L_h. The functions ψ_j constitute part of the modulus $(\phi;\psi)$.

By the sectorial normalization theorem (Theorem 3.37) we can reduce the vector field V in extended sectors T_j to V_0 and the obstructions to a univalent reduction form the collection ϕ (the Martinet–Ramis modulus), the other part of (ϕ,ψ).

We omit he other details of the proof. □

Finally we note that there exists a generalization of the latter theorem to the case of resonant saddle singularities. For details we refer the reader to the papers of A. Grintchy and S. Voronin [38] and L. Teyssier [79,80]. Besides the collection of $2p$ germs ϕ_i of local diffeomorphisms, defined in Theorem 3.44 (via Theorem 3.33) we have also collection of p germs of functions ψ_j.

3.7. *Linearization*

Now we pass to the non-resonant saddles. Their one-dimensional equivalents are the germs of non-resonant analytic diffeomorphisms f of $(\mathbb{C}, 0)$ of the form

$$z \to \mathrm{e}^{2\pi \mathrm{i}\alpha} z + \mathrm{O}(z^2),$$

where $\alpha \notin \mathbb{Q}$, $0 < \alpha < 1$. The Poincaré–Dulac theorem for diffeomorphisms says that any such f is formally linearizable (see Theorem 3.22); there exists a formal power series $h(z) \sim z + h_2 z^2 + \cdots$ such that $h^{-1} \circ f \circ h \sim \lambda z$, $\lambda = \mathrm{e}^{2\pi \mathrm{i}\alpha}$. The problem is under which assumptions f is analytically linearizable, i.e. when the series defining h is convergent.

The positive or negative answer to this question depends on the way how quickly the number α can be approximated by rational numbers. If α is slowly approximated by rationals then any such diffeomorphism is analytically linearizable. If α is quickly approximated by rationals then there should exist diffeomorphisms f for which the series h is divergent.

Let $\alpha = 1/a_1 + 1/a_2 + \cdots$ be the continued fraction expansion of the number α and let $1/a_1 + 1/a_2 + \cdots + 1/a_n = \frac{p_n}{q_n}$ be the successive rational approximations of α.

DEFINITION 3.59. We say that the number α satisfies the *Briuno condition* if

$$\sum \frac{\ln q_{n+1}}{q_n} < \infty.$$

The next theorem is the fundamental result of this section. Its first part was firstly obtained by A. Briuno, but the proof we present is due to C. Yoccoz. The second part was obtained by C. Yoccoz.

THEOREM 3.60. (See [16,87].) *If the number α satisfies the Briuno condition then any analytic diffeomorphism f is analytically linearizable and if α does not satisfy the Briuno condition then there exists an analytic diffeomorphism $f = \mathrm{e}^{2\pi \mathrm{i}\alpha} z + \cdots$ which is not analytically linearizable.*

REMARK 3.61. We should mention some previous results in this field. C. Siegel [74] proved that analyticity of the conjugating diffeomorphism holds when there exist two constants C, μ, such that for any integers p, q we have $|\alpha - p/q| > C|q|^{-\mu}$. If $\mu > 2$ then the set of those α, for which there exists a constant C with this property, is a set of full Lebesque measure.

A. Briuno [16] and H. Cremer [25] noticed that the Briuno condition is close to an optimal condition for convergence. They proved that if $\limsup \frac{\ln q_{n+1}}{q_n} = \infty$ then there exists an analytic diffeomorphism f which is not analytically linearizable.

The proofs of these results are based on estimations. The main achievement of J.-C. Yoccoz was to give a geometrical proof of a theorem of Briuno and to show that the Briuno condition is also necessary for convergence.

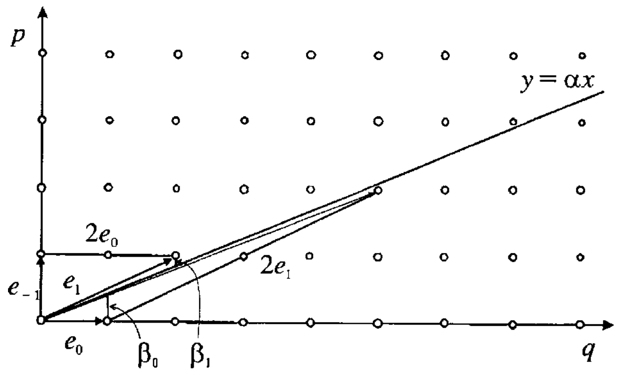

Fig. 8.

Before proving the main theorem of this section we recall some facts about *continued fractions*.

Let $0 < \alpha < 1$ be an irrational. Consider the real plane with coordinates x, y (see Fig. 8). We draw the line $y = \alpha x$. We distinguish the points with integer coordinates (q, p) in the first quadrant. They all, except $(0, 0)$, do not belong to our line. We consider the convex hulls of the sets of integer points of the quadrant, which lie at one side of our line (below and above respectively). (One can imagine the integer points as nails and our line as a thread with fixed one end at infinity; next one stretches the thread in the down direction and in the up direction.) The vertices (q, p) of the broken lines just constructed define the best approximations of the number α by rationals p/q.

We describe the construction of our broken lines in another way. Denote the basic vectors $e_{-1} = (0, 1)$, $e_0 = (1, 0)$. They lie in different sides of the line $y = \alpha x$. We shall construct a sequence of vectors $e_1, e_2, \ldots$ as follows. Let e_{n-1} and e_n be already constructed and lie in different sides of the line. We add the vector e_n to e_{n-1} as many times a_{n+1} as it is possible with the condition that the sum lies in the same side of the line as e_{n-1}.

In this way we obtain the sequence of natural numbers a_n and the sequence of integer vectors

$$e_1 = e_{-1} + a_1 e_0, \ldots, e_{n+1} = e_{n-1} + a_{n+1} e_n.$$

(Note that $a_1 = [1/\alpha]$, the integer part of $1/\alpha$.) The endpoints of the vectors $e_n = (q_n, p_n)$ are the vertices of the above two convex hulls.

We denote by β_n the 'distance' of the point e_n to the line $y = \alpha x$

$$\beta_n = (-1)^n (q_n \alpha - p_n).$$

From Fig. 8 it is seen that $\beta_{-1} = 1$, $\beta_0 = \alpha$ and $\beta_{n+1} = \beta_{n-1} - a_{n+1}\beta_n$, which implies that the ratios p_n/q_n form the best approximations to α, in the sense that $|\alpha - p_n/q_n| < |\alpha - p/q|$ for any p and any $q < q_n$. In fact, we have $\beta_n < |q\alpha - p| < \beta_{n-1}$ for $q_{n-1} < q < q_n$.

We define also the numbers $\alpha_n = \beta_n/\beta_{n-1}$. Thus $\alpha_0 = \alpha$. They lie between 0 and 1. We obtain $\alpha_{n-1}^{-1} = a_n + \alpha_n$. This means that $\alpha_0 = \{\alpha\}, \ldots, \alpha_{n+1} = \{\alpha_n^{-1}\}$ where $\{\cdot\}$ denotes the fractional part. Moreover, iterating the latter identity we recognize the continued fraction

$$\alpha = a_0 + \alpha_0 = a_0 + \frac{1}{a_1 + \alpha_1} = a_0 + \frac{1}{a_1 + \dfrac{1}{a_2 + \alpha_2}} = \cdots.$$

Note also that

$$\beta_n = \alpha_0 \cdots \alpha_n.$$

We note also that the (oriented) area of the parallelogram with sides e_n, e_{n-1} is equal to $(-1)^n$. It implies $q_{n+1}\beta_n + q_n\beta_{n+1} = 1$, or

$$\beta_n(q_{n+1} + q_n\alpha_{n+1}) = 1.$$

From this one proves the following (see also [89])

LEMMA 3.62. *The following conditions are equivalent*:

- $\sum q_n^{-1} \ln q_{n+1} < \infty$;
- $\Phi(\alpha) \stackrel{df}{=} -\sum \beta_n \ln \alpha_{n+1} < \infty$.

3.7.1. *Proof of the analytic linearizability* (We follow [65].) Recall that we have to prove the analyticity of the conjugation series under assumption of the Briuno condition.

We begin with the following criterion of analytic linearizability.

LEMMA. *f is linearizable iff it is stable, i.e. iff for any neighborhood U of* 0 *there is a neighborhood V such that $f^n(V) \subset U$ for all $n \geqslant 0$.*

PROOF. If f is linearizable then it is stable because the linear map is stable.

Assume that f is stable and let V be such that $f^n(V) \subset U = \mathbf{D}$, $n \geqslant 0$. Then the family $f^n|_V$ is normal. Define $h_n = \frac{1}{n}\sum_{i=0}^{n-1} \lambda^{-i} f^i$. It is also normal family. We have $h_n \circ f = \lambda h_n + \frac{1}{n}\lambda(\lambda^{-n} f^n - 1)$.

Now it is enough to choose a convergent subsequence from $\{h_n\}$ and the limit conjugates f with λz. □

After rescaling the variable z we can assume that we are dealing with functions which are analytic diffeomorphisms of the unit disc $\mathbf{D} = \{|z| < 1\}$ into $\mathbb{C}$ with the fixed first term $e^{2\pi i\alpha} z$. We denote the space of such functions by $S(\alpha)$. Of course, we can (and shall) assume that $0 < \alpha < 1$.

The space $S(\alpha)$ has nice properties based on the following *Theorem of Koebé*:

If $f : (\mathbf{D}, 0) \to (\mathbb{C}, 0)$ is an univalent holomorphic function with $|f'(0)| = 1$ then

$$\left|f(z)\right| < |z|\bigl(1 - |z|\bigr)^{-2}, \qquad \left|f'(z)\right| < \bigl(1 + |z|\bigr)\bigl(1 - |z|\bigr)^{-3}.$$

The important corollary of the Koebé estimates is the following.

LEMMA. *The spaces $S(\alpha)$ are compact in the topology of almost uniform convergence. In other words, the family $S(\alpha)$ is normal.*

The authorship of next lemma belongs to A. Douady and E. Ghys.

LEMMA. *The set of those α, for which any $f \in S(\alpha)$ is linearizable, is invariant with respect to the action of the group* $\mathrm{SL}(2,\mathbb{Z}) : \alpha \to \frac{a\alpha+b}{c\alpha+d}$.

PROOF. The group $\mathrm{SL}(2,\mathbb{Z})$ is generated by transformations of the form $\alpha \to \alpha + 1$ and $\alpha \to -1/\alpha$. The invariance with respect to the first transformations is obvious.

Let $0 < \alpha < 1$. We shall associate with any $f \in S(\alpha)$ a map $g \in S(\beta)$, $\beta = -1/\alpha$. If l is a ray and $f(l)$ is its image, then we add a curve l' closing-up a triangle-like region Δ. We glue l with $f(l)$ by means of f and obtain a surface, which admits a holomorphic structure of the disc $\mathbf{D}$ (see Appendix 1 below). This equivalence is of the form $z \to z_1 = z^{1/\alpha} + \cdots$.

Denote by g the map of first return to Δ. It turns out to be holomorphic with respect to the complex structure on $\Delta/\sim$. Expressed in the chart $z_1 = \rho \mathrm{e}^{\mathrm{i}\theta}$ on $\Delta/\sim$ it takes the form

$$\mathrm{e}^{2\pi\mathrm{i}(j\alpha-1)/\alpha+\mathrm{i}\theta}\rho + \cdots = \mathrm{e}^{2\pi\mathrm{i}\beta}z_1 + \cdots,$$

where $j = j(z_1)$ is the moment of the first return. $\square$

It is useful to pass to the chart, where the punctured disc $\mathbf{D}^*$ is replaced by its infinite covering; the upper half-plane $\mathbf{H} = \{\operatorname{Im}\zeta > 0\}$ with the covering map $\zeta \to z = \mathrm{e}^{2\pi\mathrm{i}\zeta}$.

The lift of the map f to $\mathbf{H}$ has the form

$$F(\zeta) = \zeta + \alpha + \sum a_n \mathrm{e}^{2\pi\mathrm{i}n\zeta},$$

i.e. it commutes with the translation $\mathrm{id}+1$. Using the compactness of the space $S(\alpha)$ (or estimates for the coefficients a_n) it is possible to get the following uniform bound for the difference of F from the pure translation.

LEMMA. *There exists a constant C, not depending on α and F, such that the inequality*

$$\left|F(\zeta) - \zeta - \alpha\right| < \alpha/4$$

holds for

$$\operatorname{Im}\zeta > t(\alpha) \overset{df}{=} (2\pi)^{-1} \cdot \ln\alpha^{-1} + C.$$

The construction from the proof of the previous lemma can be implemented for maps on $\mathbf{H}$. It is presented at Fig. 9. The line $l = \{\zeta = \mathrm{i}t, t > t(\alpha)\}$ is identified with $F(l)$. The domain $\tilde{\Delta}$ between l, $F(l)$ and the straight segment (joining $\mathrm{i}t(\alpha)$ with $F(\mathrm{i}t(\alpha))$) is replaced

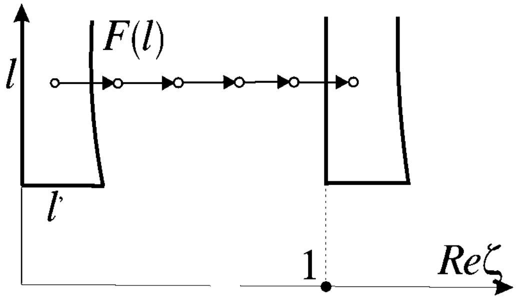

Fig. 9.

by the domain $\{0 \leqslant \operatorname{Re}\zeta_1 \leqslant 1, \operatorname{Im}\zeta_1 \geqslant 0\} \subset \mathbf{H}$: $\zeta_1 \approx \alpha^{-1}(\zeta - \mathrm{i}t(\alpha))$. The map $G \in S(\beta)$, $\beta = -1/\alpha$ is induced by the return map to $\tilde{\Delta}$ with the chart ζ_1.

Because we use the expansions of α into continuous fractions we prefer to get some map from $S(1/\alpha)$. This is done by conjugation of the map G with the map $\zeta = s + \mathrm{i}t \to -s + \mathrm{i}t$ which reverses the orientation. The new map will be denoted also by G.

Therefore we have the transformation from $S(\alpha)$ to $S(1/\alpha)$: $F \to G = G(F)$.

Note that if F is unstable, i.e. iterations of some points escape from the upper half-plane, then G is also unstable and the escape is quicker that for F. The following lemma provides a quantitative description of this.

LEMMA. *If a point ζ with $\operatorname{Im}\zeta > t(\alpha)$ escapes from* $\mathbf{H}$ *after m iterations of F, i.e. $F^m(\zeta) \notin \mathbf{H}$ and $F^i(\zeta) \in \mathbf{H}$, $i < m$, then there exists a point ζ_1 with $\operatorname{Im}\zeta_1 > \alpha^{-1}(\operatorname{Im}\zeta - t(\alpha) - C_1)$ which escapes from* $\mathbf{H}$ *after $m_1 < m$ iterations of G. Here C_1 is an universal constant such that G is well defined for* $\operatorname{Im}\zeta > t(\alpha) + C_1$.

Let us start to iterate the property of the latter lemma. Let the formulas

$$\alpha_0 = \alpha, \qquad \alpha_n^{-1} = a_{n+1} + \alpha_{n+1}$$

define the sequence of natural numbers a_n and $\alpha_n \in (0,1)$. We put $F_0 = F$, $F_1 = G(F_0) - a_1 \in S(\alpha_1), \ldots, F_{n+1} = G(F_n) - a_n$

If the point ζ_1 (from the latter lemma) satisfies the inequality $\operatorname{Im}\zeta_1 > t(\alpha_1)$, which is equivalent to

$$\operatorname{Im}\zeta > t(\alpha_0) + \alpha_0 t(\alpha_1) + C_1, \tag{3.18}$$

then $F_1^{m_1}(\zeta_1) \notin \mathbf{H}$, $F_2^{m_2}(\zeta_2) \notin \mathbf{H}$ for some ζ_1, ζ_2 and $0 < m_2 < m_1 < m$.

At the n-th repeating of this procedure the inequality (3.18) should be replaced by $\operatorname{Im}\zeta > t(\alpha_0) + \alpha_0 t(\alpha_1) + \alpha_0\alpha_1 t(\alpha_2) + \cdots + \alpha_0 \cdots \alpha_n t(\alpha_{n+1}) + C_1$. Because $t(\alpha) = -\ln\alpha/(2\pi) + C$, $\beta_n = \alpha_0 \cdots \alpha_n$ (see above), the latter finite series equals (up to constants) to the partial sum defining the series $\Phi(\alpha) = -\sum \beta_n \ln \alpha_{n+1}$ (defined in Proposition 3.62). By the Briuno condition this series is convergent.

We claim that the region $\operatorname{Im}\zeta > \sum \beta_n t(\alpha_{n+1}) + C_1$ is the region of stability of the map F. Assuming contrary, it should contain a point ζ_0 escaping from $\mathbf{H}$ after n_0 iterations

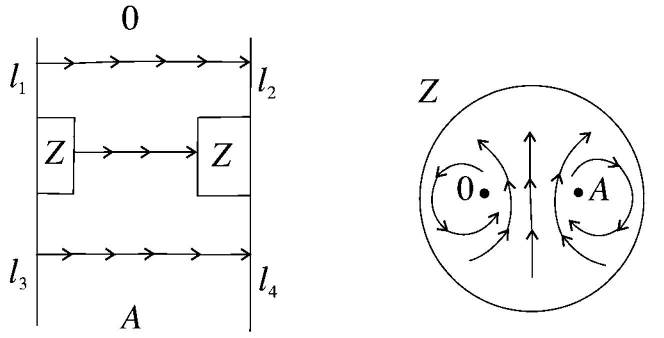

Fig. 10.

of $F = F_0$. This would imply existence of infinite sequence of points ζ_j escaping after n_j iterations of F_j. Moreover one should have infinite sequence of inequalities

$$0 < \cdots < n_2 < n_1 < n_0.$$

Of course, it is not possible.

3.7.2. *Proof of the non-linearizability* The idea of the proof of the second part of Theorem 3.60 is to construct a map from $S(\alpha)$ with a sequence of periodic orbits approaching the fixed point 0. By reversing and modifying the construction of Douady and Ghys one successively inserts fixed points near 0. In the next modification this fixed point becomes periodic and a new fixed point is inserted.

The reversing of the construction $F \to G(F)$, i.e. the transformation $G \to F(G)$, is presented at Fig. 10.

Let $G \in S(\beta)$, $0 < \beta < 1$, and let a be a positive integer. We put $\alpha = (\beta + a)^{-1}$ and let $\tilde{G} = G + a$.

We have the half-line $l_1 = \{\zeta = \mathrm{i}t, t > t_0\}$, its image $l_2 = \tilde{G}(l_1)$, the half-line $l_3 = \{\mathrm{i}t, t < t_0\}$ and the half-line $l_4 = l_3 + [\tilde{G}(\mathrm{i}t_0) - \mathrm{i}t_0]$. These curves bound a domain Λ, close to a vertical strip. Identifying the sides of this non-straight strip, we would obtain a surface $\Lambda/\sim$ with a holomorphic structure of $\mathbb{C}^*$. Because we want to have a disc, we delete form $\Lambda/\sim$ a disc Z around $\mathrm{i}t_0$ of size O(1), like at Fig. 10. Thus $(\Lambda/\sim) \setminus Z$ is the same as a doubly punctured disc $\mathbf{D} \setminus \{0, A\}$: $\mathrm{i}\infty$ is sent to 0 and $-\mathrm{i}\infty$ is sent to a point A. The set Z is sent to the outside of $\mathbf{D}$. (Of course, $(\Lambda/\sim) \setminus Z$ can be next covered by $\mathbf{H}$).

Let us define the dynamics on $(\Lambda/\sim) \setminus Z$, as induced by the translation $\mathrm{id}+1$. Because G commutes with $\mathrm{id}+1$ and l_4 is a translation of l_3 we obtain a well defined map of the quotient $(\Lambda/\sim) \setminus Z$. We denote it by $F = F(G)$. It is easy to see that $F \in S(\alpha)$ and that F has two fixed points in $\mathbf{D}$: 0 and A (see Fig. 10). Therefore we have inserted a new fixed point.

Before exploring this construction we have to perform some estimates. We want to insert the fixed point A as close to 0 as possible. However there are some restrictions implied by the geometry of the closure of $(\Lambda/\sim) \setminus Z$. When we delete from it a disc containing 0 and A then we obtain a domain Λ_1 in form of an annulus.

There is a conformal invariant of annulus, the modulus equal to $m = \ln(R/r)$ (see the Definition 3.75 below).

The modulus of the ring Λ_1 is of order $-\ln|A|$. On the other hand, this modulus can be estimated from Fig. 10. The domain Z is of order $\mathrm{O}(1)$ and the domain Λ is of order $\mathrm{O}(a) = \mathrm{O}(\alpha^{-1})$; (it is the width of Λ). This implies that the modulus of Λ_1 is of order $\ln\alpha$. The rigorous proof uses the estimate from the Theorem 3.76: $m \geqslant L^2/S$, where $S = \iint_{\Lambda-\Lambda_1} \rho$, $L = \inf \int_\gamma \rho$, where $\rho(\zeta) > 0$ is some function and γ's are paths joining the two components of the boundary of Λ_1. Yoccoz [87] uses $\rho(\zeta) = 1/|\zeta - \zeta_j|$ (in the ζ-plane).

This allows to get the estimate $|A| < \mathrm{const}\cdot\alpha$. This estimate, expressed in terms of coordinates at the upper half-plane, says that we have the fixed point ζ_0 of F and satisfying

$$\operatorname{Im}\zeta_0 > t(\alpha) - C_2$$

where $t(\beta) = -(\ln\beta)/(2\pi) + C$ is the same as in the previous proof and C_2 is some universal constant.

Notice also that, if G has some periodic orbit then F also has a periodic orbit. We have the following

LEMMA. *Let $0 < \beta < 1$, a be a positive integer, $\alpha = (\beta + a)^{-1}$ and $G \in S(\beta)$. There exists $F \in S(\alpha)$ such that*:

- *F has fixed point ζ_0 with $\operatorname{Im}\zeta_0 > t(\alpha) - C_2$.*
- *If G has a periodic orbit with the rotation number p/q (i.e. $G^q(\zeta) = \zeta + p$) then F has a periodic orbit $\{F^i(\zeta')\}$ with the rotation number $(a + p/q)^{-1}$ and such that*

$$\min \operatorname{Im} F^j(\zeta') > \alpha \cdot \min \operatorname{Im} G^i(\zeta) + t(\alpha) - \mathrm{const}.$$

Let α_n and a_n be the sequences defining the continued fraction expansion. For any natural m we construct series of maps $F_{m,m+1}, F_{m,m}, \dots, F_{m,0}$ as follows.

$F_{m,m+1} = \zeta + \alpha_{m+1} \in S(\alpha_{m+1})$, $F_{m,m} = F(F_{m,m+1})$ (with $a = a_{m+1}$) and other maps are constructed in the same way: $F_{m,l} = F(F_{m,l+1})$. Thus $F_{m,0} \in S(\alpha)$.

Moreover, each $F_{m,0}$ has periodic orbits with rotation numbers equal to the reduct p_n/q_n of the continuous fraction. The imaginary parts of such orbit are estimated from below by $\sim \sum_{i=1}^n \beta_{i-1}|\ln\alpha_i|$. Because the Briuno condition fails the latter series tends to infinity.

Now we choose a convergent subsequence from the sequence $F_{m,0}$. Its limit is just the F we are looking for. It has infinite series of periodic orbits with arbitrary large imaginary part. These periodic orbits form an obstacle to the linearizability of F. □

We shall apply the theorems of Briuno and of Yoccoz to germs of analytic planar vector fields with singular points of saddle type without resonance

$$\dot{x} = \alpha x(1 + \cdots), \qquad \dot{y} = -y(1 + \cdots), \tag{3.19}$$

where α is irrational.

As in the resonant case one associates with the analytic separatrix $x = 0$ the monodromy map $\Delta : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $\Delta(x) = \mathrm{e}^{-2\pi\mathrm{i}\alpha}x + \cdots$. Theorems 3.60 implies the following.

THEOREM 3.63. *If α satisfies the Briuno condition than any germ* (3.19) *is analytically orbitally linearizable.*

If α does not satisfy the Briuno condition then we cannot simply apply Theorem 3.60. We should have the realization theorem; any germ of non-resonant one-dimensional map is realized as a holonomy map of certain holomorphic foliation with saddle singularity. Such theorem was proved by R. Perez-Marco and C. Yoccoz [66] (see also [31]).

THEOREM 3.64. *Let α be irrational. The correspondence*:

$$\text{germ of vector field (3.19)} \to \text{the germ } \Delta \text{ of its monodromy}$$

is surjective. This implies that if α does not satisfy the Briuno condition, then there exists a germ (3.19) *of vector field which is not analytically linearizable.*

We do not present the proof of this result. It relies on gluing together certain domains with standard foliations and application the theory of almost complex structures to get a holomorphic foliation.

3.8. *Appendix 1. Complex structures and almost complex structures*

Consider the following situation. Let M^n be a n-dimensional complex manifold. Let us treat it as a $2n$-dimensional real manifold ${}^{\mathbb{R}}M^{2n}$. Assume now that the complex atlas on M is replaced by a C^1-smooth atlas on ${}^{\mathbb{R}}M^{2n}$. It turns out that the complex structure from M is not completely lost on ${}^{\mathbb{R}}M^{2n}$. There remains its trace called the almost complex structure.

DEFINITION 3.65. An *almost complex structure* on a real manifold M is a field $J = \{J_x\}$ of endomorphisms of the tangent spaces $T_x M$ satisfying the identity

$$J_x^2 + I = 0.$$

In other words, it is a field of splittings of the complexifications of tangent spaces into eigenspaces of J_x with eigenvalues $\mathrm{i} = \sqrt{-1}$ and $-\mathrm{i}$

$${}^{\mathbb{C}}T_x M = T_x M \otimes \mathbb{C} = T_x^{1,0} \oplus T_x^{0,1}$$

of equal (complex) dimension n.

The standard complex structure on a complex manifold N, with local complex coordinates $z_j = x_j + \mathrm{i}y_j$, induces the almost complex structure by the formulas $J\partial_{x_j} = \partial_{y_j}$, $J\partial_{y_j} = -\partial_{x_j}$.

DEFINITION 3.66. An almost complex structure J is called *integrable* if locally it is an image of complex structure by means of a C^1-mapping. It means that in neighborhood of

any point on M there exists a local system of C^1-coordinates x_j, y_j such that the above formulas hold.

If M is a real analytic $2n$-dimensional manifold and ${}^{\mathbb{C}}M$ is its local complexification (a complex $2n$-dimensional manifold containing M as its real part) then the integrability of the almost complex structure means simultaneous integrability of the two distributions $T^{1,0} = \bigcup T_x^{1,0}({}^{\mathbb{C}}M)$ and $T^{0,1}$. The local leaves of the corresponding foliations are $\{x_1 + \mathrm{i}y_1 = c_1, \dots, x_n + \mathrm{i}y_n = c_n\}$ and $\{x_1 - \mathrm{i}y_1 = d_1, \dots, x_n - \mathrm{i}y_n = d_n\}$, $x_j, y_j, c_j, d_j \in \mathbb{C}$.

The splitting of the complexifications of the tangent spaces induces analogous splitting of the complexifications of the cotangent spaces into forms of the type $(1, 0)$ and of the type $(0, 1)$,

$$ {}^{\mathbb{C}}T^*M = {}^{\mathbb{C}}\mathcal{E}^1 = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}; $$

ω is of the type $(1, 0)$ iff it vanishes at $T^{0,1}$. This induces splitting of all differential k-forms (with complex coefficients) into forms of the type (p, q).

As we know from Frobenius theorem (see [43]) the distributions $T^{1,0}$ and $T^{0,1}$ (of vectors of the type $(1, 0)$ and $(0, 1)$ respectively) are integrable iff they are involutive. This means that the commutator of any two vector fields from $T^{1,0}$ (respectively from $T^{0,1}$) is a vector field from $T^{1,0}$ (respectively from $T^{0,1}$). This condition can be written in the following compact form.

DEFINITION 3.67. The *torsion* of the almost complex structure J is the following bilinear form on the space of vector fields on M

$$ N(X, Y) = 2\big\{[JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\big\}. $$

Vanishing of the torsion is equivalent to the fact that the derivative of a form ω of the type $(1, 0)$ does not have component of the type $(0, 2)$. Note that, if $JX = \epsilon X$ and $JY = \epsilon Y$, $\epsilon = \pm\mathrm{i}$, then $N(X, Y) = -4\{[X, Y] + \epsilon \mathrm{i} J[X, Y]\}$.

The following result is known as the *Newlander–Nirenberg theorem*. In the case of real analytic manifold M this result follows from the above remarks and the Frobenius theorem. A. Newlander and L. Nirenberg proved it in the finitely smooth case.

THEOREM 3.68. (See [64].) *An almost complex structure is integrable iff its torsion vanishes.*

Now we pass to the 1-dimensional case. Here the problem of integrability of almost complex structures is treated differently. Assume that we are in a 2-dimensional plane (local chart) which we identify with the complex plane $\mathbb{C}$. Here the splitting of ${}^{\mathbb{C}}T^*\mathbb{C}$ can be defined by means of one 1-form

$$ \omega = \mathrm{d}z + \mu\, \mathrm{d}\bar{z} $$

where $\mu = \mu(z, \bar{z})$ is a function. We have $\mathcal{E}^{1,0} = \mathbb{C}\,\omega$ and the other space $\mathcal{E}^{0,1}$ is its conjugate.

The function μ is not defined uniquely. Notice that, when we replace the complex variable z by $w = \phi(z)$ (ϕ holomorphic), then the analogous 1-form $\tilde{\omega} = \mathrm{d}w + \tilde{\mu}\,\mathrm{d}\bar{w}$ takes the form

$$\phi'_z\,\mathrm{d}z + \tilde{\mu}\bar{\phi}'_z\,\mathrm{d}\bar{z} = \phi'_z\bigl[\mathrm{d}z + \tilde{\mu}\cdot(\bar{\phi}'_z/\phi'_z)\cdot\mathrm{d}\bar{z}\bigr]. \tag{3.20}$$

This rule of transformation of μ suggests the following definition.

DEFINITION 3.69. The quantity

$$\mu\cdot(\mathrm{d}\bar{z}/\mathrm{d}z)$$

is called the *Beltrami differential* defining the almost complex structure. In particular, $|\mu|$ is a well defined function.

The integrability of the almost complex structure, defined by the Beltrami differential, means existence of a differentiable complex function $\phi = \phi(z, \bar{z})$ which is a homeomorphism and such that the almost complex structure is defined by $\phi^*\,\mathrm{d}u = \mathrm{d}\phi = \phi_z\,\mathrm{d}z + \phi_{\bar{z}}\,\mathrm{d}\bar{z}$, i.e. the following *Beltrami equation*

$$\phi_{\bar{z}} = \mu\phi_z$$

holds. The Beltrami equation is a partial differential equation and is solved in a suitable Sobolev space. The ϕ's satisfying the Beltrami equation are called the *quasi-conformal maps* and $\mu = \mu_\phi$ is called the *modulus* of ϕ.

(Note that the torsion of any almost complex structure in $\mathbb{C}$ vanishes. It is because $\bigwedge^2 \mathcal{E}^1 \otimes \mathbb{C} = \mathcal{E}^{1,0} \wedge \mathcal{E}^{0,1}$ is one dimensional and contains only 2-forms of the type $(1, 1)$. Therefore here we have different kind of problems than in multi-dimensional case. In particular, one condition of integrability would be the preservation of the orientation by the map ϕ; it is equivalent to the property $|\mu| < 1$.)

The next theorem has many authors (L. Ahlfors, B. Bojarski, L. Bers).

THEOREM 3.70. (See [1].) *If μ is continuous and satisfies the estimate*

$$\bigl|\mu(x)\bigr| \leqslant k < 1$$

then the Beltrami equation has a solution and the almost complex structure defined by $\mu\,\mathrm{d}\bar{z}/\mathrm{d}z$ is integrable.

REMARK 3.71. If g is a conformal (i.e. analytic) mapping then $\phi \circ g$ is quasi-conformal with the modulus $\mu_{\phi\circ g} = \mu_\phi \circ g \cdot (\bar{g'}/g')$ and $g \circ \phi$ is quasi-conformal with $\mu_{g\circ\phi} = \mu_\phi$ (see (3.20)). This means that the Beltrami equation has many solutions.

If $\|\mu_\phi\|_\infty = k < 1$ then the map ϕ transforms infinitesimally small sphere around any z_0 to an infinitesimally small ellipse with the ratio of lengths of the semi-axes (longer to shorter) $K(z_0)$. We have $\sup K(z_0) = (1+k)/(1-k)$.

This implies that, if $\mu_\phi = \mu_\psi$ then the map $\phi \circ \psi^{-1}$ is conformal; ψ^{-1} maps circles to ellipses which are sent again to circles by ϕ.

Consider now the following situation. Let $f : \mathbb{C}^* = \mathbb{C} \setminus 0 \to S$ be a smooth diffeomorphism onto a Riemann surface S. Define $\mu = f'_{\bar{z}}/f'_z$. Assume that $\mu(z)$ can be continuously prolonged to $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$ with $\mu(0) = \mu(\infty) = 0$.

PROPOSITION 3.72. *In such a case there exists a conformal mapping $g : \mathbb{C}^* \to S$. This means that S has conformal type of $\mathbb{C}^*$ (not of an annulus or of a punctured disc).*

PROOF. We prolong μ to $\bar{\mathbb{C}}$ and integrate the corresponding Beltrami equation in the Riemann sphere. Using an eventual composition with a Möbius automorphism we can assume that the solution $u : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a homeomorphism such that $u(0) = 0$, $u(\infty) = \infty$.

We have $\mu_u = \mu_f$ in $\mathbb{C}^*$ and the map $u \circ f^{-1} : S \to \mathbb{C}^*$ is conformal (see the above remark). □

DEFINITION 3.73. If $Q \subset \mathbb{C}$ is a (curved) quadrangle, considered together with a pair of arcs on its boundary (b-arcs) then it can be transformed conformally onto a rectangle (with sides a, b), such that the b-arcs are sent to the vertical sides. The ratio of the lengths of sides is called the *modulus* of Q (see [1])

$$m(Q) = a/b.$$

If $P \subset \mathbb{C}$ is a domain, which can be conformally transformed to an annulus $\{r < |z| < R\}$, then we define its *modulus* as

$$m(P) = \ln(R/r).$$

$m(Q)$ is an invariant of conformal transformations; (by applying the Schwarz lemma one easily shows that two rectangles are conformally equivalent iff their moduli are the same).

The condition of quasi-conformality (for a map ϕ) says that $m(Q') < Km(Q)$ for any quadrangle Q transformed to a quadrangle Q'. Here $K = (1+k)/(1-k)$ is the constant from Remark 3.73. It means that K measures the distortion of small squares.

Note that the map $\ln z$ transforms the standard annulus, cut along a radius, into the rectangle with the sides $\ln R - \ln r$ and 2π. Thus, up to a constant, the modulus of a ring is the same as the modulus of corresponding quadrangle.

There is another definition of the modulus of quadrangle which does not make use of conformal mapping to rectangle.

THEOREM 3.74. (See [1].) *The modulus $m(Q)$ of a rectangle equals* $\sup_\rho L^2(\rho)/A(\rho)$, *where ρ are square integrable non-negative functions, $A(\rho) = \iint_Q \rho^2 \,\mathrm{d}x\,\mathrm{d}y$ and $L(\rho) = \inf \int_\gamma \rho|\mathrm{d}z|$ and γ are paths joining the b-arcs.*

3.9. *Appendix 2. Proof of the Hukuhara–Kimura–Matuda theorem*

(We follow the book [40] and also [89].) We shall present the proof only in the case $p = 1$. Therefore we assume that $x^2\,\mathrm{d}y/\mathrm{d}x = g_0(x) + y(1+\lambda x) + y^2 g_2(x, y)$ where g_0, $y^2 g_2$ have zero at $x = y = 0$ of order $\geqslant 3$ (compare Lemma 3.36).

Let $S = \{-\pi/2+\alpha < \arg x < 3\pi/2-\alpha, |x| < \epsilon\}$ be a sector in the x-plane (see Fig. 11); the case of opposite sector is considered analogously. We seek a change $(x, y) \to (x, z)$ such that $x^2\,\mathrm{d}z/\mathrm{d}x = z(1 + \lambda z^2)$ in S.

The proof is divided into two parts: existence of analytic central manifold in S_1 and analyticity of $z(x, y)$.

Existence of the center manifold. We seek it in the form of a graph

$$y = u(x)\mathrm{e}^{-1/x}, \quad x \in S,$$

where $u(x)\mathrm{e}^{-1/x} = \mathrm{O}(x)$ in the sector. We obtain the following equation onto u

$$x^2 u' = F(x, u) \tag{3.21}$$

where $F = g_0\mathrm{e}^{1/x} + \lambda x u + u^2\mathrm{e}^{-1/x} g_2(x, u\mathrm{e}^{1/x})$. We seek the solution to (3.21) in the class of functions satisfying the estimate

$$|u| < K|x|\mathrm{e}^{\mathrm{Re}(1/x)} \tag{3.22}$$

for some constant K.

The right-hand side of (3.21) satisfies the estimates: $|F(x, u_1) - F(x, u_2)| < \delta|u_1 - u_2|$ and $|F| < (K_0 + \delta K)|x|\mathrm{e}^{\mathrm{Re}(1/x)}$ for some small δ and fixed K_0. The first (Lipschitz) condition will guarantee the uniqueness of the solution.

As in other similar problems, Eq. (3.21) is replaced by the integral equation

$$u(x) = \int_0^x F\bigl(s, u(s)\bigr)s^{-2}\,\mathrm{d}s,$$

which is a fixed point equation $u = T(u)$ for a non-linear operator T. We use the following *Leray–Schauder–Tikhonov fixed point theorem*:

Any continuous map of a compact and convex subset of a locally convex space into itself contains a fixed point.

In our case the set of u's satisfying (3.22) is convex and compact in the topology of almost uniform convergence. It remains to show that the operator T maps the set defined by (3.22) into itself.

The latter statement is proved by appropriate choosing the path γ of integration in the formula defining T. This path is presented at Fig. 11.

Let $s = r\mathrm{e}^{\mathrm{i}\theta}$. We have the straight segment γ_1 joining 0 with $x_0 = r_0\mathrm{e}^{\mathrm{i}\pi/2+2\eta}$, where $\eta > 0$ is a small constant and r_0 will be defined later. In the second part γ_2 we have we have either $r(\theta) = r_0\sqrt{\sin(\theta - \pi/2 - \eta)}$ or $r_0\sqrt{\sin(\pi/2 + 3\eta - \theta)}$, depending on whether $\arg x > \pi/2 + 2\eta$ or not. If the angle α (defining the sector S, see above) is greater than 4η

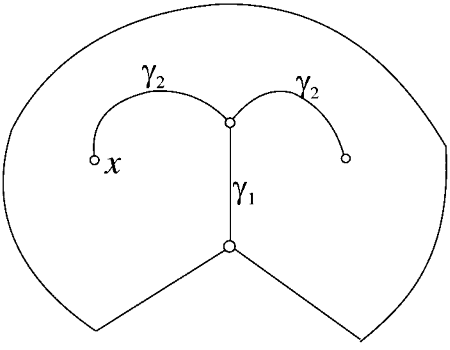

Fig. 11.

then $r_0 < r(\theta) < Mr_0$ (where $M = M(\eta)$). Now we chose r_0 such that Mr_0 is sufficiently small, $< \epsilon$.

One can show that for large K the quantity $|Fs^{-2}|$ is bounded by maximum of the absolute values of some derivatives of $K|s|\mathrm{e}^{\mathrm{Re}(1/s)}$ along $\gamma_{1,2}$ (see [40]). This gives $|T(u)| = |\int Fs^{-2}| \leqslant K|x|\mathrm{e}^{\mathrm{Re}(1/x)}$.

2. Now we can assume that $x^2\,\mathrm{d}y/\mathrm{d}x = y(1+\lambda x + yg_2)$ in S. We have to find the change $y \to z(x,y)$ giving the normal form $x^2z' = z(1+\lambda x)$.

We represent y as a function of (x,z) : $y = z + \phi(x,z)$. The function ϕ satisfies the following partial differential equation $x^2\phi_x + (1+\lambda x)z\phi_z = G(x,z,\phi)$.

We solve it using the method of characteristics: $\phi(x,z) = \psi(t)$ where $\dot{x} = x^2$, $\dot{z} = z(1+\lambda x)$, $\dot{\psi} = G(x,z,\psi)$. Here t can be replaced by x, i.e. we get $x^2\psi'_x = G(x,z,\psi)$, $z = Cx^\lambda \mathrm{e}^{-1/x}$. This is a family of equations depending on C.

Now we proceed as in the previous point. We put $\psi = w\mathrm{e}^{-1/x}$ where $w = w(x,C)$ satisfies the equation

$$x^2w' = G(x,z,w), \quad z = Cx^\lambda \mathrm{e}^{-1/x},$$

analogous to (3.21). We replace it by the integral equation

$$w(x,C) = \int_0^x G\bigl(s, Cs^\lambda \mathrm{e}^{-1/s}, w(s,C)\bigr)s^{-2}\,\mathrm{d}s,$$

which we solve in the space of w's satisfying the estimate $|w| < K|z|^2$ for some large constant K.

Using the same estimates as in the previous point (with the same choice of path of integration) one shows that this integral operator satisfies the assumption of the Leray–Schauder–Tikhonov theorem.

We refer the reader to [40] for more details. □

4. Bogdanov–Takens singularity

The subjects of investigation in this section are germs at $0 \in \mathbb{C}^2$ of analytic vector fields of the form

$$\dot{x} = y + \cdots, \qquad \dot{y} = \cdots, \tag{4.1}$$

i.e. with nilpotent linear part. Our aim is to find orbital normal forms for them (see Definition 3.1).

There was done some work on the formal orbital normal forms for the systems with nilpotent linear part. F. Takens [78] in 1974 proved that the system (4.1) can be formally reduced to

$$\dot{x} = y + a(x), \qquad \dot{y} = b(x) \tag{4.2}$$

where

$$a(x) = a_r x^r + a_{r+1} x^{r+1} + \cdots, \qquad b(x) = b_{s-1} x^{s-1} + b_s x^s + \cdots \tag{4.3}$$

are some formal power series (see Theorem 4.2 below). Some authors use the following, equivalent to (4.2), prenormal form (see Remark 4.3 below)

$$\dot{x} = y, \qquad \dot{y} = b(x) + yc(x).$$

Since these forms are not the complete normal forms, we shall call them the *Takens prenormal forms*. In order to obtain the form (4.2) Takens uses only changes of variables x, y but not the time.

DEFINITION 4.1. (See [76].) If $s < 2r$ in (4.3) then we call this case the *generalized cusp case*. The case with $s > 2r$ will be called the *generalized saddle–node case* and the case with $s = 2r$ we call the *generalized saddle case*.

4.1. *Formal orbital normal form*

We begin with the following result of Takens

THEOREM 4.2. (See [78].) *There is a formal change of the variables* (x, y) *which reduces the vector field* (4.1) *to the Takens prenormal form* (4.2).

PROOF. We use a composition of infinite series of transformations of the form $\mathrm{id} + Z$, i.e.

$$(x, y) \to (x_1, y_1) = (x, y) + Z(x, y),$$

$x_1 = x + Z_1(x, y)$, $y_1 = y + Z_2(x, y)$, where $Z_1(x, y)$ and $Z_2(x, y)$ are homogeneous polynomials of degree k. We shall treat also Z as a vector field, $Z = Z_1\partial_x + Z_2\partial_y$.

Let $V = y\partial_x + \sum_{j>1} V^{(j)}(X)$, where $V^{(j)}$ are homogeneous summands of the vector field V. Comparing the terms of degree k in the transformed vector field $(\mathrm{id}+Z)^*V = (\mathrm{id}+Z)_*^{-1}V \circ (\mathrm{id}+Z)$ we obtain the equation

$$[Z, y\partial_x] + V^{(k)} = \left(V^{(k)}\right)^{\text{Takens}} \tag{4.4}$$

where $(V^{(k)})^{\text{Takens}}$ is a term from the Takens prenormal form, $(V^{(k)})^{\text{Takens}} = a_k x^k \partial_x + b_k x^k \partial_y$.

The operator $[\cdot, y\partial_x]$, written in the components, acts on $Z_1\partial_x + Z_2\partial_y$ as follows

$$Z_1\partial_x + Z_2\partial_y \to (Z_2 - y\partial Z_1/\partial x)\partial_x - y\partial Z_2/\partial x\partial_y. \tag{4.5}$$

Equation (4.4) is the *Takens homological equation* and operator (4.5) is the *Takens homological operator*.

Applying formula (4.5) to Eq. (4.4) we see that we can cancel all the terms in the y-component of $V^{(k)}$ divisible by y and, when Z_2 is fixed, we can also cancel the terms divisible by y in the x-component. In this way we obtain the formal Takens prenormal form (4.2). □

REMARK 4.3. The equivalent to (4.1) preliminary normal forms are

$$\dot{x} = y, \qquad \dot{y} = b(x) + yc(x)$$

and

$$\dot{x} = y + \mu x c(x), \quad \dot{y} = b(x) + yc(x), \quad \mu > 0.$$

Indeed, in the first case we make the substitution $y_1 = y + a(x)$. (Note that here we do not need any restrictions).

In the second case we make the substitution $y_1 = y + d(x)$ with $d(x)$ satisfying the equation $\mu x d' = -d + a(x)$. We have $d(x) = \mu^{-1}x^{-1/\mu}\int_0^x s^{1/\mu-1}a(s)\,\mathrm{d}s$. We see that, if $\mu > 0$, then $d(x)$ is well defined; as a formal power series if $a(x)$ is formal, and as an analytic function if $a(x)$ is analytic. We see also that $d(x) = \mathrm{const}\cdot x^r + \cdots$ if $a(x) = \mathrm{const}\cdot x^r + \cdots$.

Generally, if there exist two analytic separatrices $y = u_1x^r + \cdots$ and $y = u_2x^r + \cdots$ then one firstly transforms these separatrices to $y = u_{1,2}x^r$. In such case the vector field takes the form

$$V_H + g(x, y)E_H,$$

where V_H and E_H are given in (4.8) and (4.9) below (see [47] and [76]).

PROPOSITION 4.4. (See [13].) *Let $s < \infty$. There exists a change of the variables (x, y) and of the time which reduces vector field* (4.2) *to the form*

$$\left[y + a(x)\right]\partial_x + x^{s-1}\partial_y. \tag{4.6}$$

Moreover, if vector field (4.2) *is analytic then also the vector field* (4.6) *is analytic.*

PROOF. Assume that $b(x) = x^{s-1}b_1(x)$, $b_1(0) \neq 0$. Our aim is to reduce $b_1(x)$ to 1.

We apply the change $x_1 = x\lambda(x)$, $\mathrm{d}t/\mathrm{d}t_1 = \eta(x)$, which should satisfy the equations $\mathrm{d}x_1/\mathrm{d}t_1 = y + a_1(x_1)$, $\mathrm{d}y/\mathrm{d}t_1 = x_1^{s-1}$. Comparing the terms $x_1^{s-1}\partial_y$ and $y\partial_{x_1}$ we get the conditions $\eta(x)\cdot(x\lambda(x))' = 1$, $\eta(x)b_1(x) = \lambda^{s-1}(x)$, or the Bernoulli equation $x\lambda' + \lambda = b_1(x)\lambda^{1-s}$. It has the solution

$$\lambda(x) = sx^{-1}\left[\int_0^x \tau^{s-1}b_1(\tau)\,\mathrm{d}\tau\right]^{1/s}.$$

This solution is an analytic function when $b_1(x)$ is analytic. □

The form (4.6) will be called the *Bogdanov–Takens prenormal form*.

The formal orbital classification of vector fields (4.2) requires additional partition of the space of such vector fields into some classes. Let

$$n = \min\left(r, \frac{s}{2}\right) \in \frac{1}{2}\mathbb{Z}, \tag{4.7}$$

where r, s are defined in (4.3). Introduce a quasi-homogeneous gradation $\widetilde{\deg}$ by $\widetilde{\deg}\, x = -\widetilde{\deg}\,\partial_x = 1$, $\widetilde{\deg}\, y = -\widetilde{\deg}\,\partial_y = n$. Denote

$$V_H = (y + ax^n)\partial_x + bx^{2n-1}\partial_y, \tag{4.8}$$

the lowest $\widetilde{\deg}$-part of the vector field V in (4.2) with $\widetilde{\deg}\, V_H = n - 1$, and let

$$E_H = x\partial_x + ny\partial_y \tag{4.9}$$

be the *quasi-homogeneous Euler field* with $\widetilde{\deg} E_H = 0$.

Note that one can change the pair (a, b) in (4.8) to $(\nu^2 a, \nu b)$, $\nu \in \mathbb{C}^*$.

Putting formally $z = x^n$ and dividing by x^{n-1}, we get from (4.8) the linear system

$$\dot{z} = naz + ny, \qquad \dot{y} = bz. \tag{4.10}$$

Its eigenvalues are equal $\lambda_{1,2} = \frac{n}{2}(a \mp \sqrt{a^2 + 4b/n})$.

DEFINITION 4.5. We call (4.10) the *principal linear system*.

We make a subtler division (than in Definition 4.1) of the set of Bogdanov–Takens singularities:

- C_{odd}: generalized (odd) cusp with $(a, b) = (0, 1)$ and s odd; here the ratio $\lambda_1/\lambda_2 = -1$.
- $S_{k:-l}$: generalized saddle, or generalized cusp with $s = 2n = 2r$ even, when $\lambda_1/\lambda_2 = -k/l \in \mathbb{Q}_-$.
- S_λ: generalized saddle, when $\lambda_1/\lambda_2 = \lambda \in \mathbb{R}_- \setminus \mathbb{Q}$.
- F: generalized saddle with $\lambda_1/\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$.
- $N_{k:l}$: generalized saddle with $\lambda_1/\lambda_2 = k/l \in \mathbb{Q}_+$.

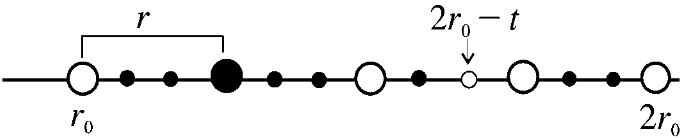

Fig. 12.

- N_λ: generalized saddle with $\lambda_1/\lambda_2 = \lambda \in \mathbb{R}_+$.
- SN: generalized saddle–node, i.e. $(a,b) = (1,0)$ and $\lambda_1/\lambda_2 = 0$.

Note that in the cases $S_\#$ (respectively F, $N_\#$, SN) the principal linear system (4.10) is a saddle (respectively focus, node, saddle–node).

In the following theorems we present a classification of vector fields from each of the above classes. These fields are of the form $V = V_H + W$, where W is of higher order with respect to the quasi-homogeneous. Each class contains the singularity V_H which is of infinite codimension (as $W \equiv 0$).

Below the notation $j = k(s)$ means $j = k \bmod s$.

THEOREM 4.6. (See [47,76].) *Any vector field from the class C_{odd} is formally orbitally equivalent either to V_H, or to $V_H + (x^r + \sum_{j \geqslant r, j \neq 0(s)} a_j x^j)\partial_x$, where $r > \frac{s}{2}$, $r \neq 0(s)$.*

EXAMPLE 4.7. In the usual cusp case, i.e. $s = 3$, equivalent to the above is the following form

$$V_H + \big[A_0(H) + xA_1(H)\big]E_H,$$

where $H = \frac{1}{2}y^2 - \frac{1}{3}x^3$ is the first integral (Hamiltonian) of V_H, used by F. Loray [48] and by M. Canalis-Durand and others [19,20].

THEOREM 4.8. (See [77].) *Any vector field from the class $S_{k:-l}$, $\gcd(k,l) = 1$, is formally orbitally equivalent either to V_H, or to*

$$V_H + x^t\big(1 + \phi(x)\big)\partial_x,$$

where $t > r$ and the series $\phi(x)$ are of the following forms (with $r_0 = (k+l)r$):

(a) $t \neq 0(r)$ *and* $\phi(x) = \sum_{j \neq 0(r_0),\ j+t \neq 0(r) \text{ or } j+t = r(r_0)} a_j x^j$ (*see Fig.* 12);

(b) $t = p_0 r_0 + r$ *and* $\phi(x) = a_{p_0 r_0} x^{p_0 r_0}$;

(c) $t = p_0 r_0 + r$ *and* $\phi(x) = a_{p_0 r_0} x^{p_0 r_0} + a_{j_0} x^{j_0} + \sum_{j > j_0, j \neq 0(r),\ j \neq j_0 + p_0 r_0} a_j x^j$, *where* $a_{j_0} \neq 0$ *for some* $j_0 > 0$, $j_0 \neq 0(r)$.

THEOREM 4.9. (See [77].) *Any vector field from the classes S_λ, F, N_λ, $N_{k:l}$ with $k, l > 1$, $\gcd(k,l) = 1$, is formally orbitally equivalent either to V_H, or to $V_H + (x^t + \sum_{j \neq 0(r),\ j > t} a_j x^j)\partial_x$ where $t \neq 0(r)$.*

THEOREM 4.10. (See [77].) *Any vector field from the class $N_{k:1}$ is formally orbitally equivalent either to V_H, or to $V = V_H + x^{kr}\partial_x$ for $k > 1$, or to $V = V_H + (cx^{kr} + x^t + \sum_{j\neq 0(r),\ j\neq t+kr,\ j>t} a_j x^j)\partial_x$ where $t > r$ and $t \neq 0(r)$, $k \geqslant 1$, and $c = 0$ for $k = 1$.*

THEOREM 4.11. (See [75].) *Any vector field from the class SN is formally orbitally equivalent either to V_H, or to*

$$V_H + x^{s-r-1}\big(1+\phi(x)\big)E_H,$$

where the series ϕ is of one of the following forms:
(a) $\phi = \sum_{j\neq 0(r)} b_j x^j$ *if $s \neq 0(r)$;*
(b) $\phi = b_{m_0 r}x^{m_0 r}$ *if $s = (m_0+2)r$;*
(c) $\phi = b_{m_0 r}x^{m_0 r} + b_{j_0}x^{j_0} + \sum_{j>j_0,\ j\neq 0(r),\ j\neq j_0+n_0 r} b_j x^j$ *where also $s = (m_0+2)r$ and $b_{j_0} \neq 0$, for $j_0 \neq 0(r)$.*

REMARK 4.12. The above theorems say nothing about the uniqueness of the presented normal forms. In fact, they are unique relative to an action of a finite group of changes. Each vector field is either of the type $V_H + x^m(1+\phi(x))\partial_x$ or of the type $V_H + x^m(1+\phi(x))E_H$. We consider the changes $x \to \alpha x$, where α is a root of unity of fixed order (e.g. $\alpha^{2r-s} = 1$ for C_{odd}) and we should have $\tilde{\phi}(x) \equiv \phi(\alpha x)$ in order that $V_H + x^m(1+\tilde{\phi}(x))\partial_x$ is equivalent to $V_H + x^m(1+\phi(x))\partial_x$, etc.

We begin the proof of Theorems 4.6, 4.8–4.10 and 4.11. They run along the same scheme. So, we present the general method, but detailed calculations will be provided only for resonant generalized saddle (Theorem 4.8).

The orbital changes rely on application of the conjugations

$$V \to \mathcal{P}_V(Z) := (\mathrm{Ad}_{\exp Z})_* V$$

and multiplications $V \to (1+\chi)V$. Here $Z = Z_1\partial_x + Z_2\partial_y$ is a vector field (formal or analytic), $\exp Z$ is the phase flow diffeomorphism (after the time 1), Ad_h is the adjoint action of a diffeomorphism h on the Lie algebra of vector fields and χ is a function (formal or analytic).

Note that if Z is parallel to V, $Z = \kappa(x,y)V$, then the map $\exp Z$ preserves the phase portrait of V and the field $\mathcal{P}_V(\kappa V)$ is also parallel to V. In order to avoid this ambiguity, one uses the notion of a bi-vector field introduced by R. Bogdanov in [13]. If $V = V_1\partial_x + V_2\partial_y$, then we define the *bi-vector field*

$$Z \wedge V = \Omega \cdot \partial_x \wedge \partial_y \tag{4.11}$$

where $\Omega(x,y) = V_2 Z_1 - V_1 Z_2$. One can say that Ω measures the component of Z transversal to V. If $\Omega = 0$ and V has isolated singularity, then $Z = \kappa V$ for some function κ.

This suggests that one should consider the map

$$Z \to \frac{\mathcal{P}_V(Z) \wedge V}{\partial_x \wedge \partial_y}, \tag{4.12}$$

from the space $(\mathbb{C}[[x,y]])^2$ of formal vector fields to the space $\mathbb{C}[[x,y]]$ of formal functions. Since the map $Z \to \mathcal{P}_V(Z)$ is non-linear, the map (4.12) cannot be factorized to a map from $\mathbb{C}[[x,y]]$ to $\mathbb{C}[[x,y]]$. But the linear part of $\mathcal{P}_V(\cdot)$ equals $-\mathrm{ad}_V(\cdot)$ and we have the following well the defined linear map

$$\mathcal{L}_V\Omega = -\frac{\mathrm{ad}_V Z \wedge V}{\partial_x \wedge \partial_y}.$$

Standard calculations give the following formula

$$\mathcal{L}_V\Omega = \dot{\Omega} - \operatorname{div} V \cdot \Omega \tag{4.13}$$

where $\dot{\Omega} = V(\Omega) = \partial\Omega/\partial V$ and div denotes the divergence.

We shall prove our theorems using the linear operator $\mathcal{L}_V$ (with different V's) and also the quasi-homogeneous gradation $\widetilde{\deg}$; we eliminate recursively terms of growing quasi-homogeneous degree. If $U = U_1\partial_x + U_2\partial_y$ is the part of V ('transversal' to V) devoted to killing, then we have the *bi-vector homological equation*

$$\mathcal{L}_V\Omega + \Theta = 0, \quad \Theta = \frac{U \wedge V}{\partial_x \wedge \partial_y}. \tag{4.14}$$

Having solved the bi-vector homological equation, we get some function Ω. Having the function Ω, we find the vector field from the equation $Z_1V_2 - Z_2V_1 = \Omega$; assuming $V_1 = y + ax^n + \cdots$, $V_2 = bx^{2n-1} + \cdots$ it is seen that the solution exists (provided that the expansion of Ω begins from terms of $\widetilde{\deg} \geqslant n+2$). The application of $(\mathrm{Ad}_{\exp Z})_*V$ destroys U, leaving only terms of the form $\sigma(x,y)V$. The latter will be reduced using the multiplication $(1-\sigma)V$. The terms non-linear in Z are of higher quasi-homogeneous degree and are eliminated in further steps of the recursive process.

The whole algorithm of reduction is divided into three general stages. Firstly we approximate the homological operator $\mathcal{L}_V$ by $\mathcal{L}_{V_H}$: we determine its kernel in $\mathbb{C}[[x,y]]$ and a subspace complementary to its image. Next, we use the operator $\mathcal{L}_{V_H + x^t\partial_x}$ to Ω's from $\ker \mathcal{L}_{V_H}$ to reduce some additional terms; here $x^t\partial_x$ is the first term not reduced in the previous step. In the third step we use the operator $\mathcal{L}_{V_H + x^t\partial_x + \mathrm{const}\cdot x^u\partial_x}$.

Denote by $\mathcal{X}_{m,i} \approx \mathbb{C}^{m+1}$ the space of polynomials Ω of the form

$$\Omega = \omega_0 x^{mn+i} + \omega_1 x^{(m-1)n+i} y + \cdots + \omega_m x^i y^m,$$

i.e. of quasi-homogeneous degree $\widetilde{\deg}\,\Omega = mn + i$.

LEMMA 4.13.

(a) *The equation $\mathcal{L}_V\Omega = 0$ means that the function Ω is the inverse integrating factor for the vector field V, i.e.* $\operatorname{div}(\Omega^{-1}V) = 0$.

(b) *In the case of generalized resonant saddle $S_{k:-l}$ the operator $\mathcal{L}_{V_H}$ acts between the spaces $\mathcal{X}_{m,i}$ and $\mathcal{X}_{m+1,i-1}$ (of dimensions $m+1$ and $m+2$ respectively) when $i > 0$, and between the spaces $\mathcal{X}_{m,0}$ and $\mathcal{X}_{m,r-1}$ (of the same dimension $m+1$) when $i = 0$.*

(c) *In the case of generalized resonant saddle* $S_{k:-l}$ *we have* $\ker \mathcal{L}_{V_H} = (y - u_1 x^r)^{pk+1} \times (y - u_2 x^r)^{pl+1} \cdot \mathbb{C}$, $r = n$, *if* $mn + i = (pk + pl + 2)r$, *i.e. in* $\mathcal{X}_{pk+pl+2,0}$, $p = 1, 2, \ldots$ (*here* $y = u_{1,2} z$ *are the equations of separatrices in* (4.10)).

PROOF. The first statement follows from the relation $\operatorname{div}(fV) = V(f) + f \cdot \operatorname{div} V$ and the second statement is rather obvious.

If Ω is the inverse integrating multiplier for V_H then the field $X_H = V_H/\Omega$ is Hamiltonian with quasi-homogeneous Hamilton function H, i.e. V_H has quasi-homogeneous first integral. The unique quasi-homogeneous first integrals take the form

$$H = (y - u_1 x^r)^{A\lambda_2} (y - u_2 x^r)^{-A\lambda_1}, \quad A \in \mathbb{C},$$

where $\lambda_{1,2}$ are the eigenvalues of the linear system (4.10). Then

$$X_H = V_H \cdot \left[(y - u_1 x^r)^{1 - A\lambda_2} (y - u_2 x^r)^{1 + A\lambda_1} \right]^{-1}.$$

We see that the only possibility that the function $\Omega = (y - u_1 x^r)^{1 - A\lambda_2} (y - u_2 x^r)^{1 + A\lambda_1}$ is a polynomial is just as in the statement (c). □

COROLLARY 4.14. *In the case* $i > 0$ *we have* $\ker \mathcal{L}_{V_H} | \mathcal{X}_{m,i} = 0$ *and* $\operatorname{Im} \mathcal{L}_{V_H} \mathcal{X}_{m,i}$ *is of codimension* 1, *with* $x^{(m+1)r+i-1} \cdot \mathbb{C}$ *as a complementary subspace to* $\operatorname{Im} \mathcal{L}_{V_H} \mathcal{X}_{m,i}$.

In the case $i = 0$ *either* $\ker \mathcal{L}_{V_H} | \mathcal{X}_{m,0} = 0$ *and* $\operatorname{coker} \mathcal{L}_{V_H} | \mathcal{X}_{m,0} = 0$, *or* $\ker \mathcal{L}_{V_H} | \mathcal{X}_{m,0} \neq 0$ *and* $\operatorname{Im} \mathcal{L}_{V_H} | \mathcal{X}_{m,i}$ *is of codimension* 1 *with* $x^{mr+r-1} \cdot \mathbb{C}$ *as a complementary subspace.*

PROOF. The only fact we have to prove is the statement about complementary subspaces. On the one hand, it follows from the Bogdanov–Takens prenormal form; i.e. the reduction to $(y + a(x))\partial_x + bx^{2r-1}\partial_y$ (see [76]).

One can see it also after composing $\mathcal{L}_{V_H}$ with the projection π of $\mathcal{X}_{m,0}$ onto the m-dimensional subspace of polynomials divisible by y. In the monomial basis in the case $i > 0$ the operator $\pi \circ \mathcal{L}_{V_H}$ becomes upper-triangular with entries $jr + i \neq 0$ on the diagonal. In the case $i = 0$ the left-down minor of dimension m is upper-triangular with entries $jr \neq 0$ on the diagonal. □

The polynomials from $\ker \mathcal{L}_{V_H}$, which are described in Lemma 4.18, form inverse integrating factors also for some vector fields which are not quasi-homogeneous.

Consider the function

$$F(x, y) = \frac{1}{(y - u_1 x^r)^{pk} (y - u_2 x^r)^{pl}} + \nu \cdot \ln \left[\frac{y x^{-r} - u_1}{y x^{-r} - u_2} \right]$$

where ν is a constant. It is the first integral of the vector field

$$V_H + \eta \cdot (y - u_1 x^r)^{pk} (y - u_2 x^r)^{pl} x^{r-1} E_H \tag{4.15}$$

where $\eta = \nu(u_2 - u_1)/p(k+l)$. The inverse integrating factor is

$$-(pk+pl)^{-1}(y-u_1x^r)^{pk+1}(y-u_2x^r)^{pl+1}, \tag{4.16}$$

the same as in Lemma 4.13(c).

Note that the vector field (4.15) consists of two quasi-homogeneous parts (of degrees $r-1$ and $r-1+rp(k+l)$). The first integral also consists of two quasi-homogeneous parts, of degrees $-(k+l)r$ and 0. Note also that the level curves $F(x,y)=f$, $f \neq \infty$ are non-algebraic, only the curve $F=\infty$ is algebraic.

It turns out that this example presents the only situations (in a sense), where a non-quasi-homogeneous vector field has a polynomial inverse integrating factor.

LEMMA 4.15. *Consider a vector field of the form $V = V_H + V_1$ where V_H has the principal linear part a $(k : -l)$-resonant saddle*, $\gcd(k,l)=1$, *and V_1 is a quasi-homogeneous field of degree $d = \widetilde{\deg} V_1 > r-1$and such that*

$$d+1 \neq 0(r).$$

Then V does not admit any inverse integrating factor of the form

$$\Omega_0 + \Omega_1 + \cdots \tag{4.17}$$

where $\Omega_0 \in \ker \mathcal{L}_{V_H}$, Ω_1 is a quasi-homogeneous polynomial of degree $d-r+1$ and "$\cdots$" means higher degree terms.

Moreover, for the vector field (4.15) *the only possible inverse integrating factor of the form* (4.17) (*with $\Omega_0 \in \ker \mathcal{L}_{V_H}$, and Ω_1 polynomial*) *is the function* (4.16).

PROOF. Following Remark 4.3 we can assume that V_1 of the form $g(x,y)E_H$. Then, after applying the quasi-homogeneous blowing-up $(x,u) \to (x,y) = (x, ux^r)$, we get the Bernoulli equation

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{u+a}{-r(u-u_1)(u-u_2)}x + \frac{\tilde{g}(u)}{(u-u_1)(u-u_2)}x^{d-r+2}$$

with $\tilde{g} = -g(1,u)/r$. This equation has the first integral

$$G = \frac{1}{(y-u_1x^r)^{\alpha}(y-u_2x^r)^{\beta}} + c\int^u \frac{\tilde{g}(u)\,\mathrm{d}u}{(u-u_1)^{\alpha+1}(u-u_2)^{\beta+1}},$$

where $\frac{\alpha}{\beta} = \frac{k}{l}$ and $(\alpha+\beta)r = d+1-r$.

The first integral G corresponds to the inverse integrating factor $M = (y-u_1x^r)^{\alpha+1}(y-u_2x^r)^{\beta+1}$ which cannot be polynomial (by our choice of d). Any other first integral for V is a functions of the primitive integral G. But taking it in the form $F = G^{\gamma}$ we find the inverse integrating factor equal $M \cdot G^{1-\gamma}$, which cannot be written in the form (4.17); it would be equal $\Omega_0 + \Omega_1 + \cdots$ where Ω_1 is not a polynomial. It is also clear that other

forms of first integral (like $F = G^{\gamma} + \text{const} \cdot G^{\delta}$) do not lead to a right inverse integrating factor.

The second statement is proved in the same way. $\square$

PROOF OF THEOREM 4.8. By Lemma 4.14 the operator $\mathcal{L}_{V_H}$ applied to the space $\mathcal{X}_{m,i}$ (of quasi-homogeneous Ω's) has non-zero kernel only in one case. It occurs when $mr + i = (pk + pl + 2)r$, i.e. $m = pk + pl + 2$, $i = 0$. This kernel is 1-dimensional and is generated by the polynomial $\Omega_0 = (y - u_1 x^r)^{pk+1}(y - u_2 x^r)^{pl+1}$.

The cokernel of $\mathcal{L}_{V_H} : \mathcal{X}_{m,0} \to \mathcal{X}_{m,r-1}$ is equal 0 for $m \neq pk + pl + 2$ and is 1-dimensional otherwise. Also $\operatorname{coker}(\mathcal{L}_{V_H} : \mathcal{X}_{m,i} \to \mathcal{X}_{m+1,i-1})$ is 1-dimensional for $i > 0$.

If $\operatorname{coker} \mathcal{L}_{V_H} \neq 0$ then we can assume that it is generated by a power $\Theta = x^{\beta}$, $\beta = (pk + pl + 2)r + r - 1$ or $\beta = (m+1)r + i - 1$ $(0 < i < r)$; it corresponds to the field (via $\Theta = W \wedge V_H / \partial_x \wedge \partial_y$):

$$W = x^{p(k+l)r+r} \partial_x \quad \text{or} \quad W = x^{(m-1)r+i} \partial_x$$

respectively. Instead of $x^{p(k+l)r+r-1} \partial_x$ in the first case we can take $W = H^p x^{r-1} E_H$, where $H = (y - u_1 x^r)^k (y - u_2 x^r)^l$. Therefore application of the homological operator, associated with the leading part of V_H, reduces V to

$$V_H + \sum b_{pj} H^p x^{r-1} E_H + \sum_m \sum_{i=1}^{r-1} a_{m,i} x^{mr+i} \partial_x \tag{4.18}$$

where b_p, $a_{m,i}$ are complex coefficients. We have three possibilities:

- $V = V_H$;
- $V = V_H + a_{m_0,i_0} x^{m_0 r + i_0} \partial_x + \cdots$, $i_0 \neq 0$, $a_{m_0,i_0} = 1$;
- $V = V_H + b_{p_0} H^{p_0} x^{r-1} E_H + \cdots$, $b_{p_0} = 1$.

In the second case dominating is the term $a_t x^t \partial_x$ with $t = m_0 r + i_0 \neq 0(r)$. Then we use the operator $\mathcal{L} = \mathcal{L}_{V_H} + \mathcal{L}_{x^t \partial_x}$ to functions

$$\Omega = \Omega_0 + \Omega_1, \quad \Omega_0 \in \ker \mathcal{L}_{V_H},$$

to reduce further terms. Here, by Lemma 4.13, $\Omega_0 = \text{const} \cdot H^j (y - u_1 x^r)(y - u_2 x^r)$, $j = 1, 2, \ldots$, and by Lemma 4.15 the corresponding operator defines an isomorphism between suitable spaces, of Ω's and of quasi-homogeneous functions Θ's (the kernel of $\mathcal{L}$ is zero). This gives the point (a) of Theorem 4.8.

In the third case dominating is the term $W_0 = H^{p_0} x^{r-1} E_H$, corresponding to $x^t \partial_x$ with

$$t = p_0 r_0 + r, \quad r_0 = (k + l)r$$

(see the cases (b) and (c) in Theorem 4.8). Thus we apply the operator $\mathcal{L} = \mathcal{L}_{V_H} + \mathcal{L}_{W_0}$ to functions of the form $\Omega_0 + \Omega_1$, where $\Omega_0 = c H^q (y - u_1 x^r)(y - u_2 x^r)$, $c \in \mathbb{C}$, and Ω_1 is quasi-homogeneous with $\widetilde{\deg} \Omega_1 - \widetilde{\deg} \Omega_0 = p_0 r_0$. Here $q = 1, 2, \ldots$.

Let $q = p_0$. By the example with the vector field (4.15) we see that the corresponding operator $\mathcal{L}$ has non-zero kernel $(\Omega_0 + \Omega_1) \cdot \mathbb{C} = (H^{p_0}(y - u_1 x^r)(y - u_2 x^r) + 0) \cdot \mathbb{C}$ and the

operator is not surjective. This means that the term $x^{t+p_0r_0}\partial_x = x^{r+2p_0r_0}\partial_x$ (corresponding to $H^{2p_0}x^{r-1}E_H$ in (4.18)) cannot be killed.

Let $q \neq p_0$. Then Lemma 4.15 asserts that the corresponding kernel is 0. Therefore all the terms $x^{t+qr_0}\partial_x$ with $q \neq p_0$, corresponding to $H^{p_0+q}x^{r-1}E_H$ in (4.18), can be killed.

In the case (b) form the thesis of Theorem 4.8 we have the situation where all other terms $a_{m,i}x^{mr+i}\partial_x$ in (4.18) are equal zero; (there remain only two terms $b_pH^px^{r-1}E_H$ with $p = p_0, 2p_0$).

If there exists a term $W_1 = a_{j_0}x^{j_0+t}\partial_x$, with j_0 not divisible by r, then we use the operator $\mathcal{L} = \mathcal{L}_{V_H} + \mathcal{L}_{W_0} + \mathcal{L}_{W_1}$ applied to

$$\Omega = \text{const} \cdot H^{p_0}(y - u_1x^r)(y - u_2x^r) + \Omega_1 + \Omega_2 = \Omega_0 + \Omega_2,$$

where $\Omega_0 \in \ker \mathcal{L}_{V_H+W_0}$, in order to kill additional term. Here $\ker \mathcal{L} = 0$ and the operator $\mathcal{L}$ is a surjection. This gives the case (c) of Theorem 4.8.

The remaining part of Theorem 4.8 is standard. □

Other theorems of this subsection are proved along the same lines as in the latter proof.

REMARK 4.16. Let us say some words about the classification of the Bogdanov–Takens singularities with respect to conjugation, not orbital equivalence (see Definition 3.1).

Firstly, A. Baider and J. Sanders [7,8] showed that in the generalized cusp case, i.e. $s < 2r$, the unique (non-orbital) normal form is following

$$V_H + \left(\alpha_{2s-1}x^{2s-1} + \sum_{j>s-1,\ j\neq -2,-1,r-2(s)} \alpha_j x^j\right)\partial_y + \left(\sum_{j\geqslant r-1,\ j\neq -1(s)} \beta_j x^j\right)(x\partial_x + y\partial_y).$$

They obtained also a unique normal form in the generalized saddle case, but it is more complicated (and we do not present it here).

Next, H. Kokubu, H. Oka and D. Wang [44] considered the case of non-resonant generalized saddle S_λ for $r = 2$ and $s = 4$. They reduced the system to so-called first order normal form (which is unique)

$$V_H + \left(\sum_{j\geqslant 4} \alpha_j x^j\right)\partial_y.$$

X. Wang, G. Chen and D. Wang [84] considered the generalized cusp singularity with $s = 3$ and $r \geqslant 2$. They obtained the following unique normal forms

$$V_H + \alpha_3x^3\partial_y + \bigl(xf(x^3) + x^3g(x^3)\bigr)y\partial_y, \qquad f(0) \neq 0,$$

and

$$V_H + (\alpha_3x^3 + \alpha_5x^5)\partial_y + \bigl(x^4f(x^3) + x^3g(x^3)\bigr)y\partial_y,$$

provided $110f(0) \neq 183\alpha_3 g(0)$.

Also A. Algaba, E. Freire and E. Gamaro [2] studied the problem and obtained some list of (non-orbital) normal forms.

Nevertheless the problem of classification of the nilpotent singularities with respect to formal conjugation is still not finished. Probably the list of formal orbital normal forms given above would be helpful to complete this classification.

4.2. *Analytic Takens prenormal form*

The main result of this subsection solves the old problem of analyticity of the prenormal form obtained by F. Takens.

THEOREM 4.17. (See [76,49].) *The Takens prenormal form $\dot{x} = y + a(x)$, $\dot{y} = b(x)$ of the singularity $\dot{x} = y + \cdots$, $\dot{y} = x^{s-1}\cdots$, $s < \infty$, obtained in the proof of Theorem* 4.2 *is analytic. This implies also that the Bogdanov–Takens prenormal form* (4.6) *is analytic.*

REMARK 4.18. The theorem holds true also when $s = 2$. Then the singular point is a saddle with the $1 : -1$ resonance and Theorem 4.18 gives an analytic prenormal form for such saddles.

The theorem holds true also when $s = \infty$, i.e. in the case of generalized saddle–node of infinite codimension. Here the vector field has formal first integral, $H(x, y) = y$ in the formal Takens coordinates. It is easy to get analyticity of this integral (see [75] and [86]).

Theorem 4.18 has two proofs. One proof from [76] relies on estimates of the successive approximations from the proof of the Takens theorem. It seems that it can be generalized to the cases of higher dimensional nilpotent singularities of vector fields. Another proof from [49] is geometrical and uses a preparation theorem for holomorphic foliations. Below we present the latter proof with details.

PROOF OF THEOREM 4.17. Following [49] we shall prove even more general statement:

if the linear part of a holomorphic germ $V = V_0$ of vector field at $0 \in \mathbb{C}^2$ is not radial, i.e. not of the form $\mu(x\partial_x + y\partial_y)$, then there exist local analytic coordinates x, y in which $V_0 = (y + a(x))\partial_x + b(x)\partial_y$ for some analytic germs a, b.

PROOF. By the non-radiality assumption it is easy to find linear coordinates in which $dV_0(0) = \left(\begin{smallmatrix} 0 & 1 \\ \alpha & \beta \end{smallmatrix}\right)$. Moreover, we can assume that $V_0 = f\partial_x + g\partial_y$ with $f(0, y) = y$.

In $\mathbb{C} \times \bar{\mathbb{C}}$ we consider the vertical line $L = 0 \times \bar{\mathbb{C}}$ together with its covering defined by $\Delta_0 = 0 \times \{|y| < r\}$ and $\Delta_\infty = 0 \times \{|y| > r/2\}$. Also we denote by $C = \Delta_0 \cap \Delta_\infty$ the intersection *corona* (see [49]).

If $r > 0$ is small enough then the vector field V is well defined on a neighborhood of the closure of Δ_0 and is transverse to this disc outside $y = 0$. By the flow-box theorem there exists a unique germ of a diffeomorphism $\Phi : (\mathbb{C}^2, C) \to (\mathbb{C}^2, C)$, such that $\Phi(0, y) = (0, y)$ and Φ conjugates V_0 to the horizontal vector field $V_\infty = y\partial_x$.

Let us glue the germs of holomorphic surfaces $(\mathbb{C} \times \bar{\mathbb{C}}, \Delta_0)$ and $(\mathbb{C} \times \bar{\mathbb{C}}, \Delta_\infty)$ along the corona by means of the diffeomorphism Φ. We obtain a germ of smooth complex surface

M along a rational curve L with a meromorphic vector field V. Since the x-component of V_0 at C agrees with $y\partial_x$, it follows that the Jacobian of the gluing map Φ takes the form $\mathrm{d}\Phi(0,y) = \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$ and the embedded rational curve L has zero self-intersection.

V. Savelev [69] proved that in this situation the line L is a regular fiber of some trivial fibration on M, i.e. there exist global coordinates $(x,y):(M,L) \to (\mathbb{C},0) \times \bar{\mathbb{C}}$ sending L onto $\{x=0\}$.

The vector field V has a simple pole along a curve, which we may assume still given by $\{y=\infty\}$. Since $\partial_y = -(1/y)^2 \partial_{1/y}$ has second order zero at $\{y=\infty\}$ we can write

$$V = \left[a_0(x) + a_1(x)y\right]\partial_x + \left[b_0(x) + b_1(x)y + b_2(x)y^2 + b_3(x)y^3\right]\partial_y. \qquad (4.19)$$

The vector field V has exactly one isolated zero, say at $x=y=0$.

The tangency divisor Σ between the foliation $\mathcal{F}$ induced by V and the fibration $\{x = \text{const}\}$ is a smooth curve $\Sigma = \{a_0(x) + a_1(x)y = 0\}$ intersecting transversally the line $L = \{x=0\}$. Indeed, the Jacobian of the change from the first chart to the global coordinates at the singular point fixes the y direction, so that the linear part of the vector field takes the form $\left(\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right)$, $l \neq 0$. Therefore $a_1(0) \neq 0$ in (4.19).

We can further simplify (4.19). Note that the admissible changes of the coordinates in $(\mathbb{C},0) \times \widehat{\mathbb{C}}$ are of the form $(x,y) \to (\phi(x), \frac{\alpha(x)y+\beta(x)}{\gamma(x)y+\delta(x)})$.

By straightening one leaf of the foliation $\mathcal{F}$ (i.e. through $(0,\infty)$) we can assume that $\{y=\infty\}$ is a leaf of $\mathcal{F}$. Thus $b_3(x) \equiv 0$ in (4.19).

Next, the change $y \to (\mathrm{e}^{\int b_2/a_1})y$ reduces $b_2(x)$ in (4.19) to zero.

By a change of the x coordinate we can assume that $a_1(x) \equiv 1$. The change $y \to y - a_1(x)$ straightens the curve Σ to $\{y=0\}$. Hence we can put $a_0(x) \equiv 0$.

We obtain the vector field $V = y\partial_x + [b_0(x) + b_1(x)y]\partial_y$ which is equivalent to the Takens field (4.2) (compare Remark 4.3). $\square$

Next to consider is the question of analyticity of the final orbital normal forms given in Theorems 4.6, 4.8–4.10 and 4.11.

But, as we noticed in Remark 4.18, in the case $s=2$ we have a $1:-1$ resonant saddle, where the formal normal form is not analytic (see Subsection 3.5). So there is no hope to expect that for $s \geqslant 3$ one could prove analyticity of the final formal orbital normal form.

This question for the standard cusp singularity, i.e. $s=3$ and $r=2$, was firstly considered in [19]. There the authors calculated numerically some initial terms in the Loray's normal form

$$V_H + \left(x + A_0(H)\right)E_H, \quad H = y^2/2 - x^3/3, \qquad (4.20)$$

i.e. the coefficients a_{3j} in the series $A_0(H) = a_3 H + a_6 H^2 + \cdots$ (compare Remark 4.7). These coefficients turned out growing like the Gevrey series of order 2, i.e.

$$a_{3j} \sim CA^{6j}\Gamma(\alpha + 6j) \qquad (4.21)$$

(compare Definition 3.48). Approximate values of the constants in (4.21) are $C \approx 7678015$, $A \approx 0.1803$ and $\alpha \approx -37.27$.

In [20] M. Canalis-Durand and R. Schäfke gave a rigorous proof of the divergence of the form (4.21). They proved the following

THEOREM 4.19. (See [20].) *For an analytic vector field*

$$V = V_H + \Delta(x, y) E_H, \tag{4.22}$$

with $H = \frac{1}{2}y^2 - \frac{1}{3}x^3$ *and analytic function* $\Delta = x + \cdots$, *there exists a unique formal series* $\Phi(x, y) = 1 + \cdots$ *such that the formal transformation* $(x, y) \to (x\Phi^2, y\Phi^3)$ *and division by* Φ *reduces* V *to the form* (4.20) *such that* $|a_{3j}| < CA^{6j}(6j)!$ *for some constants* C, A. *The series* $A_0(z^6)$ *is* 1*-summable if* $\arg z$ *is not congruent to* $\pi/6$ *modulo* $\pi/3$.

Moreover, for typical germs Δ, *i.e. outside a hypersurface in the space of such germs, the series* $A_0(H)$ *is divergent.*

REMARK 4.20. (About the proof of Theorem 4.19.) A general vector field which is tangent to the cusp $\{H(x, y) = 0\}$ is reduced to the form (4.22), after division by a non-zero function. The reduction used in the first statement of Theorem 4.19 comes from the Loray's algorithm [47].

The proof of Theorem 4.19 given in [20] is very short. However we cannot present it here with details. It uses so-called Borel transform $\mathcal{B}$ (see Definition 3.47), or its generalization to two variables, and its inverse (so-called Laplace transform).

The main idea of the proof is to solve the suitable conjugation equation in the 'Borel plane'. It turns out that the Borel transform of the series $\Phi(x, y)$ is a meromorphic function with poles. Namely, these poles are responsible for the divergence of the series Φ.

Finally, we note that in the sequent paper [21] Canalis-Durand and Schäfke proved an analogous result when $\Delta(x, y)$ in (4.22) starts from higher order terms and in (4.20) we have $V_H + (A_0(H) + xA_1(H))E_H$. Then the order of summability of the series A_0, A_1 depends on their first non-zero terms. □

REMARK 4.21. Recall that in Subsection 3.5 we defined certain holonomy map associated with a saddle or a saddle–node singularity of planar vector field. This holonomy map (or monodromy map) was associated with a loop in a separatrix of the singularity. Also with the nilpotent singularities one can associate certain 1-dimensional object. It is the so-called hidden holonomy group.

Namely, in the resolution π of the singularity one obtains some special divisor E with 3 singular points p_0, p_1, p_2 of the induced foliation $\mathcal{G} = \pi^*\mathcal{F}$. (In the case of ordinary cusp it is the divisor E_3 from Fig. 4.) The *hidden holonomy group* G is formed by the monodromy maps $\Delta_\gamma : D \to D$, of a small disc $D \approx (\mathbb{C}, 0)$ transversal to a point $q \in E^* = E \setminus \{p_0, p_1, p_2\}$. The loops γ run through the fundamental group $\pi_1(E^*, q)$.

In many situations the (analytic) orbital classification of the vector fields V is reduced to the classification of finitely generated subgroups $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ (obeying some natural relations) with respect to conjugation in the group $\mathrm{Diff}(\mathbb{C}, 0)$.

This is an interesting and advanced theory . The interested reader is referred to [23,31, 48,50,61,76,75] and [89] for details.

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